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# THE NON-EXTENSIBILITY OF $D(4k)$-TRIPLES $\{1, 4k(k-1), 4k^2+1\}$ WITH $|k|$ PRIME

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Abstract. For a nonzero integer $n$, a set of $m$ distinct positive integers $\{a_1, \ldots, a_m\}$ is called a $D(n)$-$m$-tuple if $a_i a_j + n$ is a perfect square for each $i, j$ with $1 \le i < j \le m$. Let $k$ be an integer with $|k|$ prime. Then we show that the $D(4k)$-triple $\{1, 4k(k-1), 4k^2+1\}$ cannot be extended to a $D(4k)$-quadruple.

## 1. Introduction

Diophantus raised the problem of finding four (positive rational) numbers $a_1, a_2, a_3, a_4$ such that $a_i a_j + 1$ is a square of a rational number for each $i, j$ with $1 \le i < j \le 4$ and gave a solution $\{1/16, 33/16, 68/16, 105/16\}$. The first example of such a set containing only positive integers was found by Fermat: $\{1, 3, 8, 120\}$. Replacing "+1" by "$+n$" leads to the following definition.

Definition 1.1. *Let $n$ be a nonzero integer. A set of $m$ distinct positive integers $\{a_1, \ldots, a_m\}$ is called a $D(n)$-$m$-tuple (or a Diophantine $m$-tuple with the property $D(n)$, or a $P_n$-set of size $m$) if $a_i a_j + n$ is a perfect square for each $i, j$ with $1 \le i < j \le m$.*

In the case of $n = 1$, it has been known that any $D(1)$-triple can be extended to a $D(1)$-quadruple ([2]), while the folklore conjecture says that there does not exist a $D(1)$-quintuple. There are several results supporting the validity of this conjecture ([3, 6, 10, 12, 17, 22]), and it has been almost completely settled. More precisely, there does not exist a $D(1)$-sextuple and there exist only finitely many $D(1)$-quintuples ([12]). The case of $n = 4$ can be treated in a similar way to that of $n = 1$ ([18, 21]).

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Define

$$M_n := \sup\{|\mathcal{S}| : \mathcal{S} \text{ has the property } D(n)\}.$$

Considering congruences modulo 4, one may easily find that $M_n = 3$ if $n \equiv 2 \pmod 4$ ([4, 24, 28]). On the other hand, Dujella showed that if $n \not\equiv 2 \pmod 4$ and if $n \notin S := \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then $M_n \geq 4$ ([5]), and conjectured the following.

CONJECTURE 1.2 ([7]). If $n \in S$, then we have $M_n = 3$.

In the case of $n = -1$, there are several results supporting the validity of this conjecture ([1, 9, 14, 15, 19, 20]), and it has been almost completely settled. More precisely, there does not exist a $D(-1)$-quintuple ([15]) and there exist only finitely many $D(-1)$-quadruples ([14]). Suppose now that there exists a $D(-4)$-quadruple $\{a_1, a_2, a_3, a_4\}$. Then it induces a $D(-1)$-quadruple $\{a_1', a_2', a_3', a_4'\}$ such that $a_i = 2a_i'$ for each $i$ ([5, Remark 3]); hence, if Conjecture 1.2 is true for $n = -1$, then the same is true for $n = -4$.

As for general $n$, Dujella first gave an upper bound, depending only on $n$, for $M_n$ ([11]) and then improved it to obtain the following ([13]):

if $|n| \leq 400$, then $M_n \leq 31$;
if $|n| > 400$, then $M_n < 15.476 \log |n|$.

Moreover, in case $|n|$ is prime, Dujella and Luca gave an absolute upper bound for $M_n$: $M_n < 3 \cdot 2^{168}$ ([16]).

We are interested in the extensibility of $D(n)$-triples for large $n$ with $n \not\equiv 2 \pmod 4$. Brown showed that if $\{a_1, a_2, a_3, a_4\}$ is a $D(n)$-quadruple with $n \equiv 5 \pmod 8$, then $a_i \equiv 2 \pmod 4$ for each $i$ ([4]). For example, the $D(5)$-triple $\{L_{2k+1}, L_{2k+3}, L_{2k+5}\}$, where $k \geq 1$ and $L_m$ denotes the $m$-th Lucas number, cannot be extended to a $D(5)$-quadruple ([25]). By enhancing the results of Mootha and Berzsenyi [27], Dujella examined in detail possible types of $D(n)$-quadruples and $D(n)$-quintuples modulo 4 ([8]). In general, unless an argument using congruences modulo 4, 8 or 16 works, for large $n$ it seems to be more difficult to find a $D(n)$-triple $\{a, b, c\}$ which cannot be extended to a $D(n)$-quadruple than to find a large set with the property $D(n)$ (e.g., Gibbs found a $D(255104784)$-sextuple $\{3267, 11011, 17680, 87120, 234256, 1683715\}$; see [23]). The difficulty of finding such a triple comes from the necessity of finding the fundamental solutions of each of the Pell equations

$$ay^2 - bx^2 = 1, \quad az^2 - cx^2 = 1, \quad bz^2 - cy^2 = 1. \tag{1.1}$$

In this paper, we will give a $D(n)$-triple which cannot be extended to a $D(n)$-quadruple, for each of infinitely many $n$'s with $n \equiv 0 \pmod 4$; this is the first non-trivial complete non-extensibility result for large $n$.

We now consider the $D(4k)$-triple $\{1, 4k(k-1), 4k^2+1\}$ with $k \neq 0, 1$ an integer. Then the fundamental solutions of (1.1) are easily found, since each of $ab$, $ac$ and $bc$ is of Richaud-Degert type ([29, Section 3.2]). Moreover, in case $|k|$ is prime, we can completely determine the fundamental solutions of the Pell equation

$$z^2 - (4k^2+1)x^2 = -16k^3$$

(Proposition 2.4), which is the key to show our main theorem, that is:

THEOREM 1.3. *Let $k$ be an integer with $|k|$ prime. Then the $D(4k)$-triple $\{1, 4k(k-1), 4k^2+1\}$ cannot be extended to a $D(4k)$-quadruple.*

Note that this theorem contains the examples $\{1, 8, 17\}$, $\{1, 24, 37\}$, $\{1, 80, 101\}$ supporting the validity of Conjecture 1.2 for $n = 8, 12, 20$. We will divide the proof of Theorem 1.3 into four cases:

$$k < -2, \;\; k = -2, \;\; k > 2, \;\; k = 2.$$

In fact, we will show that certain simultaneous Pell equations have no solutions in each case of the above $k$'s (propositions 3.3, 3.4, 3.6 and 3.7), except for the case of $k = 2$; in this case, we will show that the system of equations has only the trivial solutions $(x, y, z) = (\pm 3, \pm 2, \pm 5)$. This implies the non-extensibility of the $D(8)$-triple $\{1, 8, 17\}$ (Remark 3.8). Theorem 1.3 immediately follows from these four propositions.

REMARK 1.4. Professor Andrej Dujella told us that if $k$ is an integer such that $k \equiv 3 \pmod 4$, then one may easily prove the non-extensibility of the $D(4k)$-triple $\{1, 4k(k-1), 4k^2+1\}$; hence, the main result is only interesting in the case where $k = \pm 2$ or $k = 4l + 1$ ($l \in \mathbb{Z}$) with $|k|$ prime. Indeed, each of the elements of the triple is congruent to 1, 8 or 5 modulo 16, while $4k$ is congruent to 12 modulo 16. This immediately leads to the non-existence of the fourth element, by noting that a square of an integer is congruent to 0, 1, 4 or 9 modulo 16. We would like to express our sincere thanks to Professor Dujella for pointing out this remark.

## 2. The key proposition

Assume that $\{1, 4k(k-1), 4k^2+1, d\}$ is a $D(4k)$-quadruple with $k \neq 0, 1$ an integer. Then there exist integers $x, y, z$ such that

$$d + 4k = x^2, \;\; 4k(k-1)d + 4k = 4y^2, \;\; (4k^2+1)d + 4k = z^2.$$

Eliminating $d$, we obtain the simultaneous Pell equations

$$y^2 - k(k-1)x^2 = -k(4k^2 - 4k - 1), \tag{2.1}$$

$$z^2 - (4k^2+1)x^2 = -16k^3, \tag{2.2}$$

$$k(k-1)z^2 - (4k^2+1)y^2 = -k(4k+1). \tag{2.3}$$

In this section, we will determine the fundamental solutions of the Pell equation (2.2) in case $|k|$ is an odd prime, which is crucial for the proof of Theorem 1.3. For this purpose, we need the following lemmas.

Lemma 2.1 ([31, Theorem 22, p. 209]). *Let $N \neq 0$ and $D > 0$ be integers with $D$ not a square. If $(X_0, Y_0)$ is a primitive solution of the Pell equation*

$$X^2 - DY^2 = N, \tag{2.4}$$

*then there exists an integer $j$ such that*

$$X_0 \equiv jY_0 \pmod N \text{ and } j^2 \equiv D \pmod N. \tag{2.5}$$

Here, a solution $(X_0, Y_0)$ is called primitive if $\gcd(X_0, Y_0) = 1$. We say that a solution $(X_0, Y_0)$ of (1.1) belongs to an integer $j$ if (2.5) holds. Note that the primitive solutions belonging to the same integer (modulo $N$) are in the same class ([31, Theorem 23, p. 209]).

Lemma 2.2 ([31, Theorem 25, p. 212]). *Let $N$ and $D$ be as in Lemma 2.1. Let $j$ be an integer such that $j^2 \equiv D \pmod N$ and put $M := (j^2 - D)/N$. The Pell equation (2.4) has a positive solution belonging to $\pm j$ if and only if the Pell equation*

$$T^2 - DU^2 = M$$

*has a positive solution $(T_0, U_0)$ belonging to $\pm j$.*

Lemma 2.3. *Let $N$ and $K$ be integers with $1 < |N| \leq K$. Then the Pell equation*

$$Z^2 - (K^2 + 1)X^2 = N$$

*has no primitive solution.*

Proof. Let $P_n/Q_n$ be the $n$-th convergent of $\sqrt{K^2+1}$. Then it is well known that the continued fraction expansion of $\sqrt{K^2+1}$ is $[K, \overline{2K}]$ and that

$$P_n^2 - (K^2 + 1)Q_n^2 = (-1)^n \quad \text{for all } n \geq 1, \tag{2.6}$$

which contradicts the assumption $1 < |N| \leq K$ and the fact that in case $|N| < \sqrt{K^2+1}$, the primitive solutions are obtained from the partial quotients ([31, Theorem 20, p. 204]). This completes the proof of Lemma 2.3. □

Now we are ready to prove our key tool which is the following proposition.

Proposition 2.4. *Let $k$ be an integer with $|k|$ an odd prime. Then the Pell equation (2.2) has exactly four classes of solutions to which the following (fundamental) solutions belong, respectively:*

$$(z, x) = (\pm 2k(2k-1), 2|k|),\ (\pm 2(2k^2 + k + 1), 2|k+1|).$$

Remark 2.5. Let $\{1, 4k^2 + 1, c\}$ be a $D(4k)$-triple. The former solutions $(z, x) = (\pm 2k(2k-1), 2|k|)$ correspond to $c = 4k(k-1)$, which leads us to consider the $D(4k)$-triple $\{1, 4k(k-1), 4k^2 + 1\}$.

PROOF. Finding the solutions of the equation (2.2) is equivalent to finding the primitive solutions of those six Pell equations given by dividing both sides of (2.2) by the squares of divisors of $4k$.

(i) Dividing (2.2) by $16k^2$, we have

$$Z^2 - (4k^2+1)X^2 = -k. \tag{2.7}$$

It follows from Lemma 2.3 that the Pell equation (2.7) has no (primitive) solution.

(ii) Dividing (2.2) by $4k^2$, we have

$$Z^2 - (4k^2+1)X^2 = -4k. \tag{2.8}$$

By Lemma 2.1, a primitive solution $(Z_0, X_0)$ of (2.8) belongs to an integer $j$ such that

$$Z_0 \equiv jX_0 \pmod{4k} \text{ and } j^2 \equiv 4k^2+1 \equiv 1 \pmod{4k}.$$

Since $|k|$ is an odd prime, the latter relation implies that $j \equiv \pm 1 \pmod{2k}$; hence we have

$$j \equiv \pm 1 \text{ or } \pm(2k-1) \pmod{4k}.$$

Suppose that $j \equiv \pm 1 \pmod{4k}$. By Lemma 2.2, if (2.8) has a positive solution belonging to $j$, then the equation

$$(Z')^2 - (4k^2+1)(X')^2 = k$$

must have a (primitive) solution, which contradicts Lemma 2.3.

Hence, the primitive solutions of (2.8) must belong to $\pm(2k-1)$. The equation (2.8) certainly has primitive solutions $(Z, X) = (\pm(2k-1), 1)$ belonging to $\pm(2k-1)$, respectively. Therefore, (2.8) has exactly two classes of solutions. The above solutions correspond to the solutions

$$(z, x) = (\pm 2|k|(2k-1), 2|k|)$$

of (2.8), respectively.

(iii) Dividing (2.2) by $k^2$, we have

$$Z^2 - (4k^2+1)X^2 = -16k. \tag{2.9}$$

By Lemma 2.1, a primitive solution $(Z_0, X_0)$ of (2.9) belongs to an integer $j$ such that

$$Z_0 \equiv jX_0 \pmod{16k} \text{ and } j^2 \equiv 4k^2+1 \pmod{16k}.$$

The latter relation implies that $j^2 \equiv 5 \pmod{16}$, which is impossible. Therefore, the equation (2.9) has no primitive solution.

(iv) Dividing (2.2) by 16, we have

$$Z^2 - (4k^2+1)X^2 = -k^3. \tag{2.10}$$

By Lemma 2.1, a primitive solution $(Z_0, X_0)$ of (2.10) belongs to an integer $j$ such that

$$Z_0 \equiv jX_0 \pmod{k^3} \text{ and } j^2 \equiv 4k^2 + 1 \pmod{k^3}.$$

The latter relation implies that $j \equiv \pm 1 \pmod{k^2}$; hence we have $j \equiv \alpha k^2 \pm 1 \pmod{k^3}$ for some integer $\alpha$. It follows that $(\alpha k^2 \pm 1)^2 \equiv 4k^2 + 1 \pmod{k^3}$, which implies that $\alpha \equiv \pm 2 \pmod{k}$. Hence we obtain

$$j \equiv \pm(2k^2 + 1) \pmod{k^3}.$$

The equation (2.10) certainly has primitive solutions $(Z, X) = (\pm(k^2 + (k+1)/2), |k+1|/2)$ belonging to $\pm(2k^2+1)$, respectively. Therefore, (2.10) has exactly two classes of solutions. The above solutions correspond to the solutions

$$(z, x) = (\pm 2(2k^2 + k + 1), 2|k+1|)$$

of (2.2), respectively.

(v) Dividing (2.2) by 4, we have

$$Z^2 - (4k^2 + 1)X^2 = -4k^3. \tag{2.11}$$

By Lemma 2.1, a primitive solution $(Z_0, X_0)$ of (2.11) belongs to an integer $j$ such that

$$Z_0 \equiv jX_0 \pmod{4k^3} \text{ and } j^2 \equiv 4k^2 + 1 \pmod{4k^3}.$$

The latter relation implies that $j \equiv \pm 1 \pmod{2k^2}$; hence we have $j \equiv 2\alpha k^2 \pm 1 \pmod{4k^3}$ for some integer $\alpha$. It follows that $(2\alpha k^2 \pm 1)^2 \equiv 4k^2 + 1 \pmod{4k^3}$, which implies that $\alpha \equiv \pm 1 \pmod{k}$. Hence we have

$$j \equiv \pm(2k^2 + 1) \text{ or } \pm(2k^3 - 2k^2 - 1) \pmod{4k^3}.$$

Suppose that $j \equiv \pm(2k^2 + 1) \pmod{4k^3}$. Lemma 2.2 implies that if the equation (2.11) has a positive solution belonging to $j$, then the equation

$$(Z')^2 - (4k^2 + 1)(X')^2 = -k$$

must have a (primitive) solution, which contradicts Lemma 2.3.

Suppose that $j \equiv \pm(2k^3 - 2k^2 - 1) \pmod{4k^3}$. Lemma 2.2 implies that if the equation (2.11) has a positive solution belonging to $j$, then the equation

$$(Z')^2 - (4k^2 + 1)(X')^2 = -k^3 + 2k^2 - k + 1 \tag{2.12}$$

also has a positive solution belonging to $j \equiv \pm(2k^2 - 2k + 1) \pmod{(k^3 - 2k^2 + k - 1)}$. Lemma 2.2 again implies that if (2.12) has a positive solution belonging to $j$, then the equation

$$(Z'')^2 - (4k^2 + 1)(X'')^2 = -4k \tag{2.13}$$

also has a positive solution belonging to $j \equiv \pm 1 \pmod{4k}$. If we repeat this process once again, we will find that the equation

$$(Z''')^2 - (4k^2 + 1)(X''')^2 = k$$

must have a (primitive) solution, which contradicts Lemma 2.3. Consequently, the equation (2.11) has no primitive solution.

(vi) Finally, we consider the equation (2.2) itself. By Lemma 2.1, a primitive solution $(z_0, x_0)$ of (2.2) belongs to an integer such that

$$z_0 \equiv jx_0 \pmod{16k^3} \text{ and } j^2 \equiv 4k^2+1 \pmod{16k^3}.$$

The latter relation implies that $j^2 \equiv 5 \pmod{16}$, which is impossible. Therefore, the equation (2.2) has no primitive solution. This completes the proof of the proposition. ∎

## 3. Proof of Theorem 1.3

In this section, we will show that the simultaneous Pell equations (2.1), (2.2), (2.3) have no solutions in case $|k|$ is prime, except for the case of $k = 2$ (in this case, we will show that they have only the solutions $(x, y, z) = (\pm 3, \pm 2, \pm 5)$), from which Theorem 1.3 will follow immediately.

Lemma 3.1. *Let $k$ be an integer with $|k| \geq 2$. Let $(y, x)$, $(z, x)$ and $(z, y)$ be positive solutions of* (2.1), (2.2) *and* (2.3)*, respectively. Then there exist non-negative integers $l, m, n$ and* (*fundamental*) *solutions $(y_0, x_0)$, $(z_1, x_1)$, $(z_2, y_2)$ of* (2.1), (2.2), (2.3) *respectively such that*

$$y + x\sqrt{k(k-1)} = (y_0 + x_0\sqrt{k(k-1)})(|2k-1| + 2\sqrt{k(k-1)})^l, \tag{3.1}$$

$$z + x\sqrt{4k^2+1} = (z_1 + x_1\sqrt{4k^2+1})(8k^2+1+4|k|\sqrt{4k^2+1})^m, \tag{3.2}$$

$$\begin{aligned} z\sqrt{k(k-1)} + y\sqrt{4k^2+1} &= (z_2\sqrt{k(k-1)} + y_2\sqrt{4k^2+1}) \\ &\quad \times(|8k^3-8k^2+2k-1| + 2|2k-1|\sqrt{k(k-1)(4k^2+1)})^n, \end{aligned} \tag{3.3}$$

*where in case $k \leq -2$ we may take*

$$|x_0| < -2k; \tag{3.4}$$

*in case $k \geq 2$ we may take*

$$|z_2| \leq \sqrt{(4k^2+1)(4k+1)} < 4.4k\sqrt{k}. \tag{3.5}$$

Proof. Note that by [30, Theorem 105],

$$(y, x) = (|2k-1|, 2), \ \ (z, x) = (8k^2+1, 4|k|),$$
$$(z, y) = (|8k^3-8k^2+2k-1|, 2|2k-1|)$$

are the fundamental solutions of the Pell equations

$$y^2 - k(k-1)x^2 = 1, \ \ z^2 - (4k^2+1)x^2 = 1, \ \ z^2 - k(k-1)(4k^2+1)y^2 = 1,$$

respectively. Hence, there exist an integer $m \geq 0$ and a solution $(z_1, x_1)$ of (2.2) such that (3.2) holds.

Suppose that $k \leq -2$. Then [30, Theorem 108] implies that in each class of solutions there exists a solution $(y_0', x_0')$ of (2.1) such that $0 \leq x_0' < -2k$

and with $|y_0'|$ minimal (in the class of solutions). For any positive solution $(y, x)$ of (2.1), the minimality of $|y_0'|$ allows us to write

$$y + x\sqrt{k(k-1)} = \pm(y_0' + x_0'\sqrt{k(k-1)})(-2k+1+2\sqrt{k(k-1)})^l$$

for some integer $l \geq 0$ and some sign. Putting $y_0 = \pm y_0'$ and $x_0 = \pm x_0'$, we see that (3.1) holds with (3.4). We may prove the statement for (3.3) and (3.5) in the same way as above, using [30, Theorem 108a] instead of [30, Theorem 108]. This completes the proof of Lemma 3.1. $\square$

We first consider the case of $k \leq -2$. By (3.1) and (3.2), we may write $x = p_l$, where

$$p_0 = x_0, \ p_1 = (-2k+1)x_0 + 2y_0, \ p_{l+2} = 2(-2k+1)p_{l+1} - p_l,$$

and $x = q_m$, where

$$q_0 = x_1, \ q_1 = (8k^2+1)x_1 - 4kz_1, \ q_{m+2} = 2(8k^2+1)q_{m+1} - q_m.$$

From these recursive sequences, we easily see by induction the following. Note that if $k$ is square-free, then $y_0 \equiv 0 \pmod{k}$, since $(y_0, x_0)$ is a solution of (2.1).

LEMMA 3.2. *Let $k \leq -2$ be a square-free integer. Then we have $p_l \equiv x_0 \pmod{2k}$ for all $l$ and $q_m \equiv x_1 \pmod{2k}$ for all $m$.*

PROPOSITION 3.3. *Let $-k > 0$ be an odd prime. Then the simultaneous Pell equations* (2.1) *and* (2.2) *have no solutions.*

PROOF. It suffices to show that $p_l \neq q_m$ for all $l, m$. Suppose that $p_l = q_m$ for some $l, m$. Lemma 3.2 implies that $x_0 \equiv x_1 \pmod{2k}$. Since Proposition 2.4 implies that $x_1 = \pm 2k$ or $\pm 2(k+1)$, it follows from (3.4) that $x_0 = 0$, $\pm 2$ or $\pm 2(k+1)$. If $x_0 = 0$, then $y_0^2 = -k(4k^2-4k-1) \equiv k \pmod{k^2}$, which contradicts the assumption that $-k$ is prime. If $x_0 = \pm 2$, then $y_0^2 = -k(2k-1)(2k-3)$, which is also a contradiction. If $x_0 = \pm 2(k+1)$, then

$$y_0^2 = k(k-1) \cdot 4(k+1)^2 - k(4k^2-4k-1) = 4k^4 - 3k \equiv -3k \pmod{k^2},$$

which means that $k = -3$. Hence we must have

$$y_0^2 = 4 \cdot 81 + 9 = 9 \cdot 37,$$

which is impossible. This completes the proof of Proposition 3.3. $\square$

Assume that $k = -2$, and consider the equations (2.1) and (2.3):

$$y^2 - 6x^2 = 46, \tag{3.6}$$
$$6z^2 - 17y^2 = -14. \tag{3.7}$$

PROPOSITION 3.4. *The simultaneous Pell equations* (3.6) *and* (3.7) *have no solutions.*

PROOF. From [30, Theorem 108a] it follows that the positive solutions of (3.7) are given by

$$z\sqrt{6} + y\sqrt{17} = (\pm 3\sqrt{6} + 2\sqrt{17})(101 + 10\sqrt{102})^n.$$

Hence, by (3.1) and the last formula we may write $y = \alpha_l$, where

$$\alpha_0 = 10,\ \alpha_1 = 50 \pm 36,\ \alpha_{l+2} = 10\alpha_{l+1} - \alpha_l,$$

and $y = \beta_n$, where

$$\beta_0 = 2,\ \beta_1 = 202 \pm 180,\ \beta_{n+2} = 202\beta_{n+1} - \beta_n.$$

This gives a contradiction, since by induction it follows that $\alpha_l \equiv 0, \pm 1 \pmod 5$ and $\beta_n \equiv 2 \pmod 5$. This completes the proof of Proposition 3.4. □

Secondly, we consider the case of $k \geq 2$. By (3.2) and (3.3), we may write $z = v_m$, where

$$v_0 = z_1,\ v_1 = (8k^2+1)z_1 + 4k(4k^2+1)x_1,\ v_{m+2} = 2(8k^2+1)v_{m+1} - v_m,$$

and $z = w_n$, where

$$\begin{aligned} &w_0 = z_2,\ w_1 = (8k^3 - 8k^2 + 2k - 1)z_2 + 2(2k-1)(4k^2+1)y_2, \\ &w_{n+2} = 2(8k^3 - 8k^2 + 2k - 1)w_{n+1} - w_n. \end{aligned}$$

From these recursive sequences, we easily see by induction the following.

LEMMA 3.5. *Let $k \geq 2$ be an integer. Then we have $v_m \equiv (-1)^m z_1 \pmod{2(4k^2+1)}$ for all $m$ and $w_n \equiv z_2 \pmod{2(4k^2+1)}$ for all $n$.*

PROPOSITION 3.6. *Let $k > 0$ be an odd prime. Then the simultaneous Pell equations (2.2) and (2.3) have no solutions.*

PROOF. It suffices to show that $v_m \neq w_n$ for all $m, n$. Suppose that $v_m = w_n$ for some $m, n$. Since Proposition 2.4 implies that $|z_1| = 2k(2k-1)$ or $2(2k^2+k+1)$, it follows from (3.5) that

$$\begin{aligned} |z_1 \pm z_2| \leq |z_1| + |z_2| &< 2(2k^2+k+1) + 4.4k\sqrt{k} \\ &= 4k^2\left(1 + \frac{1.1}{\sqrt{k}} + \frac{1}{2k} + \frac{1}{2k^2}\right) \\ &< 8k^2 < 2(4k^2+1); \end{aligned}$$

hence from Lemma 3.5 we obtain $|z_1| = |z_2|$. However, we know that

$$|z_1| \geq 2k(2k-1) = 4k^2\left(1 - \frac{1}{2k}\right) \geq \frac{10}{3}k^2 > 5k\sqrt{k} > |z_2|,$$

which is a contradiction. This completes the proof of Proposition 3.6. □

Assume that $k = 2$, and consider the equations (2.2) and (2.3):

$$z^2 - 17x^2 = -128, \tag{3.8}$$

$$2z^2 - 17y^2 = -18. \tag{3.9}$$

PROPOSITION 3.7. *The simultaneous Pell equations* (3.8) *and* (3.9) *have no solutions except* $(x, y, z) = (\pm 3, \pm 2, \pm 5)$.

REMARK 3.8. The solutions $(x, y, z) = (\pm 3, \pm 2, \pm 5)$ correspond to $d = 1$, where $d$ is as at the beginning of Section 2. Hence, Proposition 3.7 implies that the $D(8)$-triple $\{1, 8, 17\}$ cannot be extended to a $D(8)$-quadruple.

PROOF. One can prove this proposition using the standard method due to Baker and Davenport ([3]). By lemmas 3.1 and 3.5, the positive solutions of (3.8) and (3.9) are given by

$$z + x\sqrt{17} = (\pm 5 + 3\sqrt{17})(33 + 8\sqrt{17})^m,$$
$$z\sqrt{2} + y\sqrt{17} = (\pm 5 + 2\sqrt{17})(35 + 6\sqrt{34})^n,$$

respectively. Hence, we may write $z = v_m$, where

$$v_0 = \pm 5,\ v_1 = 408 \pm 165,\ v_{m+2} = 66v_{m+1} - v_m,$$

and $z = w_n$, where

$$w_0 = \pm 5,\ w_1 = 204 \pm 175,\ w_{n+2} = 70w_{n+1} - w_n.$$

We can transform the equation $v_m = w_n$ into an inequality for a linear form in three logarithms of algebraic numbers. Then, applying Baker's theory, e.g., a theorem of Matveev ([26]), we obtain $m < 2 \cdot 10^{15}$. Thus, the reduction method due to Dujella and Pethő ([17, Lemma 5a)]), based on the Baker-Davenport lemma ([3]), completes the proof of Proposition 3.7. In fact, in the first step of reduction we obtain $m \leq 4$; in the second step we obtain $m \leq 1$, which is a contradiction. ∎

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