GLASNIK MATEMATIČKI
Vol. 41(61)(2006), 217 – 221

ON A FAMILY OF QUARTIC EQUATIONS AND A DIOPHANTINE PROBLEM OF MARTIN GARDNER

P. G. Walsh
University of Ottawa, Canada

Abstract. Wilhelm Ljunggren proved many fundamental theorems on equations of the form $aX^2 - bY^4 = \delta$, where $\delta \in \{\pm 1, 2, \pm 4\}$. Recently, these results have been improved using a number of methods. Remarkably, the equation $aX^2 - bY^4 = -2$ remains elusive, as there have been no results in the literature which are comparable to results proved for the other values of δ. In this paper we give a sharp estimate for the number of integer solutions in the particular case that $a = 1$ and b is of a certain form. As a consequence of this result, we give an elementary solution to a Diophantine problem due to Martin Gardner which was previously solved by Charles Grinstead using Baker's theory.

1. Introduction

Ljunggren [6, 7] proved a number of fundamental theorems on Diophantine equations of the form $aX^2 - bY^4 = c$, with $c \in \{\pm 1, 2, \pm 4\}$. Noticeably absent from this list is the value $c = -2$. Therefore, it seems worthwhile to pursue the Diophantine equation

$$aX^2 - bY^4 = -2. \tag{1.1}$$

In so doing, we necessarily restrict our attention to the case that a and b are odd integers for which ab is nonsquare, and for which the quadratic equation $aX^2 - bY^2 = -2$ is solvable in odd integers X, Y.

All positive integer solutions to the equation $aX^2 - bY^2 = -2$ arise from a minimal solution as follows. Assume that $(X, Y) = (T, U)$ is the smallest

2000 *Mathematics Subject Classification.* 11D25.
Key words and phrases. Diophantine equation, integer point, elliptic curve, Pell equation.

positive integer solution to $aX^2 - bY^2 = -2$, and let

$$\alpha = \frac{T\sqrt{a} + U\sqrt{b}}{\sqrt{2}}.$$

For $i \geq 1$ and odd, define sequences $\{T_i\}, \{U_i\}$ by

$$\alpha^i = \frac{T_i\sqrt{a} + U_i\sqrt{b}}{\sqrt{2}}.$$

Then all positive integer solutions to $aX^2 - bY^2 = -2$ are given by $(X, Y) = (T_i, U_i)$.

The following is the main result of this note, and to this author's knowledge, it represents the only result known on equation (1.1). We will see that a special case of this result yields an elementary solution to a problem of Martin Gardner [2], which was later solved by Grinstead [3] using Baker's method.

THEOREM 1.1. *Assume that* $d \equiv 0 \pmod 3$ *is a positive nonsquare integer for which the equation* $X^2 - dY^2 = -2$ *is solvable in odd integers* X, Y*, and assume further that the minimal solution* $(X, Y) = (T, U)$ *satisfies the property that* 3 *does not divide* U*. Then for any odd positive integer* A*, for which the Legendre symbol* $((2d/3)/p) = -1$ *for all primes* p *dividing* A*, and* $D = 9A^2d$*, the equation*

$$X^2 - DY^4 = -2 \tag{1.2}$$

has at most one solution in positive integers X, Y*.*

The proof will make use of Ljunggren's theorem on $X^2 - 2Y^4 = -1$ in [5] (see also [8]), and also a result of Chen and Voutier in [1] on the equation $X^2 - DY^4 = -1$. In particular, Ljunggren showed that the only positive integer solutions to $X^2 - 2Y^4 = -1$ are $(X, Y) = (1, 1), (239, 13)$, while for $D > 2$, Chen and Voutier proved that there is at most one positive integer solution to $X^2 - DY^4 = -1$.

2. PROOF

Let (x, y) denote a positive integer solution to (1.2). Then there is an odd positive k for which $U_k = 3Ay^2$. By the assumptions that d is divisible by 3 and U is not, it follows from the binomial theorem that 3 divides k, say $k = 3l$. By the divisibility properties of terms in the sequence $\{U_i\}$ (see [4]), it follows that $U_l = A_1w^2$ for some integer A_1 dividing A, and some positive integer w. The identity

$$U_{3l} = U_l(2U_l^2d - 3)$$

is readily verified from the equation

$$\frac{T_{3l} + U_{3l}\sqrt{d}}{\sqrt{2}} = \left(\frac{T_l + U_l\sqrt{d}}{\sqrt{2}}\right)^3.$$

From this we deduce that there is a positive integer z for which

$$2U_l^2 d - 3 = 3(A/A_1)z^2,$$

which can be rewritten as

$$(2d/3)A_1^2 w^4 - A_2 z^2 = 1,$$

where $A_2 = A/A_1$. The assumption on the prime factors of A implies that $A_2 = 1$, and hence that $A_1 = A$. Therefore, the above equation can be rewritten as

$$(2d/3)A^2 w^4 - z^2 = 1. \tag{2.1}$$

In the case that $A = 1$ and $d = 3$ (i.e. $D = 27$), Ljunggren's theorem implies that either $(z, w) = (1, 1)$, or $(z, w) = (239, 13)$. The latter pair of integers does not lead to a solution to $X^2 - 27Y^4 = -2$, while the pair $(z, w) = (1, 1)$ leads to the solution $(X, Y) = (5, 1)$. We therefore see that the complete solution to the quartic equation $X^2 - 27Y^4 = -2$ is an elementary consequence of Ljunggren's theorem on $X^2 - 2Y^4 = -1$.

In the case that $A > 1$ or $d > 3$, the result of Chen and Voutier implies that equation (2.1) has at most one solution in positive integers z, w, from which it immediately follows that there is at most one positive integer solution to equation (1.2).

Remark 2.1. We first remark that the bound of one solution in Theorem 1.1 is best possible. In particular, if $D = 9(m^2 + 2)$ for a positive integer m not divisible by 3, then D satisfies the hypotheses of Theorem 1.1, with $A = 1$ and $d = m^2 + 2$. If m is chosen to be a positive integer arising from a solution to $3n^2 - 2m^2 = 1$ (of which there are infinitely many), then $(X, Y) = (2m^3 + 3m, n)$ satisfy $X^2 - DY^4 = -2$.

3. A Diophantine problem of Martin Gardner

Martin Gardner [2] posed the problem of determining all those positive integers which are simultaneously triangular, square, and centered hexagonal. In other words, find all positive integer solutions to the system

$$k^2 + k = 2m^2, \qquad 3n^2 - 3n + 1 = m^2.$$

By the transformation (and relabelling) $x = 2k + 1, y = m, z = 2n - 1$, this system can evidently be rewritten as

$$x^2 - 8y^2 = 1, \qquad 4y^2 - 3z^2 = 1. \tag{3.1}$$

The system of simultaneous Pell equations in (3.1) was completely solved by Grinstead in [3], wherein he proved that the only solution in positive integers is $(x, y, z) = (3, 1, 1)$. The proof however uses deep results from Transcendance Theory in order to obtain a bound for the size of solutions, and then a very clever modular argument to show that any solution other than the known one must be extraordinarily large.

By subtracting the second equation in (3.1) from the first, we see that

$$x^2 - 12y^2 + 3z^2 = 0.$$

Furthermore, as 3 evidently divides x, putting $x_1 = x/3$, we obtain

$$4y^2 - z^2 = 3x_1^2,$$

which can be rewritten as

$$(2y - z)(2y + z) = 3x_1^2.$$

The two factors on the left are coprime, and so there are odd positive integers u and v, with $x_1 = uv$, for which

$$2y \pm z = 3u^2, \qquad 2y \mp z = v^2.$$

Therefore, $y = (v^2 + 3u^2)/4$, and since $x = 3x_1 = 3uv$, substituting x, y into the first equation in (3.1), and simplifying, gives

$$v^4 - 12u^2v^2 + 9u^4 = -2.$$

By completing the square, this equation can be rewritten as

$$(v^2 - 6u^2)^2 - 27u^4 = -2.$$

The remarks contained in the proof of Theorem 1.1 show that $u = v = 1$, which lead to the unique solution $(x, y, z) = (3, 1, 1)$ to equation (3.1), and hence the unique solution $(k, m, n) = (1, 1, 1)$ to Gardner's problem.

We emphasize here that the particular case of Theorem 1.1 required to solve Gardner's problem is a consequence of Ljunggren's theorem only, and not the aforementioned result of Chen and Voutier. Therefore, as Ljunggren's proof was of an elementary nature, the above proof provides an elementary solution to Gardner's problem.

Acknowledgement.

The author gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada.

References

[1] J. H. Chen and P. Voutier, *Complete solution of the Diophantine equation $x^2+1 = dy^4$ and a related family of quartic Thue equations,* J. Number Theory **62** (1997), 71-99.

[2] M. Gardner, *On the patterns and the unusual properties of figurate numbers,* Sci. Amer. **231** (1974), 116-121.

[3] C. M. Grinstead, *On a method of solving a class of Diophantine equations,* Math. Comp. **32** (1978), 936-940.

[4] D. H. Lehmer, *An extended theory of Lucas' functions,* Ann. of Math. (2) **31** (1930), 419-448.

[5] W. Ljunggren, *Zur Theorie der Gleichung $x^2 + 1 = Dy^4$,* Avh. Norske Vid. Akad. Oslo (1942), 1-27.

[6] W. Ljunggren, *Ein Satz über die Diophantische Gleichung $Ax^2 - By^4 = C$* ($C = 1, 2, 4$), in: Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, Lunds Universitets Matematiska Inst., Lund, 1954, 188–194.

[7] W. Ljunggren, *On the diophantine equation* $Ax^4 - By^2 = C (C = 1, 4)$, Math. Scand. **21** (1967), 149-158.
[8] W. Ljunggren, Collected Papers of Wilhelm Ljunggren, edited by Paulo Ribenboim, Queen's University, Kingston, 2003.

P. G. Walsh
Department of Mathematics
University of Ottawa
585 King Edward St.
Ottawa, Ontario
Canada
K1N-6N5
E-mail: gwalsh@mathstat.uottawa.ca

Received: 27.2.2006.
Revised: 16.4.2006.