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FINITE 2-GROUPS G WITH $|\Omega_2(G)| = 16$

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Abstract. It is a known fact that the subgroup $\Omega_2(G)$ generated by all elements of order at most 4 in a finite 2-group G has a strong influence on the structure of the whole group. Here we determine finite 2-groups G with $|G| > 16$ and $|\Omega_2(G)| = 16$. The resulting groups are only in one case metacyclic and we get in addition eight infinite classes of non-metacyclic 2-groups and one exceptional group of order 2^5. All non-metacyclic 2-groups will be given in terms of generators and relations.

In addition we determine completely finite 2-groups G which possess exactly one abelian subgroup of type $(4, 2)$.

1. Introduction

In Berkovich [1, Lemma 42.1] finite 2-groups G have been determined with the property $|\Omega_2(G)| \leq 8$. All the resulting groups (with $|G| > 8$) turn out to be metacyclic.

In this paper we shall determine finite 2-groups G with $|G| > 16$ and $|\Omega_2(G)| = 16$. We recall that $\Omega_2(G) = \langle x \in G \mid x^4 = 1\rangle$. The resulting groups are only in one case metacyclic and we get in addition eight infinite classes of non-metacyclic 2-groups and one exceptional group of order 2^5. All non-metacyclic 2-groups will be given in terms of generators and relations.

In section 2 we study the title groups G with the additional property that G does not possess an elementary abelian normal subgroup of order 8. Using the main theorem of Janko [4], we obtain three classes of such 2-groups and an exceptional group of order 2^5 (Theorem 2.1).

In section 3 we study the title groups G with the property that G is nonabelian and has a normal elementary abelian subgroup E of order 8. It is

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easy to see that G/E is either cyclic or a generalized quaternion group Q_{2^n} of order 2^n, $n \geq 3$. If G/E is cyclic, then we get three classes of 2-groups (Theorem 3.1). If G/E is generalized quaternion and $E \not\leq \Phi(G)$, then we get one class of 2-groups (Theorem 3.3). If G/E is generalized quaternion and $E \leq \Phi(G)$, we get two classes of 2-groups (Theorem 3.4).

In section 4 we investigate 2-groups G with $|G| > 16$ which possess exactly one subgroup of order 16 and exponent 4. It turns out that a 2-group G has this property if and only if $|G| > 16$ and $|\Omega_2(G)| = 16$ (Theorem 4.1). It is interesting to note that there are exactly five possibilities for the structure of $\Omega_2(G)$:

$$Q_8 * C_4,\ Q_8 \times C_2,\ D_8 \times C_2,\ C_4 \times C_2 \times C_2,\ \text{and } C_4 \times C_4.$$

In fact, if $\Omega_2(G) \cong Q_8 \times C_2$ or $D_8 \times C_2$ and $|G| > 16$, then we get exactly one 2-group (of order 2^5) in each case (Theorem 2.1 (d) and Theorem 3.1 (a) for $n = 2$). In other three cases for the structure of $\Omega_2(G)$ we get infinitely many 2-groups.

In section 5 we consider a similar problem. In Berkovich [2, Sect. 48] the 2-groups G have been considered which possess exactly one abelian subgroup of type $(4, 2)$. It was shown that either $|\Omega_2(G)| = 8$ (and then G is isomorphic to one of the groups in Proposition 1.4) or G possesses a self-centralizing elementary abelian subgroup of order 8 . We improve this result by determining completely the groups of the second possibility (Theorem 5.1). Finally, Theorem 5.2 shows that our result also slightly improves the classification of 2-groups with exactly two cyclic subgroups of order 4 (Berkovich[1, Theorem 43.4]).

We state here some known results which are used in this paper. The notation is standard (see Berkovich [1] or Janko [3]) and all groups considered are finite.

PROPOSITION 1.1. *(Janko [4]) Let G be a 2-group which does not have a normal elementary abelian subgroup of order* 8. *Suppose that G is neither abelian nor of maximal class. Then G possesses a normal metacyclic subgroup N such that $C_G(\Omega_2(N)) \leq N$ and G/N is isomorphic to a subgroup of D_8. In addition, $\Omega_2(N)$ is either abelian of type* $(4, 4)$ *or N is abelian of type* $(2^j, 2)$, $j \geq 2$.

PROPOSITION 1.2. *(Janko [4]) Let G be a 2-group which does not have a normal elementary abelian subgroup of order* 8. *Suppose that G has two distinct normal 4-subgroups. Then $G = D * C$ with $D \cong D_8$, $D \cap C = Z(D)$ and C is either cyclic or of maximal class distinct from D_8.*

PROPOSITION 1.3. *(Berkovich [1]) Let G be a minimal nonabelian p-group. If G is metacyclic, then*

$$G = \langle a, b \mid a^{p^m} = b^{p^n} = 1,\ a^b = a^{1+p^{m-1}} \rangle,$$

where $m \geq 2,\ n \geq 1,\ |G| = p^{m+n}$ *or* $G \cong Q_8$. *If* G *is non-metacyclic, then*

$$G = \langle a, b \mid a^{p^m} = b^{p^n} = c^p = 1,\ [a,b] = c,\ [a,c] = [b,c] = 1\rangle,$$

where $m \geq 1,\ n \geq 1,\ |G| = p^{m+n+1}$ *and if* $p = 2$, *then* $m + n > 2$.

PROPOSITION 1.4. *(Berkovich [1]) Let* G *be a group of order* $2^m,\ m \geq 4$, *satisfying* $|\Omega_2(G)| \leq 8$. *Then one of the following holds:*

(a) $G \cong M_{2^m}$.
(b) G *is abelian of type* $(2^{m-1}, 2)$.
(c) G *is cyclic.*
(d) $G = \langle a, b \mid a^{2^{m-2}} = b^8 = 1,\ a^b = a^{-1},\ a^{2^{m-3}} = b^4\rangle$, *where* $m \geq 5$.

PROPOSITION 1.5. *(Berkovich [2, Sect. 48]) Suppose that* G *is a metacyclic 2-group possessing a nonabelian subgroup* H *of order* 8. *Then* G *is of maximal class.*

PROOF. If $C_G(H) \not\leq H$, then $HC_G(H)$ is not metacyclic. Hence $C_G(H) \leq H$ and G is of maximal class by Proposition 1.12. □

PROPOSITION 1.6. *(Berkovich [1]) Let* G *be a nonabelian 2-group such that* $|G : G'| = 4$. *Then* G *is of maximal class.*

PROPOSITION 1.7. *(Berkovich [1]) Let* A *be an abelian subgroup of index* p *of a nonabelian* p*-group* G. *Then* $|G| = p|G'||Z(G)|$.

PROPOSITION 1.8. *(Berkovich [1]) Suppose that a* p*-group* G *has only one nontrivial proper subgroup of order* p^n *for some natural number* n. *Then* G *is cyclic unless* $p = 2$, $n = 1$ *and* G *is generalized quaternion.*

PROPOSITION 1.9. *(Janko [5, Proposition 1.10]) Let* τ *be an involutory automorphism acting on an abelian 2-group* B *so that* $C_B(\tau) = W$ *is contained in* $\Omega_1(B)$. *Then* τ *acts invertingly on* $\mho_1(B)$ *and on* B/W.

PROPOSITION 1.10. *(Berkovich [1]) If* G *is a 2-group such that* $\Omega_2(G)$ *is metacyclic, then* G *is also metacyclic.*

PROPOSITION 1.11. *(Berkovich [2, Sect.48]) Let* G *be a metacyclic 2-group of order* $> 2^4$. *If* $|\Omega_2(G)| = 2^4$, *then* $\Omega_2(G) \cong C_4 \times C_4$.

PROOF. Let G be a metacyclic group of order $2^m,\ m > 4$. Suppose that $|\Omega_2(G)| = 2^4$ and $\Omega_2(G) = H$ is nonabelian. If H is not minimal nonabelian, then Proposition 1.5 implies that G is of maximal class. But then $\Omega_2(G) = G$, a contradiction. Suppose that H has a cyclic subgroup of index 2. Since H is minimal nonabelian, $H \cong M_{2^4}$. But then $\Omega_2(H)$ is abelian of type $(4, 2)$ which contradicts the fact that $H = \Omega_2(G)$. Proposition 1.3 implies

$$H = \langle a, b \mid a^4 = b^4 = 1,\ a^b = a^{-1}\rangle.$$

It follows that $Z(H) = \langle a^2, b^2\rangle = \Omega_1(H) = \Omega_1(G)$, $H' = \langle a^2\rangle$, and $Z = \langle b^2\rangle$ is a characteristic subgroup of order 2 in H. Indeed, b^2 is a square in H,

a^2b^2 is not a square in H, and $Z(H) - H' = \{b^2, a^2b^2\}$. Thus Z is central in G. Since $H/Z \cong D_8$, Proposition 1.5 implies that G/Z is of maximal class. All involutions in G/Z lie in H/Z and so $G/Z \cong SD_{16}$ and $|G| = 2^5$. But there are elements of order 4 in $(G/Z) - (H/Z)$ whose square lies in $Z(G/Z) = \langle a^2, b^2\rangle/Z$. Hence there is an element $x \in G - H$ such that $x^2 \in \langle a^2, b^2\rangle$ and so $o(x) \leq 4$, which is the final contradiction. We have proved that H must be abelian and so $H \cong C_4 \times C_4$. □

Proposition 1.12. *(Berkovich [1]) Let G be a p-group with a nonabelian subgroup P of order p^3. If $C_G(P) \leq P$, then G is of maximal class.*

2. G has no normal E_8

In this section we prove the following result.

Theorem 2.1. *Let G be a 2-group of order $> 2^4$ satisfying $|\Omega_2(G)| = 2^4$. Suppose in addition that G has no normal elementary abelian subgroup of order 8. Then we have one of the following four possibilities:*

(a) *$G = D * C$ (the central product) with $D \cong D_8$, $D \cap C = Z(D)$, and C is cyclic of order ≥ 8. Here we have $\Omega_2(G) \cong Q_8 * C_4 \cong D_8 * C_4$.*
(b) *G is metacyclic of order $\geq 2^5$ with $\Omega_2(G) \cong C_4 \times C_4$.*
(c) *$G = QS$, where $Q \cong Q_8$, Q is normal in G, S is cyclic of order ≥ 16, $Q \cap S = Z(Q)$, and if C is the subgroup of index 2 in S, then $C_G(Q) = C$. Setting $Q = \langle a, b\rangle$ and $S = \langle s\rangle$, we have $a^s = a^{-1}$ and $b^s = ba$. Here we have $\Omega_2(G) \cong Q_8 * C_4$.*
(d) *The exceptional group G of order 2^5 has a maximal subgroup $M = \langle u\rangle \times Q$, where u is an involution, $Q \cong Q_8$, $C_G(u) = M$ and G is isomorphic to the group $A2(a)$ from Janko [3, Theorem 2.1]. We have $\Omega_2(G) = M \cong Q_8 \times C_2$.*

Proof. We may assume that G is nonabelian. If G were of maximal class, then $\Omega_2(G) = G$, a contradiction. Since G is neither abelian nor of maximal class, we may apply Proposition 1.1. Then G possesses a normal metacyclic subgroup N such that $C_G(\Omega_2(N)) \leq N$, G/N is isomorphic to a subgroup of D_8, and $W = \Omega_2(N)$ is either abelian of type $(4, 4)$ or N is abelian of type $(2^j, 2)$, $j \geq 2$. However, if $W \cong C_4 \times C_4$, then $W = \Omega_2(G)$ and Proposition 1.10 implies that G is metacyclic. This gives the case (b) of our theorem. We assume in the sequel that N is abelian of type $(2^j, 2)$, $j \geq 2$ and so $W = \Omega_2(N)$ is abelian of type $(4, 2)$.

Suppose that G has (at least) two distinct normal 4-subgroups. Then Proposition 1.2 implies that $G = D * C$ with $D \cong D_8$ and C is either cyclic or of maximal class and $D \cap C = Z(D)$. Since $|\Omega_2(G)| = 16$, C must be cyclic of order ≥ 8 and this gives the case (a) of our theorem. In what follows we shall assume that G has the unique normal 4-subgroup $W_0 = \Omega_1(W) = \Omega_1(N)$. Also we have $C_G(W) = N$.

Set $\tilde{W} = \Omega_2(G)$ so that $\tilde{W} \cap N = W$, $|\tilde{W} : W| = 2$, and $\tilde{W}$ is nonabelian. If $\tilde{W}$ is metacyclic, then by Proposition 1.10 the group G is metacyclic. But then Proposition 1.11 gives a contradiction. Thus $\tilde{W}$ is nonabelian non-metacyclic. In particular, $\exp(\tilde{W})$=4. Proposition 1.6 gives $|\tilde{W}/\tilde{W}'| \geq 8$ and so $|\tilde{W}'| = 2$. By Proposition 1.7, $|\tilde{W}| = 2|\tilde{W}'||Z(\tilde{W})|$ and so $|Z(\tilde{W})| = 4$ and $Z(\tilde{W}) < W$.

First assume that $Z(\tilde{W}) = \langle v \rangle \cong C_4$ and set $z = v^2$, $W_0 = \langle z, u \rangle = \Omega_1(W)$. Since $\tilde{W}' \leq \langle v \rangle$, we have $\tilde{W}' = \langle z \rangle$. For any $x \in \tilde{W} - W$, $x^2 \in \langle v \rangle$ since $x^2 \in Z(\tilde{W})$. But $\tilde{W}$ has no elements of order 8 and so $x^2 \in \langle z \rangle$ and $u^x = uz$. It follows that $W_0\langle x \rangle \cong D_8$ and so (since D_8 has five involutions) we may assume that x is an involution. Then $\tilde{W}$ is the central product of $\langle u, x \rangle \cong D_8$ and $\langle v \rangle \cong C_4$ with $\langle v \rangle \cap \langle u, x \rangle = \langle z \rangle$ and $(ux)^2 = z$. We set $Q = \langle ux, vx \rangle \cong Q_8$ and see that $\tilde{W} = Q * \langle v \rangle$ is also the central product of Q_8 and C_4. All six elements in $\tilde{W} - (Q \cup \langle v \rangle)$ are involutions and therefore Q is the unique quaternion subgroup of $\tilde{W} = \Omega_2(G)$ and so Q is normal in G. There are exactly three 4-subgroups in $\tilde{W}$ and one of them is $W_0 = \langle z, u \rangle$ and this one is normal in G. Since G has exactly one normal 4-subgroup, it follows that the other two 4-subgroups in $\tilde{W}$ are conjugate to each other in G. Set $C = C_G(Q)$. Obviously C is normal in G and $C \cap \tilde{W} = \langle v \rangle$. But $\langle v \rangle$ is the unique subgroup of order 4 in C and Proposition 1.8 implies that C is cyclic and $\tilde{W}C = Q * C$ with $Q \cap C = \langle z \rangle$. Set $Q = \langle a, b \rangle$, where we choose the generator a so that $u = av$. Since $W_0 = \langle z, u \rangle$ and $\langle v \rangle = Z(\tilde{W})$ are both normal in G, it follows that $\langle a \rangle$ is normal in G. The 4-subgroups $\langle bv, z \rangle$ and $\langle bav, z \rangle$ must be conjugate in G. Therefore $|G : (Q * C)| = 2$ and $G/(Q * C)$ induces an "outer" automorphism α on Q normalizing $\langle a \rangle$ and sending $\langle b \rangle$ onto $\langle ba \rangle$. A Sylow 2-subgroup of $Aut(Q)$ is isomorphic to D_8. It follows that $G/C \cong D_8$ and so there is an element $s \in G - (Q * C)$ such that $s^2 \in C$ and $b^s = ba$. Since $(ba)^s = b$, it follows that $b^s a^s = b$ and so $a^s = a^{-1}$. It remains to determine s^2. Set $\langle s \rangle C = S$ and we see that C is a cyclic subgroup of index 2 in S. Since $o(s) \geq 8$, we have $o(s^2) \geq 4$ and therefore $|Z(S)| \geq 4$. If S is nonabelian, then $S \cong M_{2^n}$ and there is an involution in $S - C$, a contradiction. Hence S is abelian. There are no involutions in $S - C$ and so S is cyclic. We may set $s^2 = c$, where $\langle c \rangle = C$. We compute

$$(sb)^2 = sbsb = s^2 b^s b = cbab = cabzb = ca.$$

If $|C| = 4$, then $C = \langle v \rangle$ and so ca is an involution. But then $o(sb) = 4$, a contradiction. Thus $|C| \geq 8$ and so $o(s) \geq 16$. We have obtained the groups stated in part (c) of our theorem.

Assume now $Z(\tilde{W}) \cong E_4$ so that $Z(\tilde{W}) = W_0 = \Omega_1(W) = \Omega_1(N) = \langle z, u \rangle$, where $z = v^2$ and $W = \langle u \rangle \times \langle v \rangle \cong C_2 \times C_4$.

If $\tilde{W}$ is minimal nonabelian, then the fact that $\tilde{W}$ is non-metacyclic implies with the help of Proposition 1.3 that

$$\tilde{W} = \langle a, b \mid a^4 = b^2 = c^2 = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle.$$

We see that $Z(\tilde{W}) = \Phi(\tilde{W}) = \langle a^2, c\rangle \cong E_4$ and $\tilde{W}$ has exactly three maximal subgroups $\langle a^2, c, b\rangle \cong E_8$, $\langle a, c\rangle \cong C_4 \times C_2$, and $\langle ba, c\rangle \cong C_4 \times C_2$ since $(ba)^2 = a^2c$. Hence $\langle a^2, c, b\rangle$ is normal in G, which is a contradiction.

We have proved that $\tilde{W}$ is not minimal nonabelian. Let D be a nonabelian subgroup of $\tilde{W}$ of order 8. If $u \in Z(\tilde{W}) - D$, then $\tilde{W} = \langle u\rangle \times D$ and $\Phi(\tilde{W}) = \tilde{W}' = \Phi(D) = \langle z\rangle$ since $z = v^2$.

Suppose first $D \cong D_8$. Let t be an involution in $\tilde{W} - W$. Since $[v, t] = z$, t inverts $\langle v\rangle$ and so we may set $D = \langle v, t\rangle$, where $\tilde{W} = \langle u\rangle \times D$. Since t acts invertingly on $\Omega_2(N) = W$, it follows that $C_N(t) = W_0 = \langle z, u\rangle$. Let x be an element of order 8 in $N - W$. Then Proposition 1.9 implies $x^t = x^{-1}w_0$ with $w_0 \in W_0$. But then

$$(tx)^2 = (txt)x = x^{-1}w_0x = w_0$$

and therefore $o(tx) \leq 4$ with $tx \notin \tilde{W}$, a contradiction. This proves that $N = W$ is abelian of type $(4, 2)$. Since t acts invertingly on $N = W$, all eight elements in $\tilde{W} - N$ are involutions. In particular, no involution in $\tilde{W} - N$ is a square in G and so $G/N \cong E_4$ and $|G| = 2^5$. All elements in $G - \tilde{W}$ must be of order 8 and so for any $y \in G - \tilde{W}$, $y^2 \in N - W_0$. Replacing $\langle uv\rangle$ with $\langle v\rangle$ and v with v^{-1} (if necessary), we may assume that $y^2 = v$. Since $C_G(N) = N$, $u^y = uz$ and so $y^u = yz$. Thus $\langle y, u\rangle \cong M_{2^4}$ and $\langle y, u\rangle > N$. The cyclic group $\langle v\rangle = Z(\langle y, u\rangle)$ of order 4 is normal in G and $C_G(v) = \langle y\rangle N \cong M_{2^4}$. Since ty is also an element of order 8, we have $(ty)^2 \in N - W_0$. If $(ty)^2 \in \langle v\rangle$, then $C_G(v) \geq \langle N, y, ty\rangle = G$, a contradiction. Thus $(ty)^2 \in \langle uv\rangle$. If $(ty)^2 = (uv)^{-1} = (uz)v$, then replacing u with uz, we may assume from the start that $(ty)^2 = uv$. This relation implies $t^y = tuz$ and so y normalizes the elementary abelian subgroup $E = \langle z, u, t\rangle$ of order 8. Since E is normal in $\tilde{W}$, E is also normal in $G = \langle \tilde{W}, y\rangle$, a contradiction.

Finally, suppose $D \cong Q_8$ so that $\tilde{W} = \langle u\rangle \times D$, $Z(D) = \langle z\rangle$, $z = v^2$, and $D \cap N = \langle v\rangle$. Let r be an element in $D - \langle v\rangle$. We have $r^2 = z$ with $z \in Z(G)$ and so r induces an involutory automorphism on the abelian group N of type $(2^j, 2)$, $j \geq 2$. Since r acts invertingly on $W = \langle u, v\rangle = \Omega_2(N)$, it follows that $C_N(r) = W_0 = \langle u, z\rangle$. Then Proposition 1.9 implies that r acts invertingly on N/W_0. Suppose that $W \neq N$ and let $x \in N - W$. We get $x^r = x^{-1}w_0$ with $w_0 \in W_0$ and so

$$(rx)^2 = rxrx = r^2x^rx = zx^{-1}w_0x = zw_0 \in W_0.$$

Since $rx \notin \tilde{W} = \Omega_2(G)$ and $o(rx) \leq 4$, we get a contradiction. We have proved that $N = W = C_G(N)$ is abelian of type $(4, 2)$.

Set $S = C_G(W_0)$ so that $S \geq \tilde{W}$ and S/N stabilizes the chain $N > W_0 > 1$. Hence S/N is elementary abelian of oder ≤ 4. Suppose $S > \tilde{W}$ so that S/N is a 4-group. Let $\tilde{S}/N$ be a subgroup of order 2 in S/N distinct from $\tilde{W}/N$. Thus $\tilde{S} \cap \tilde{W} = N$. Take an element $y \in \tilde{S} - N$ so that $o(y) = 8$ and therefore $y^2 \in N - W_0$. But then y centralizes $\langle y^2, W_0 \rangle = N$, a contradiction. Hence $S = \tilde{W} = C_G(W_0)$, $|G : \tilde{W}| = 2$ and $|G| = 2^5$. In particular, $C_G(u) = \langle u \rangle \times D$ with $D \cong Q_8$. Note that the involution u is contained in the unique normal 4-subgroup W_0 of G and $C_G(u) \neq G$. Then our group G is isomorphic to the group A2(a) of the Theorem 2.1 of Janko [3]. Our theorem is proved. □

3. G has a normal E_8

Throughout this section we assume that G is a nonabelian 2-group of order $> 2^4$ satisfying $|\Omega_2(G)| = 2^4$. We assume in addition that G has a normal elementary abelian subgroup E of order 8. Since G/E has exactly one subgroup $\Omega_2(G)/E$ of order 2, it follows that G/E is either cyclic of order ≥ 4 or generalized quaternion (see Proposition 1.8). If G/E is cyclic, then G will be determined in Theorem 3.1. If G/E is generalized quaternion and $E \not\leq \Phi(G)$, then G is determined in Theorem 3.3. Finally, if G/E is generalized quaternion and $E \leq \Phi(G)$, then G is determined in Theorem 3.4.

Theorem 3.1. *Let G be a nonabelian 2-group of order $> 2^4$ satisfying $|\Omega_2(G)| = 2^4$. Suppose in addition that G has a normal elementary abelian subgroup E of order 8 such that G/E is cyclic of order 2^n, $n \geq 2$. Then we have one of the following three possibilities:*

(a)
$$G = \langle a, e_1 \mid a^{2^{n+1}} = e_1^2 = 1,\ n \geq 2,$$
$$a^{2^n} = z,\ [a, e_1] = e_2,\ e_2^2 = [e_1, e_2] = 1,\ [a, e_2] = z \rangle$$
is a group of order 2^{n+3}, $Z(G) = \langle a^4 \rangle$, $G' = \langle z, e_2 \rangle$, $\Phi(G) = \langle a^2, e_2 \rangle$, $E = \langle z, e_1, e_2 \rangle$ is a normal elementary abelian subgroup of order 8, and $\Omega_2(G) = E\langle a^{2^{n-1}} \rangle$ is abelian of type $(4, 2, 2)$ for $n \geq 3$, and $\Omega_2(G) \cong C_2 \times D_8$ for $n = 2$.

(b) *$G \cong M_{2^{n+2}} \times C_2$ with $n \geq 2$, and $\Omega_2(G)$ is abelian of type $(4, 2, 2)$.*

(c)
$$G = \langle a, e_1 \mid a^{2^{n+1}} = e_1^2 = 1,\ n \geq 2,$$
$$[a, e_1] = e_2,\ e_2^2 = [e_1, e_2] = [a, e_2] = 1 \rangle$$
is minimal nonabelian, $|G| = 2^{n+3}$, $\Phi(G) = \langle a^2, e_2 \rangle$, $G' = \langle e_2 \rangle$, and $\Omega_2(G) = E\langle a^{2^{n-1}} \rangle$ is abelian of type $(4, 2, 2)$, where $E = \langle a^{2^n}, e_1, e_2 \rangle$ is a normal elementary abelian subgroup of order 8.

Proof. Let a be an element in $G - E$ such that $\langle a \rangle$ covers the cyclic group G/E of order 2^n, $n \geq 2$. Obviously G does not split over E and so $o(a) = 2^{n+1}$ and $\langle a \rangle \cap E = \langle z \rangle$ is of order 2, where $z = a^{2^n}$.

Suppose that a induces an automorphism of order 4 on E. In that case $C_E(a) = \langle z \rangle$ so that $Z(G) = \langle a^4 \rangle \geq \langle z \rangle$ and $C_G(E) = Z(G)E$. There are involutions $e_1, e_2 \in E$ such that $\langle z, e_1, e_2 \rangle = E$ and $e_1^a = e_1 e_2$, $e_2^a = e_2 z$. Then $C_E(a^2) = \langle z, e_2 \rangle = G'$, $G = \langle a, e_1 \rangle$, $\Phi(G) = \langle a^2, e_2 \rangle$, and $\Omega_2(G) = E\langle a^{2^{n-1}} \rangle$. We have determined the groups stated in part (a) of our theorem.

Suppose that a induces an involutory automorphism on E. We have $C_E(a) = E_1 = \langle z, e_2 \rangle \cong E_4$, $G' = [a, E]$ is a subgroup of order 2 in E_1, and $Z(G) = \langle a^2, E_1 \rangle$. If $G' = \langle z \rangle$, then $\langle a \rangle$ is normal in G. Let $e_1 \in E - E_1$. Then $\langle a, e_1 \rangle \cong M_{2^{n+2}}$ and $G = \langle e_2 \rangle \times \langle a, e_1 \rangle$ which is the group stated in part (b) of our theorem. If $G' = \langle e_2 \rangle$, where $e_2 \in E_1 - \langle z \rangle$ and $e_1 \in E - E_1$, then $[a, e_1] = e_2$ and this group is stated in part (c) of our theorem. □

In the rest of this section we assume $G/E \cong Q_{2^n}$, $n \geq 3$. Then $C_G(E) > E$ since Q_{2^n} cannot act faithfully on $E \cong E_8$. Therefore $\tilde{W} = \Omega_2(G)$ centralizes E, where $\tilde{W}/E = Z(G/E)$, $\tilde{W}$ is abelian of type $(4, 2, 2)$ and $\langle z \rangle = \mho_1(\tilde{W})$ is of order 2. All elements in $\tilde{W} - E$ are of order 4 and so no involution in $E - \langle z \rangle$ is a square in G. We first prove the following useful result.

Lemma 3.2. *We have $C_G(E) \geq \tilde{W} = \Omega_2(G)$, $C_G(E)/E$ is cyclic of order ≥ 2, and $G/C_G(E)$ is isomorphic to C_2, E_4 or D_8.*

Proof. Suppose that $C_G(E)$ contains a subgroup $H > E$ such that $H/E \cong Q_8$. Then $Z(H/E) = \tilde{W}/E$. Let A/E and B/E be two distinct cyclic subgroups of order 4 in H/E. Then A and B are abelian maximal subgroups of H. It follows that $A \cap B = \tilde{W} \leq Z(H)$ and therefore $\tilde{W} = Z(H)$ is of order 2^4. Using Proposition 1.7, we get

$$2^6 = |H| = 2|Z(H)||H'|,$$

which gives $|H'| = 2$. Since $\Omega_1(H) = E$, we get $H' \leq E$ and H/E is abelian, a contradiction. It follows that $C_G(E)/E$ must be a normal cyclic subgroup of G/E and so $G/C_G(E)$ is isomorphic to a nontrivial subgroup of D_8. However $G/C_G(E) \cong C_4$ is not possible since $(G/E)/(G/E)' \cong E_4$. □

Theorem 3.3. *Let G be a 2-group of order $> 2^4$ satisfying $|\Omega_2(G)| = 2^4$. Suppose in addition that G has a normal elementary abelian subgroup E of order 8 such that $G/E \cong Q_{2^2}$, $n \geq 3$, and $E \not\leq \Phi(G)$. Then we have:*

$$G = \langle a, b, e \mid a^{2^n} = b^8 = e^2 = 1,\ n \geq 3,\ a^{2^{n-1}} = b^4 = z,$$
$$a^b = a^{-1},\ [e, b] = 1,\ [e, a] = z^\mu,\ \mu = 0, 1 \rangle.$$

Here G is a group of order 2^{n+3}, $G' = \langle a^2 \rangle$, $\Phi(G) = \langle a^2, b^2 \rangle$ is abelian of type $(2^{n-1}, 2)$. The subgroup $E = \langle e,\ a^{2^{n-2}} b^2,\ z \rangle$ is a normal elementary abelian subgroup of order 8 in G and $G/E \cong Q_{2^n}$. If $\mu = 0$, then $Z(G) = \langle e,\ b^2 \rangle$ is abelian of type $(4, 2)$. If $\mu = 1$, then $Z(G) = \langle b^2 \rangle \cong C_4$. Finally, $\Omega_2(G) = E\langle b^2 \rangle$ is abelian of type $(4, 2, 2)$.

Proof. There is a maximal subgroup M of G such that $E \not\leq M$. We have $E_0 = E \cap M \cong E_4$, $E_0 > \langle z \rangle$, $\tilde{W}_0 = \tilde{W} \cap M$ is abelian of type $(4, 2)$, $\tilde{W}_0 = \Omega_2(M)$, and $M/E_0 \cong Q_{2^n}$, $n \geq 3$. By Proposition 1.4, M is isomorphic to a group (d) in that proposition. Hence M is metacyclic with a cyclic normal subgroup $\langle a \rangle$ of order 2^n and the cyclic factor-group $M/\langle a \rangle$ of order 4 so that there is an element b of order 8 in $M - (E_0\langle a \rangle)$ with $b^2 \in \tilde{W}_0 - E_0$, $b^4 = z = a^{2^{n-1}}$, $\langle b \rangle \cap \langle a \rangle = \langle z \rangle$, and $\tilde{W}_0 = \langle b^2 \rangle \langle a^{2^{n-2}} \rangle \leq \Phi(M)$. We have $\Phi(M) = \langle a^2, b^2 \rangle \geq \tilde{W}_0$, $M' = \langle a^2 \rangle$, and $a^b = a^{-1}$. Also, $Z(M) = \langle b^2 \rangle$ is cyclic of order 4 and so $E_0\langle a \rangle = \langle b^2, a \rangle$ is abelian of index 2 in M, and $C_M(E_0) = E_0\langle a \rangle$. The element b induces an involutory automorphism on E (because b^2 centralizes E) and since b acts nontrivially on E_0, there is an element $e \in E - E_0$ such that b centralizes e. Set $u = a^{2^{n-2}} b^2 \in E_0 - \langle z \rangle$ and we have $u^b = uz$. We act with the cyclic group $\langle a \rangle$ on E. We see that $\langle a \rangle$ stabilizes $E > E_0 > 1$ and so a^2 centralizes E and $e^a = ey$ with $y \in E_0$. This gives $a^e = ay$. We may set $a^b = a^{-1}$. Then we act on $b^{-1}ab = a^{-1}$ with e. It follows $b^{-1}ayb = (ay)^{-1}$ and so $a^{-1}y^b = a^{-1}y$ which gives $y^b = y$ and $y \in \langle z \rangle$. If $y = 1$, then $G = \langle e \rangle \times M$. We set $a^e = az^{\mu}$, where $\mu = 0, 1$. The group G is completely determined. ☐

Theorem 3.4. *Let G be a 2-group of order $> 2^4$ satisfying $|\Omega_2(G)| = 2^4$. Suppose in addition that G has a normal elementary abelian subgroup E of order 8 such that $G/E \cong Q_{2^2}$, $n \geq 3$, and $E \leq \Phi(G)$. Then we have one of the following two possibilities:*

(a)

$$\begin{aligned} G = \langle a, b \,|\, a^{2^n} &= b^8 = 1,\ a^{2^{n-1}} = b^4 = z, b^2 = a^{2^{n-2}} e, a^b = a^{-1} u e^{\epsilon}, \\ &\epsilon = 0, 1, u^2 = e^2 = [e, u] = [e, z] = [u, z] = 1, \\ &e^a = ez,\ u^a = u,\ e^b = ez,\ u^b = uz,\ n \geq 3, \\ &\textit{and if } \epsilon = 1, \textit{ then } n \geq 4 \rangle. \end{aligned}$$

Here G is a group of order 2^{n+3}, $G' = \langle ue^{\epsilon} \rangle \times \langle a^2 \rangle$ is abelian of type $(2^{n-1}, 2)$, $\Phi(G) = E\langle a^2 \rangle$, where $E = \langle z, e, u \rangle$ is a normal elementary abelian subgroup of order 8 in G with $G/E \cong Q_{2^n}$. Finally, $Z(G) = \langle z \rangle$ is of order 2, and $\Omega_2(G) = E\langle a^{2^{n-2}} \rangle$ is abelian of type $(4, 2, 2)$.

(b)

$$\begin{aligned} G = \langle a, b \,|\, a^{2^n} &= b^8 = 1,\ n \geq 3,\ a^{2^{n-1}} = b^4 = z,\ b^2 = a^{2^{n-2}} u, \\ &a^b = a^{-1} e, u^2 = e^2 = [e, u] = [e, z] = [u, z] = [a, e] = \\ &= [a, u] = [b, e] = 1,\ u^b = uz \rangle. \end{aligned}$$

Here G is a group of order 2^{n+3}, $G' = \langle a^2 e \rangle$ is cyclic of order 2^{n-1}, $\Phi(G) = E\langle a^2 \rangle$, where $E = \langle z, e, u \rangle$ is a normal elementary abelian *subgroup of order* 8 *in* G *with* $G/E \cong Q_{2^n}$ *and* $C_G(E) = E\langle a\rangle$. *Finally,* $Z(G) = \langle e, b^2\rangle$, *and* $\Omega_2(G) = E\langle b^2\rangle$ *is abelian of type* $(4,2,2)$.

Proof. We have in this case $\Phi(G) \geq \tilde{W} = \Omega_2(G)$, $\tilde{W}/E = Z(G/E)$ and $\tilde{W}$ is abelian of type $(4,2,2)$. Hence $\langle z\rangle = \mho(\tilde{W})$ is a central subgroup of order 2 contained in E. By Lemma 3.2, there is a maximal subgroup M of G such that M/E is cyclic and $C_G(E) \leq M$. (If $n > 3$, then M is unique. However, if $n = 3$ and $C_G(E) = \tilde{W}$, then we have three possibilities for M.) Let a be an element in $M - E$ such that $\langle a\rangle$ covers M/E. Then $o(a) = 2^n$ since $|M/E| = 2^{n-1}$ and M does not split over E. Hence $a_0 = a^{2^{n-2}} \in \tilde{W} - E$, $a_0^2 = z$, and $M = E\langle a\rangle$. By the structure of $G/E \cong Q_{2^n}$, $n \geq 3$, we have for each $x \in G - M$, $x^2 \in \tilde{W} - E$, $x^4 = z$, x induces an involutory automorphism on E (since $C_G(E) \leq M$) and so $C_E(x) \cong E_4$, $C_E(x) > \langle z\rangle$, and $C_{\tilde{W}}(x) = \langle x^2\rangle C_E(x)$ is abelian of type $(4,2)$. We have $M' < E$, $|M'| \leq 4$, and $M' = [\langle a\rangle, E]$. For each $a^i e \in M$, where $e \in E$ and i is an integer, we get

$$(a^i e)^2 = a^i e a^i e = a^{2i}(a^{-i} e a^i)e = a^{2i}[a^i, e]$$

and so $\Phi(M) = \langle a^2\rangle M'$. We have exactly the following four cases for the action of $\langle a\rangle$ on E:

(1) a induces an automorphism of order 4 on E.
(2) a induces an automorphism of order 2 on E and $[E,a] = \langle u\rangle$, where $u \in E - \langle z\rangle$.
(3) a induces an automorphism of order 2 on E and $[E,a] = \langle z\rangle$.
(4) a centralizes E.

Case (1) Since a induces an automorphism of order 4 on E, we get $n \geq 4$ and $C_E(a) = \langle z\rangle$. In this case M' is a 4-subgroup contained in E, $M' > \langle z\rangle$ and for each $\tilde{e} \in E - M'$, $\tilde{e}^a = \tilde{e}\tilde{u}$ with some $\tilde{u} \in M' - \langle z\rangle$ and $\tilde{u}^a = \tilde{u}z$. Also, $C_E(a^2) = M'$. It follows that $C_G(M') = E\langle a^2, b'\rangle$, where $b' \in G - M$ and $E\langle a^2, b'\rangle$ is a maximal subgroup of G. Set $b = b'a$ so that $\tilde{u}^b = \tilde{u}z$ for each $\tilde{u} \in M' - \langle z\rangle$ and therefore $C_E(b) = \langle e, z\rangle$ for some $e \in E - M'$, $e^a = eu$ with $u \in M' - \langle z\rangle$, $u^a = uz$, and $u^b = uz$. Since $G/E \cong Q_{2^n}$, $b^{-1}ab = a^{-1}y$ with $y \in E$. We have two subcases according to $y \in E - M'$ or $y \in M'$.

Case (1a) Here $y = e' \in E - M'$ and so $b^{-1}ab = a^{-1}e'$. We compute

$$a^{b^2} = (a^b)^b = (a^{-1}e')^b = (a^{-1}e')^{-1}(e')^b = e'a(e')^b = a(e')^a(e')^b = au.$$

Indeed, if $e' = ez^{\mu}\,(\mu = 0,1)$, then $(e')^a = e'u$, $(e')^b = e'$, and if $e' = euz^{\mu}\,(\mu = 0,1)$, then $(e')^a = e'uz$, $(e')^b = e'z$. On the other hand, $b^2 = a_0 x$ with $x \in E$. But $M'\langle a\rangle = \langle u, a\rangle$ is isomorphic to $M_{2^{n+1}}$ and so $a^{b^2} = au = a^{a_0 x} = a^x$ implies that $x \in E - M'$. We may set $x = eu'$ with $u' \in M'$ and so $b^2 = a_0 eu'$. We compute

$$b^{-1}a^2 b = (a^{-1}e')^2 = a^{-1}e'a^{-1}e' = (a^{-1}e'a)a^{-2}e' = (e')^a(e')^{a^2}a^{-2} = u_0 z a^{-2},$$

since $(e')^a = e'u_0\,(u_0 \in M' - \langle z\rangle)$ and $(e')^{a^2} = e'z$. Therefore

$$(a^4)^b = ((a^2)^b)^2 = (u_0 z a^{-2})^2 = a^{-4}$$

because $C_E(a^2) = M'$. Since $a_0 = a^{2^{n-2}} \in \langle a^4\rangle$, we get $a_0^b = a_0^{-1}$. Obviously, b centralizes $b^2 = a_0 e u'\,(u' \in M')$ and so

$$a_0 e u' = (a_0 e u')^b = a_0^{-1} e (u')^b = a_0 z e (u')^b$$

which gives $(u')^b = u'z$ and $u' \in M' - \langle z\rangle$. Therefore $u' = uz^\epsilon\,(\epsilon = 0, 1)$ and $b^2 = a_0 e u z^\epsilon$. On the other hand,

$$au = a^{b^2} = a^{a_0 e u z^\epsilon} = a^{eu} = a(a^{-1}eua)eu = a(eu)(uz)eu = auz,$$

which is a contradiction.

Case (1b) Here $b^{-1}ab = a^{-1}y$ with $y \in M'$. This gives

$$\begin{aligned}(a^2)^b &= (a^{-1}y)^2 = a^{-1}ya^{-1}y = (a^{-1}ya)a^{-2}y = y^a y^{a^2} a^{-2}\\ &= yz^\nu y a^{-2} = z^\nu a^{-2}\,(\nu = 0, 1)\end{aligned}$$

and so $(a^4)^b = (z^\nu a^{-2})^2 = a^{-4}$. Since $a_0 \in \langle a^4\rangle$, we get $a_0^b = a_0^{-1}$. If $b^2 \in M'\langle a_0\rangle$, then $S = M'\langle a, b\rangle$ is a maximal subgroup of G not containing E since $S \cap E = M'$, a contradiction. Thus $b^2 = a_0 e_0$ with $e_0 \in E - M'$. Since b centralizes b^2, we get

$$a_0 e_0 = (a_0 e_0)^b = a_0^{-1} e_0^b = a_0 z e_0^b$$

and so $e_0^b = e_0 z$. Hence $e_0 = euz^\epsilon\,(\epsilon = 0, 1)$ and $b^2 = a_0 e u z^\epsilon$. From $b^{-1}ab = a^{-1}y$ with $y \in M'$, we get

$$b^{-2}ab^2 = (a^{-1}y)^b = (a^b)^{-1}y^b = yay^b = a(a^{-1}ya)y^b = a(yz^\eta)(yz^\eta) = a,$$

where $\eta = 0$ if $y \in \langle z\rangle$ and $\eta = 1$ if $y \in M' - \langle z\rangle$. On the other hand, we compute

$$a = a^{b^2} = a^{a_0 e u z^\epsilon} = a^{eu} = euaeu = a(a^{-1}eua)eu = a(eu)(uz)eu = auz,$$

which is a contradiction.

Case (2) Here a induces an automorphism of order 2 on E and $M' = [E, a] = \langle u\rangle$, where $u \in E - \langle z\rangle$. It follows $\langle u, z\rangle \le Z(G)$. For each $\tilde{e} \in E - \langle u, z\rangle$, $\tilde{e}^a = \tilde{e}u$. Since $\Phi(M) = \langle u\rangle \times \langle a^2\rangle$ and $\Phi(G) \ge E$, there exists exists $b \in G - M$ such that $b^2 = a_0 e$, where $e \in E - \langle u, z\rangle$. We have $a^b = a^{-1}y$ with $y \in E$. This gives

$$a^{b^2} = (a^{-1}y)^b = (a^b)^{-1}y^b = yay^b = ay^a y^b = a^{a_0 e} = a^e =$$

$$= eae = a(a^{-1}ea)e = aeue = au.$$

Hence $y^a y^b = u$. If $y \in \langle u, z\rangle \le Z(G)$, then $y^2 = 1 = u$, a contradiction. Therefore $y = es$ with $s \in \langle u, z\rangle$. But then

$$u = y^a y^b = (es)^a (es)^b = (eus)(e^b s)$$

and so $e^b = e$. Hence b centralizes E, a contradiction.

Case (3) Here a induces an automorphism of order 2 on E and $M' = [E,a] = \langle z\rangle$. It follows $\Phi(M) = \langle a^2\rangle$ and so $\langle a_0\rangle$ is normal in G. Since $\Phi(G) \geq \tilde{W} = E\langle a_0\rangle$, it follows $\langle a_0, x^2 \,|\, x \in G - M\rangle = \tilde{W}$. Since $\langle x^2\rangle$ is normal in G for each $x \in G - M$ and $x^4 = z$, we get $\tilde{W}/\langle z\rangle \leq Z(G/\langle z\rangle)$. Set $U = C_E(a) \cong E_4$ and obviously $Z(M) = U\langle a^2\rangle$. Also, $U = Z(M) \cap E$ is normal in G. Suppose $U \leq Z(G)$. Then the 4-group $G/E\langle a^2\rangle$ acts as the full stability group of the chain $E > U > 1$ since $G/E\langle a^2\rangle$ acts faithfully on E. (Note that $C_G(E) \leq M$.) This is a contradiction since $E/\langle z\rangle \leq Z(G/\langle z\rangle)$. Hence for each $x \in G - M$ and each $\tilde{u} \in U - \langle z\rangle$, we have $\tilde{u}^x = \tilde{u}z$. There exists $b \in G - M$ such that $b^2 = a_0 e$ with $e \in E - U$. Also, $a^b = a^{-1}y$ with $y \in E$. Then we compute

$$\begin{aligned} a^{b^2} &= (a^{-1}y)^b = yay^b = a(a^{-1}ya)y^b = ay^ay^b = a^{a_0 e} = eae \\ &= a(a^{-1}ea)e = aeze = az. \end{aligned}$$

This gives

(1) $y^a y^b = z$.

On the other hand, b centralizes b^2 and so $a_0 e = (a_0 e)^b = a_0^b e^b$ which gives

(2) $a_0 e = a_0^b e^b$.

We compute further

$$b^{-1}a^2 b = (a^{-1}y)(a^{-1}y) = (a^{-1}ya)(a^{-2}ya^2)a^{-2} = y^a y a^{-2}$$

and so we get

(3) $(a^2)^b = y^a y a^{-2}$.

There are two subcases according to $y \in U$ or $y \in E - U$.

Case (3a) Suppose $y \in U = C_E(a)$. Then (1) implies $y^b = yz$ and so $y = u \in U - \langle z\rangle$ and $b^{-1}ab = a^{-1}u$. From (3) we get $(a^2)^b = a^{-2}$ and so $a_0^b = a_0^{-1}$. From (2) follows $e^b = ez$ and we have determined the group stated in part (a) of our theorem for $\epsilon = 0$.

Case (3b) Suppose $y \in E - U$, where $a^b = a^{-1}y$. From (1) we get $yzy^b = z$ and so $y^b = y$. The relation (3) implies $(a^2)^b = a^{-2}z$. We compute for each $x \in E$ and each integer i

$$(b(a^2)^i x)^2 = b^2 z^\xi \textit{ and } (ba(a^2)^i x)^2 = b^2 yz^\eta,$$

where $\xi, \eta = 0, 1$. Hence the square of each element in $G - M$ lies in $\langle b^2\rangle = \langle a_0 e\rangle$ and in $\langle a_0 ey\rangle$. Since $\Phi(G) \geq E$, we must have $y = eu$ with some $u \in U - \langle z\rangle$. From $y^b = y$ follows $eu = e^b uz$ (since $u^b = uz$) and so $e^b = ez$. Then (2) implies $a_0 e = a_0^b ez$ and so $a_0^b = a_0 z = a_0^{-1}$. From $(a^2)^b = a^{-2}z$ follows that $\langle a^2\rangle > \langle a_0\rangle$ which gives $n \geq 4$. We have obtained the group stated in part (a) of our theorem for $\epsilon = 1$ and $n \geq 4$.

Case (4) Here a centralizes E and so M is abelian of type $(2^n, 2, 2)$, $n \geq 3$, and $\langle a_0\rangle$ is normal in G, where $a_0 = a^{2^{n-2}}$. Since $\Phi(G) \geq \tilde{W} = E\langle a_0\rangle$, there exists $b \in G - M$ such that $b^2 = a_0 u$, where u is an involution in $E - \langle z\rangle$.

Since $C_G(b^2) \geq \langle M, b\rangle = G$, we get $b^2 \in Z(G)$, $\langle a_0, b^2\rangle = \langle a_0\rangle \times \langle u\rangle$ is abelian of type $(4,2)$ and $\Phi(G) \geq \langle a^2, u\rangle$. Again $\Phi(G) \geq \tilde{W}$ implies that there exists $c \in G - M$ such that $c^2 \in \tilde{W} - (E \cup \langle a_0, b^2\rangle)$ and (as before) $c^2 \in Z(G)$. If $a_0 \in Z(G)$, then $\langle a_0, b^2, c^2\rangle = \tilde{W}$ lies in $Z(G)$ contrary to Lemma 3.2. Hence $a_0 \notin Z(G)$ and therefore $a_0^b = a_0^{-1}$ and $u^b = uz$. We have $Z(G) = \langle b^2, c^2\rangle$ is abelian of type $(4,2)$. Proposition 1.7 implies $|G| = 2^{n+3} = 2|Z(G)||G'|$ and so $|G'| = 2^{n-1}$. Since $|(G/E)'| = 2^{n-2}$ and $G' \geq \langle z\rangle$, G' is cyclic of order 2^{n-1}. Set $a^b = a^{-1}e$ with $e \in E$. If $e \in \langle a_0, u\rangle = \langle a_0, b^2\rangle$, then $\langle a, b\rangle$ is a maximal subgroup of G which does not contain E, a contradiction. Thus $e \in E - \langle z, u\rangle$ and we compute

$$(ba)^2 = baba = b^2(b^{-1}ab)a = b^2(a^{-1}e)a = b^2e.$$

Since both b^2 and b^2e lie in $Z(G)$, it follows $e \in Z(G)$ and therefore $Z(G) = \langle e, b^2\rangle$. Finally, $G' = \langle a^2e\rangle$ is cyclic of order 2^{n-1}. We have obtained the group stated in part (b) of our theorem. □

4. 2-groups with exactly one subgroup of order 2^4 and exponent 4

Theorem 4.1. *The following two statements for a 2-group G of order $> 2^4$ are equivalent:*

(a) $|\Omega_2(G)| = 2^4$.

(b) *G has exactly one subgroup of order 2^4 and exponent 4.*

Proof. Suppose that (a) holds. The results of sections 2 and 3 imply that $\Omega_2(G)$ is isomorphic to one of the following groups:

$$Q_8 * C_4,\ Q_8 \times C_2,\ D_8 \times C_2,\ C_4 \times C_2 \times C_2,\ \text{and } C_4 \times C_4.$$

In particular, $exp(\Omega_2(G)) = 4$ and so (b) holds.

Assume now that (b) holds. Let H be the unique subgroup of order 2^4 and exponent 4 in G, where $|G| > 2^4$. We want to show that $H = \Omega_2(G)$. Suppose that this is false. Then there exist elements of order ≤ 4 in $G - H$, where H is a characteristic subgroup of G. In particular, there is an element $a \in G - H$ such that $o(a) \leq 4$ and $a^2 \in H$. Set $\tilde{H} = H\langle a\rangle$ so that $|\tilde{H}| = 2^5$. Since H is neither cyclic nor a 2-group of maximal class, there exists a G-invariant 4-subgroup W_0 contained in H.

Let x be any element in $\tilde{H} - H$ with $o(x) \leq 4$. Then $\langle x, W_0\rangle$ is a subgroup of order $\leq 2^4$ in $\tilde{H}$ and so there exists a maximal subgroup M of $\tilde{H}$ containing $\langle x, W_0\rangle$. Since $exp(M \cap H) \leq 4$ and $M = \langle M \cap H, x\rangle$, we have $\Omega_2(M) = M$. But $|M| = 2^4$ and $M \neq H$, so M is either elementary abelian or $exp(M) = 8$. Suppose that $exp(M) = 8$. Then M is a nonabelian group of order 2^4 with a cyclic subgroup of index 2. Since M possesses the normal 4-subgroup W_0, it follows that M is not of maximal class. Thus $M \cong M_{2^4}$. But $\Omega_2(M_{2^4})$ is abelian of type $(4,2)$ which contradicts the above fact that $\Omega_2(M) = M$.

We have proved that M is elementary abelian of order 2^4. In particular, x must be an involution and $exp(\tilde{H}) = 4$ because for each $y \in \tilde{H}$, $y^2 \in M$. Hence all elements in $\tilde{H} - H$ are of order ≤ 4 and so (by the above) all these elements must be involutions. It follows that for each $h \in H$, $(hx)^2 = 1$ and so $h^x = h^{-1}$. Therefore the involution $x \in \tilde{H} - H$ acts invertingly on H. Let k be any element of order 4 in H. Since $k^x = k^{-1}$, $\langle k, x \rangle = D \cong D_8$. Let $\tilde{M}$ be a maximal subgroup of $\tilde{H}$ containing D. Then $|\tilde{M}| = 2^4$, $exp(\tilde{M}) = 4$, $\tilde{M} \neq H$, and this is a final contradiction. Our theorem is proved. $\square$

5. A theorem of Berkovich on 2-groups with exactly one abelian subgroup of type $(4, 2)$

Here we improve results of Berkovich [1, Theorem 43.4] and Berkovich [2, Sect.48].

Theorem 5.1. *Let G be a 2-group containing exactly one abelian subgroup of type $(4, 2)$. Then one of the following holds:*

(a) *$|\Omega_2(G)| = 8$ and G is isomorphic to one of the metacyclic groups (a),(b) or (d) in Proposition 1.4.*

(b) *$G \cong C_2 \times D_{2^{n+1}}$, $n \geq 2$.*

(c)
$$G = \langle b, t \mid b^{2^{n+1}} = t^2 = 1,\, b^t = b^{-1+2^{n-1}}u,\, u^2 = [u, t] = 1,$$
$$b^u = b^{1+2^n},\, n \geq 2 \rangle.$$

Here $|G| = 2^{n+3}$, $Z(G) = \langle b^{2^n} \rangle$ is of order 2, $\Phi(G) = \langle b^2, u \rangle$, $E = \langle b^{2^n}, u, t \rangle \cong E_8$ is self-centralizing in G, $\Omega_2(G) = \langle u \rangle \times \langle b^2, t \rangle \cong C_2 \times D_{2^{n+1}}$, $G' = \langle b^{2^n}, u \rangle \cong E_4$ in case $n = 2$, and $G' = \langle b^2 u \rangle \cong C_{2^n}$ for $n \geq 3$. Finally, the group G for $n = 2$ (of order 2^5) is isomorphic to the group (a) in Theorem 3.1 for $n = 2$ (since $\Omega_2(G) \cong C_2 \times D_8$).

Proof. Let G be a 2-group possessing exactly one abelian subgroup A of type $(4, 2)$. Obviously, A is normal (even characteristic) in G. Set $C = C_G(A)$. Then C is normal in G and G/C is isomorphic to a subgroup of $Aut(A) \cong D_8$. We claim that $\Omega_2(C) = A$. Indeed, let $y \in C - A$ with $o(y) \leq 4$ and $y^2 \in A$. Then $A\langle y \rangle$ is an abelian subgroup of order 2^4 and exponent 4. But then $A\langle y \rangle$ contains an abelian subgroup of type $(4, 2)$ distinct from A, a contradiction. Thus $\Omega_2(C) = A$, as claimed. Proposition 1.4 implies that C must be abelian of type $(2^n, 2)$, $n \geq 2$. If $\Omega_2(G) = \Omega_2(C) = A$, then G is metacyclic and G is isomorphic to one of the groups (a), (b) or (d) in Proposition 1.4.

We assume from now on that $\Omega_2(G) > A$. Set $U = \Omega_1(A)$ and $\langle z \rangle = \Phi(A)$ so that $z \in Z(G)$ and U is a normal 4-subgroup of G. Let $a \in G - C$ such that $o(a) \leq 4$ and $a^2 \in C$. Obviously, $a^2 \in U$ since $U = \Omega_1(C)$. We consider the subgroup $D = \langle a \rangle C$ and we have $|D : C| = 2$. Since D is a nonabelian group (of order $\geq 2^4$) containing a normal 4-subgroup U, it follows that D

is not of maximal class. If $o(a) = 4$, then a does not centralize U (otherwise $U\langle a\rangle$ would be abelian of type $(4, 2)$ distinct from A) and so $U\langle a\rangle \cong D_8$. But then there exists an involution in $(U\langle a\rangle) - U$. In any case the coset Ca contains involutions and let t be one of them. If $C_C(t)$ contains an element s of order 4, then $\langle s, t\rangle$ is an abelian subgroup of type $(4, 2)$ distinct from A, a contradiction. Hence $C_C(t)$ is elementary abelian and (since D is not of maximal class) $C_C(t) = U$ and $E = C_D(t) = U \times \langle t\rangle \cong E_8$. For each $x \in D - C = Ct$, $C_C(x) = U$. Hence $x^2 \in U$ and $\langle U, x\rangle$ is elementary abelian (of order 8). Thus, all elements in $D - C$ are involutions and therefore t acts invertingly on C. The involution t induces on A the central involutory automorphism of $Aut(A) \cong D_8$ and so D is normal in G. We have $D \cong C_2 \times D_{2^{n+1}}$, $n \geq 2$.

Suppose that G/C has a cyclic subgroup K/C of order 4. Then $K > D$ and let $k \in K$ be such that $\langle k\rangle$ covers K/C. We have $k^2 \in D - C$ and so k^2 is an involution. It follows $o(k) = 4$ and so $\langle k, z\rangle$ is an abelian subgroup of type $(4, 2)$ distinct from A, a contradiction. We have proved that G/C is elementary abelian. If $|G/C| = 2$, then $G = D$ and we are done.

It remains to study the possibility $G/C \cong E_4$. By the above, if $y \in G-D$ and $o(y) \leq 4$, then $y^2 \in C$ and y is an involution acting invertingly on C. But then $yt \notin C$ and yt centralizes C, a contradiction. Hence, for each element $y \in G - D$, $o(y) \geq 8$ and $y^2 \in C$. This gives $\Omega_2(G) = D$, $C_G(t) = E$ and so E is a self-centralizing elementary abelian subgroup of order 8 in G.

For each $y \in G - D$, we have $y^2 \in C$ and $o(y^2) \geq 4$ and so y centralizes a cyclic subgroup of order 4 in A. Since y does not centralize A, it follows that y does not centralize $U = \Omega_1(A)$. Hence $D = C_G(U)$ and so $Z(G) = \langle z\rangle = \Phi(A)$ is of order 2. Since $\Omega_2(\langle y\rangle C) = A$ and $\langle y\rangle C$ is nonabelian, Proposition 1.4 implies that $\langle y\rangle C$ is isomorphic to a group (a) or (d) of that proposition.

Assume that there is $b \in G-D$ such that $\langle b\rangle C$ is isomorphic to a group (d) of Proposition 1.4. In particular, $|\langle b\rangle C| \geq 2^5$, C is the unique abelian maximal subgroup of $\langle b\rangle C$ (since $Z(\langle b\rangle C) \cong C_4$), $o(b) = 8$, $Z(\langle b\rangle C) = \langle b^2\rangle < A$, $\langle b^4\rangle = \langle z\rangle = \Phi(A)$, C contains a cyclic subgroup $\langle a\rangle$ of index 2 such that $\langle a\rangle$ is normal in $\langle b\rangle C$, $o(a) \geq 2^3$, $A \cap \langle a\rangle \cong C_4$, $\langle b\rangle \cap \langle a\rangle = \langle z\rangle$, and $a^b = a^{-1}z^{\epsilon}$ with $\epsilon = 0, 1$. Also, we see that for each $y \in (\langle b\rangle C) - C$, $C_C(y) = \langle b^2\rangle \cong C_4$, and y is of order 8. Since t acts invertingly on C, we get $a^{bt} = a$ and so $|C_C(bt)| \geq 2^3$. This implies that $\langle bt\rangle C \cong M_{2^{n+2}}$ because $C_C(bt) \leq Z(\langle bt\rangle C)$.

Replacing b with bt (if necessary), we may assume from the start that there is an element $b \in G - D$ such that $\langle b\rangle C \cong M_{2^{n+2}}$ $(n \geq 2)$, which is a group in part (a) of Proposition 1.4. We have $o(b) = 2^{n+1}$, $\langle b\rangle$ is a cyclic subgroup of index 2 in $\langle b\rangle C$, $z = b^{2^n}$, $U = \langle z, u\rangle = \Omega_1(\langle b\rangle C)$, and $u^b = uz$. This gives also $b^u = bz = b^{1+2^n}$, $Z(\langle b\rangle C) = \langle b^2\rangle \cong C_{2^n}$, $A = \langle b^{2^{n-1}} = v, u\rangle$, and $C = \langle b^2\rangle \times \langle u\rangle$. We see that $C_C(bt) = \langle vu\rangle \cong C_4$ and so $Z(\langle bt\rangle C) = \langle vu\rangle$. It follows that $\langle bt\rangle C \cong M_{2^4}$ for $n = 2$ and $\langle bt\rangle C$ is isomorphic to a group (d) of Proposition 1.4 for $n > 2$. In any case, $(bt)^2 \in \langle vu\rangle - \langle z\rangle$ and so

$(bt)^2 = vu$ or $(bt)^2 = v^{-1}u = vuz$. Replacing u with uz (if necessary), we may assume $(bt)^2 = vu$, where $v = b^{2^{n-1}}$. From this crucial relation follows $b^t = b^{-1+2^{n-1}}u$. The structure of $G = \langle b, t\rangle$ is determined and our theorem is proved. □

THEOREM 5.2. *The following two statements for a 2-group G are equivalent:*

(a) *G has exactly two cyclic subgroups of order 4.*
(b) *G has exactly one abelian subgroup of type $(4, 2)$.*

PROOF. Suppose that G is a 2-group having exactly two cyclic subgroups U, V of order 4. Then $|G : N_G(U)| \leq 2$ and so $V \leq N_G(U)$. Similarly, $U \leq N_G(V)$. We get $[U, V] \leq U \cap V$. Since $A = \langle U, V\rangle$ has exactly two cyclic subgroups of order 4, A must be abelian of type $(4, 2)$. But A is generated by its two cyclic subgroups of order 4 and so (b) holds.

Assume that (b) holds. Then the group G is completely determined by Theorem 5.1. Looking at $\Omega_2(G)$, we see that G has exactly two cyclic subgroups of order 4. □

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