GLASNIK MATEMATIČKI
Vol. 38(58)(2003), 285 – 298

# PERIODIC SOLUTIONS OF A FIRST ORDER DIFFERENTIAL EQUATION

Lavoslav Čaklović
University of Zagreb, Croatia

Abstract. The first order dynamical system $\dot{z} = F(t, z)$ is considered, where $F$ is $T$-periodic in time and sub-linear at infinity. Existence of $T$–periodic solution is proved, using degree theory, and applications to non-convex Hamiltonian systems is given as well.

## 1. Introduction and main results

*General type equation.* We consider the following first order differential equation

$$\dot{z} = F(t, z) \tag{F}$$

where $F \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$, $N \in \mathbb{N}$, is $T$-periodic in time. Our goal is to find a $T$-periodic solution of (F) under the following `sub-linearity` condition at infinity

$$\limsup_{|z|\to+\infty} \frac{|F(t,z)|}{|z|} = 0, \text{ uniformly in } t, \tag{SL}$$

in Sobolev space $H^1_T := \{z \in H^1(0, T; \mathbb{R}^N) \mid z(0) = z(T)\}$ of $T$-periodic functions. A standard technique is to decompose the space in orthogonal sum

$$H^1_T = E^1 \oplus \mathbb{R}^N, \; z = u + m,$$

where $E^1 = \{u \in H^1 \mid ⨍ u = 0\}$, and $⨍ z = \frac{1}{T}\int_0^T z(t)dt$ is the mean value of function $z(t)$. According to the above decomposition we split equation

2000 *Mathematics Subject Classification.* 34C25, 46N20, 47H15, 55M25, 58C30, 58F22.
*Key words and phrases.* Differential equations, periodic solution, degree, Hamiltonian system.

(F) to obtain $E^1$-component of (F), an infinite dimensional equation, and $\mathbb{R}^N$-component of (F), finite dimensional one:

$$\begin{aligned} \dot u &= F(t,u+m) - ⨍ F(t,u+m)\\ 0 &= ⨍ F(t,u+m). \end{aligned} \tag{F$_1$}$$

We would like to introduce a deformation parameter $\tau$, $0 \le \tau \le 1$ to obtain a homotopicaly equivalent uncoupled system which has a solution. Following this idea we introduce

$$\begin{aligned} \dot u &= \tau\big[F(t,\tau u+m) - ⨍ F(t,\tau u+m)\big]\\ 0 &= ⨍ F(t,\tau u+m), \end{aligned} \tag{F$_\tau$}$$

for $\tau = 1$ ($\mathrm{F}_\tau$) reduces to ($\mathrm{F}_1$) and for $\tau = 0$ we obtain an uncoupled system

$$\begin{aligned} \dot u &= 0\\ 0 &= ⨍ F(t,m). \end{aligned} \tag{F$_0$}$$

Solving ($\mathrm{F}_\tau$) is equivalent to finding a zero of the function

$$(m,u) \mapsto \left( ⨍ F(t,m+\tau u), u - \left(\frac{d}{dt}\right)^{-1}\Big(\tau\big[F(t,m+\tau u) - ⨍ F(t,m+\tau u)\big]\Big)\right)$$

defined on $\mathbb{R}^N \times E^1$. Some a priori bounds are needed and invertibility of $\frac{d}{dt}$ should be justified.

As it was kindly pointed out by the referee, it seems that instead the homotopy defined in formula ($\mathrm{F}_\tau$) the homotopy in the proof of Theorem IV.3 in the Mawhin's book [9] can be used.

*A priori bound.* To obtain an a priori bound on the solution let us rewrite the sub-linearity condition (SL) in equivalent form:

$$\begin{aligned} &\forall\varepsilon > 0,\ \exists C_\varepsilon > 0,\ \text{such that}\\ &|F(t,z)|\ \le \varepsilon|z| +\ C_\varepsilon,\ z\in\mathbb{R}^N,\ t\in\mathbb{R}. \end{aligned} \tag{$\varepsilon$SL}$$

As shown in the next proposition some restrictions on $\varepsilon$ are essential for obtaining a priori bound on solution. See also an example in the proof of Theorem 1.3.

PROPOSITION 1.1. *Assume that $F$ is sub-linear at infinity, $m \in \mathbb{R}^n$ and $u \in E^1$ is a solution of the first equation in* ($\mathrm{F}_\tau$). *If $\varepsilon$, in inequality ($\varepsilon$SL), is such that $\varepsilon T < \sqrt{3}/2$ in then*

$$\|u\|_{L^\infty} \le \delta|m| + \gamma \tag{1.1}$$

*where*

$$\delta := \frac{\varepsilon T}{\sqrt{T} - \varepsilon T} \text{ and } \gamma := \frac{C_\varepsilon T}{\sqrt{T} - \varepsilon T}.$$

We prove the proposition in section 3 on page 291. Moreover, for any $r > 0$ we introduce

$$R_\varepsilon(r) := \frac{r+\gamma}{1-\delta}$$

As we shall see in Lemma 3.1, inequality (1.1) implies that $z = m + u(t)$ is localized in the ball $B(|m|, R_\varepsilon(r) - r)$ whenever $|m| \geq R_\varepsilon(r)$.

There are two additional properties, we call them 'guiding function' and 'half space localization'. Each of them assures a priori bound on the solution.

`Guiding function.`

(1.2) There exist a guiding function $W(z)$, $W \in C^1(\mathbb{R}^N, \mathbb{R})$, and positive $r > 0$ such that $|z| \geq r \Rightarrow F(t,z) \cdot W'(z) > 0$ uniformly on $t$.

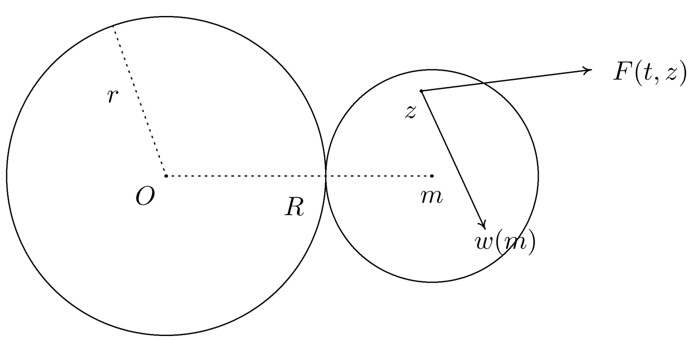
As shown in Lemma 3.2, if $F$ has a guiding function then $|m| \geq R_\varepsilon(r)$ implies that the second equation $(\mathrm{F}_\tau)$ has no solution.

`Half space localization.`

(1.3) There exists $r > 0$ and a continuous function $w : \mathbb{R}^N \setminus B(0, R_\varepsilon(r)) \to \mathbb{R}^N$, such that $|m| \geq R_\varepsilon(r) \Rightarrow \big(w(m) \cdot m \geq 0 \ \text{ and } \ F(t,z) \cdot w(m) > 0\big)$ for all $z \in B(m, R_\varepsilon(r) - r)$ uniformly on $t$.

If $z \in B(m, R_\varepsilon(r) - r)$ and $F$ satisfies half space localization property then $F(t,z)$ belongs to the half space $\{z \in \mathbb{R}^N \mid w(m) \cdot z > 0\}$. Specially, this implies that the second equation $(\mathrm{F}_\tau)$ has no solution.

In both cases, i.e. if $F$ has guiding function or satisfies half space localization property, then, if there exists a solution $z = u + m$ of $(\mathrm{F}_\tau)$ it should satisfy $|m| \leq R_\varepsilon(r)$. Evidently, some additional property of $F$ is needed to prove the existence of solution. This is the non-triviality of degree as stated in next theorem.




A priori bound

Theorem 1.2 (Krasnoselski). *Assume that* $F(t,z)$ *satisfies* (SL) *(or* ($\varepsilon$SL)*). If* $F$ *has guiding function and* $d := \deg(W', B(0,r), 0) \neq 0$ *then* (F) *has a* $T$*-periodic solution.*

A simple argument for introducing condition $\deg(W', B(0,r), 0) \neq 0$ is the situation when $F : \mathbb{R} \to \mathbb{R}$ is strictly positive. Then, equation $\dot{z} = F(z)$ has no periodic solutions, $\deg(W', B, 0) = 0$ for any interval $B = (-r, r)$, $r > 0$ and condition (1.2) is fulfilled with $W' = F$. Evidently, the degree $d$ has the same value for greater $r$ because of the non-vanishing derivative $W'$ in (1.2).

Applications to some types of Hamiltonian systems are given in section 5. The theorem is a particular case of [6, Lema 6.5, Ch. 2] and we are not going to prove it here. Its proof is inspirative for more general statement in the next theorem.

Theorem 1.3. *Assume that* $F(t,z)$ *satisfies* (SL) *(or* ($\varepsilon$SL)*). If* $F$ *satisfies half space localization property and* $\deg(w, 0, R) \neq 0$ *then* (F) *has a* $T$*-periodic solution.*

An application to radial-like Hamiltonian is given in Theorem 1.4, with $w(m) = Jm$.

*Radial-like Hamiltonians.* We consider the first order Hamiltonian system

$$\dot{z} = JH'(t,z) \tag{H}$$

where $H(t,z) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is $C^1$ function and $T$-periodic in time (prime denotes partial derivative with respect to $z$).

We say that Hamiltonian $H$ is `strongly sub-quadratic` at infinity if there exists $r > 0$ and $1 < p < 2$ such that

$$|H'(t,z)| \leq \Theta_2 |z|^{p-1} \quad \text{whenewer } |z| \geq r. \tag{SS}$$

We say that Hamiltonian $H$ is `radial-like` if there exists $\mu > 0$ such that

$$\frac{H'(t,z) \cdot z}{|H'(t,z)||z|} \geq \mu > 0 \text{ for } |z| \geq r > 0 \text{ uniformly in } t. \tag{Rad}$$

The following theorem is a consequence of Theorem 1.3.

Theorem 1.4. *Suppose that the Hamiltonian* $H$ *is radial-like an strongly sub-quadratic at infinity. Then:*

i) $F = JH'$ *is sub-linear at infinity and satisfies* (1.3) *with* $w(m) = Jm$.

ii) *Hamiltonian system (H) has a* $T$*-periodic solution.*

The same conclusion as in Theorem 1.4 can be proved using variational methods under additional hypothesis on Hamiltonian

$$|H'(t,z)| \cdot |z| \geq \beta \geq 0, \quad |z| \geq r \geq 0 \text{ uniformly in } t.$$

The proof can be found in [5].

A simple test for radial-like Hamiltonian is given in the following theorem.

THEOREM 1.5. *Assume that Hamiltonian $H$ is strongly sub-quadratic at infinity and satisfies*

$$\Theta_1|z|^p \le H'(t,z)z \tag{1.4}$$

*where $0 < \Theta_1 \le \Theta_2$ and $1 < p < 2$. Then $H(t,z)$ is radial-like Hamiltonian.*

OPEN PROBLEM. Is it possible to prove Theorem 1.4 under weaker condition

$$H'(t,z)\cdot z > 0, \quad |z| \ge r \ge 0 \text{ ( uniformly in } t)$$

instead of (Rad)?

*Almost convex Hamiltonians.* We also consider Hamiltonians that are `weakly sub-quadratic` in the sense

$$\limsup_{|z|\to+\infty} \frac{|H(t,z)|}{|z|^2} = 0, \qquad \text{uniformly on } t. \tag{WS}$$

It seems that weak subquadraticity is not sufficient for existence of $T$-periodic solutions for (H) even for radial-like Hamiltonians. We need an additional assumption

$$H(t,z) = \hat{H}(t,z) - \frac{k}{2}|z|^2 \tag{AC}$$

where $\hat{H}$ is strictly convex for some positive number $k$. Hamiltonian which satisfies (AC) we call `almost convex`. In other words, $H$ is almost convex if adding a quadratic term makes it strictly convex.

The following theorem is then easy to prove.

THEOREM 1.6. *Assume that the Hamiltonian $H(t,z)$ is radial-like, weakly sub-quadratic, and almost convex for $0 < k < \frac{2}{T\sqrt{3}}$. Then, the Hamiltonian system* (H) *has a $T$-periodic solution.*

Evidently, without proof, we have the following corollary.

COROLLARY 1.7. *Assume that Hamiltonian $H$ is radial-like, weakly sub-quadratic and convex. Then* (H) *has a $T$-periodic solution.*

## 2. SOME TECHNICAL RESULTS

The framework for our problem is the space $H^1_T := H^1(S_T, \mathbb{R}^N)$ of $T$-periodic functions from $\mathbb{R}$ to $\mathbb{R}^N$, here $S_T$ denotes the sphere $\mathbb{R}/[0,T]$, with a standard Hilbert space structure and norm

$$\|z\|_{H^1} = \left(\int_0^T |z(t)|^2 dt + \int_0^T |\dot{z}(t)|^2 dt\right)^{1/2}.$$

From now on we shall use shorthand notation $\int_0^T f$ for the integral $\int_0^T f(t)dt$.

LEMMA 2.1. *For all $u \in E^1$ we have*

$$\|u\|_{L^\infty} \le \sqrt{\frac{T}{12}}\|\dot{u}\|_{L^2}. \tag{2.1}$$

*Moreover, the constant $\sqrt{T/12}$ is the best Sobolev constant in* (2.1).

PROOF. Let $u = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} u_k e^{i\,k\frac{2\pi}{T}t}$ be Fourier expansion for $u$, where $u_k \in \mathbb{R}^N$. Then

$$\begin{aligned} \dot{u}(t) &= \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} ik\frac{2\pi}{T}u_k e^{i\,k\frac{2\pi}{T}t} \\ \|\dot{u}\|_{L^2} &= \frac{2\pi}{T}(\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} Tk^2|u_k|^2)^{1/2}. \end{aligned}$$

On the other hand

$$\begin{aligned} |u(t)| &\le \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |u_k|k\frac{1}{k} \le \left(\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |u_k|^2k^2\right)^{1/2} \left(\sum_{\substack{k\in\mathcal{Z}\\ k\neq 0}} \frac{1}{k^2}\right)^{1/2} \\ &= \frac{\pi}{\sqrt{3}}\frac{\sqrt{T}}{2\pi}\|\dot{u}\|_{L^2} = \sqrt{\frac{T}{12}}\|\dot{u}\|_{L^2}. \end{aligned}$$

This proves inequality. To see that $\sqrt{T/12}$ is the best Sobolev constant in inequality (2.1) we take

$$u = \sum_{k\neq 0}\frac{1}{k^2}e^{i\,k\frac{2\pi}{T}t},$$

with $\|u\|_{L^\infty} = u(0) = \sum_{k\neq 0}\frac{1}{k^2} = \frac{\pi^2}{3}$. On the other side,

$$\|\dot{u}\|_{L^2} = \frac{2\pi}{T}\left(\sum_{k\neq 0} T\frac{1}{k^2}\right)^{1/2} = \frac{2\pi}{\sqrt{T}}\frac{\pi}{\sqrt{3}},$$

which proves the claim. □

The following lemma speaks about invertibility of $\frac{d}{dt}$.

LEMMA 2.2.

i) *$L := \frac{d}{dt}$ is a bounded linear operator from $H^1$ to $L^2 := L^2(S_T;\mathbb{R}^N)$*
ii) *$L$ is bijective from $E^1$ onto $E = \{u \in L^2 | ⨍ u = 0\}$ and $L^{-1} : E \to E^1$ is an isomorphism of Banach spaces.*

iii) $L^{-1} : E \to E$ *and* $L^{-1} : E \to L^\infty := L^\infty(0,T;\mathbb{R}^N)$ *are compact operators, and*

$$\|L^{-1}\|_{\mathcal{L}(E,L^\infty)} \le \sqrt{\frac{T}{12}}.$$

Proof. i) follows from the definition of the norm on $H^1$. ii) Injectivity is clear: $N(L) = \{u \in E^1 | \dot{u} = 0\} = \{0\}$. To prove surjectivity let us take $v \in E$, i.e. $⨍ v = 0$. Then $z(t) = \int_0^t v(\tau)d\tau$ belongs to $H^1, \dot{z} = v$ and $z(t)$ is $T$-periodic. Put $z_0 = z - ⨍ z$. Then $z_0 \in E^1$ and $\dot{z}_0 = v$.

That $L^{-1} : E \to E^1$ is an isomorphism follows from open mapping theorem.

iii) is a consequence of the well-known theorem of Rellich and Kondrachov (see H. Brezis [3]). The inequality follows from Lemma 2.1. ∎

## 3. A priori bounds

Proof of proposition 1.1. Because $0 \le \tau \le 1$ it is sufficient to prove the proposition for $\tau = 1$. Using inequality $\|u + m\|_{L^\infty} \le \|u\|_{L^\infty} + |m|$ and inequality ($\varepsilon$SL), one gets from ($\mathrm{F}_\tau$), that

$$\|\dot{u}\|_{L^2} \le 2\varepsilon T^{1/2}(\|u\|_{L^\infty} + |m|) + 2C_\varepsilon T^{1/2}.$$

Using Lemma 2.1 we obtain

$$2\sqrt{3}T^{-1/2}\|u\|_{L^\infty} \le 2\varepsilon T^{1/2}(\|u\|_{L^\infty} + |m|) + 2C_\varepsilon T^{1/2}$$

and finally

$$\|u\|_{L^\infty} \le \frac{\varepsilon T}{\sqrt{3} - \varepsilon T}|m| + \frac{C_\varepsilon T}{\sqrt{3} - \varepsilon T} =: \delta|m| + \gamma.$$

which proves the inequality. ∎

Lemma 3.1. *Assume that $F$ is sub-linear at infinity. For a given $m \in \mathbb{R}^N$ let $u(t)$ is a solution of the first equation in ($\mathrm{F}_\tau$) and $r > 0$. Then, for $|m| \ge R_\varepsilon(r)$ the following inequalities take place:*

$$|u(t)| \le |m| - r \quad \textit{and} \quad |u(t) + m| \ge r.$$

Proof. i) Let us prove first that $|m| = R_\varepsilon$. Then, because of Proposition 1.1,

$$\begin{aligned} |u(t)| &\le \delta\frac{r+\gamma}{1-\delta} + \gamma = \frac{\delta r + \gamma}{1-\delta} = \frac{r+\gamma}{1-\delta} - r \\ &= R_\varepsilon(r) - r = |m| - r \end{aligned}$$

ii) If $|m| = R > R_\varepsilon$, let us denote $r(R) = R(1-\delta) - \gamma$. Obviously $r(R) > r$ and, as above,

$$|u(t)| \le R - r(R) \le |m| - r.$$

To prove the second inequality in lemma let us calculate

$$\begin{aligned}|u(t)+m| &\geq |m|-|u(t)| \geq |m|-\delta|m|-\gamma\\ &=(1-\delta)|m|-\gamma \geq r.\end{aligned}$$

□

The following lemma is already proved in the book of Krasnoselski [6]. Because of its importance and simplicity we are giving a sketch of the proof.

Lemma 3.2 (Krasnoselski). *Assume that function $F$ is sub-linear at infinity and has a guiding function for $|z| \geq r$. Then, equation ($F_\tau$) has no solution with mean $m$ such that $|m| \geq R_\varepsilon(r)$.*

Sketch of the proof. Let $u \in E^1$ is a solution of the first equation ($\mathrm{F}_\tau$). Then,

$$0 \neq ⨍ F(t, \tau u(t)+m).$$

Otherwise,

$$\dot{u} = \tau F(t, \tau u + m)$$

and for $z(t) = \tau u(t) + m$ we have

$$\dot{z} = \dot{u} = F(t, z).$$

Using Lemma 3.1 and (1.2) and we finally have

$$\frac{d}{dt} W(z(t)) = \tau^2 F(t, z(t)) \cdot W'(z(t)) > 0$$

which is impossible since $z$ is $T$-periodic. □

## 4. Proof of Theorem 1.3

Let us consider a model Hamiltonian of the form

$$H(z) = \frac{1}{p}|z|^p, \quad 1 < p < 2, \quad z \in \mathbb{R}^2.$$

Using the fact that the corresponding energy is constant we can solve it explicitly using a substitution $z = re^{i\phi(t)}$. But Theorem 1.2 cannot be applied. If we look carefully why this method fails, we see that the choice of function $W(z)$ is the cause of difficulties. The equation is

$$\dot{z} = |z|^{p-2} Jz.$$

If we take $w(z) = Jz$ and multiply both sides of the equation, we get

$$\dot{z} \cdot Jz = |z|^{p-2} Jz \cdot Jz = |z|^p.$$

The right-hand side is positive, but the left-hand side cannot be written in the form $\frac{d}{dt}U(z)$ and we cannot prove an à priori bound on the solution. To overcome this difficulty we introduced half space localization property (1.3).

Proof of theorem 1.3. Let us denote by $\varphi_\tau(u,m)$ a function from $E \times \mathbb{R}^N$ to $E$ defined by

$$\varphi_\tau(u,m) = L^{-1}\Big\{\tau\Big[F(t,\tau u+m) - ⨍ F(t,\tau u+m)\Big]\Big\}.$$

The function $\varphi_\tau$ is continuous in $(\tau,u,m)$ and compact. Solving $(\mathrm{F}_\tau)$ is equivalent to finding a zero of the function $\chi_\tau : E \times \mathbb{R}^N \to E \times \mathbb{R}^N$ defined by

$$\chi_\tau(u,m) = \big(u - \varphi_\tau(u,m), ⨍ F(t,\tau u(t)+m)dt\ \big).$$

Because of inequality

$$\|u\|_{L^2} \leq T^{1/2}\|u\|_{L^\infty} \leq T^{1/2}(\delta|m|+\gamma)$$

it is more convenient to study solvability of equation $\chi_\tau(u,m) = 0$ in the subset $\Omega = B_1 \times B_2$ in $E \times \mathbb{R}^n$ where $B_1 = B\Big(0, \dfrac{\delta R+\gamma}{1-\delta}T^{1/2}\Big) \subset E$ and $B_2 = B(0,R) \subset \mathbb{R}^N$, with $R = \dfrac{r+\gamma}{1-\delta}$. To prove the existence of solution, it suffices to show that the degree

$$d_\tau := \deg(\chi_\tau, B_1 \times B_2, (0,0)), \qquad \tau \in [0,1]$$

is different from zero. Because of the half space localization property (1.3) and Lemma 3.1 the degree is well defined, because $⨍ F(t,\tau u+m) \neq 0$ for $|m| = R$, and does not depend on $\tau$. We calculate it for $\tau = 0$:

$$\begin{aligned} d_0 &= \deg(u \times ⨍ F(t,m), B_1 \times B_2, (0,0)) \\ &= \deg(id_E, B_1, 0) \cdot \deg(⨍ F(t,m), B_2, 0) \\ &= \deg(⨍ F(t,m), B_2, 0) \end{aligned}$$

where we have used $\deg(id_E, B_1, 0) = 1$. Let us define $\bar{F}(m) = ⨍ F(t,m)$. Then $|m| = R$ implies that

$$\bar{F}(m) \cdot w(m) = ⨍ F(t,m) \cdot w(m)dt > 0.$$

We conclude that $\bar{F}/_{\partial B_2}$ and $w/_{\partial B_2}$ are homotopic and

$$\deg(w, B_2, 0) = 1 \neq 0.$$

This proves the theorem. ∎

## 5. Some consequences of Theorems 1.2 and 1.3

Here are some examples of the first order Hamiltonian systems for which Theorem 1.2 is applicable.

Corollary 5.1. *Suppose $F(t,z) = JH'(t,z)$ sub-linear at infinity and*

$$H'_x x - H'_y y > 0 \qquad (\text{or} < 0).$$

*Then, the equation (F) has a $T$-periodic solution.*

Here $W(z) = \frac{1}{2}(x^2 - y^2) = \frac{1}{2}\mathrm{Re}|z|^2$, $w(z) = (x, -y)$ and $\deg(w, B_R, 0) \neq 0$ for $R > 0$. In fact, $\deg(w, B_R, 0) = -1$ which is a consequence of excision and multiplicative properties of the degree.

Corollary 5.2. *A special case of the previous corollary is the following one:*

$$H(t,z) = f(t)\Big(\frac{1}{p}|x|^p - \frac{1}{p}|y|^p\Big), \quad 1 < p < 2$$

*where $f(t) > 0$ and is $T$-periodic. Then*

$$H'_x x - H'_y y = \Big(|x|^p + |y|^p\Big) f(t) > 0$$

*and Corollary 5.1 can be applied.*

Corollary 5.3. *Let $H(t,z) = \frac{1}{p} f(t)|z|^p + g(t)|z|$ where $1 < p < 2$ and $f, g$ are real $T$-periodic continuous functions and $0 < \alpha \leq \min f(t)$. Then the Hamiltonian system $\dot{z} = JH'(t,z)$ has a $T$-periodic solution.*

Proof. Indeed, $F(t,z) := JH'(t,z)$ is sub-linear at infinity. It is sufficient to prove that $F$ satisfies the half space localization property. Let us define $w(m) := Jm$ and choose $r > 0$ such that

$$\alpha r^{p-1} > \rho := \max_{t \in [0,T]} |g(t)|.$$

Then,

$$w \cdot m = Jm \cdot m = 0$$

and if $|m| = R_\varepsilon(r) =: R$ and $z \in B(m, R - r)$ then $|z| > r$ by Lemma 3.1 and

$$\begin{aligned} F(t,z) \cdot Jm = JH'(t,z) \cdot Jm &= H'(t,z) \cdot m \\ &= f(t)|z|^{p-2} z \cdot m + g(t)\frac{z}{|z|} \cdot m \\ &\geq \alpha r^{p-1} R - \rho R \geq (\alpha r^{p-1} - \rho) R > 0 \end{aligned}$$

which implies (1.3). □

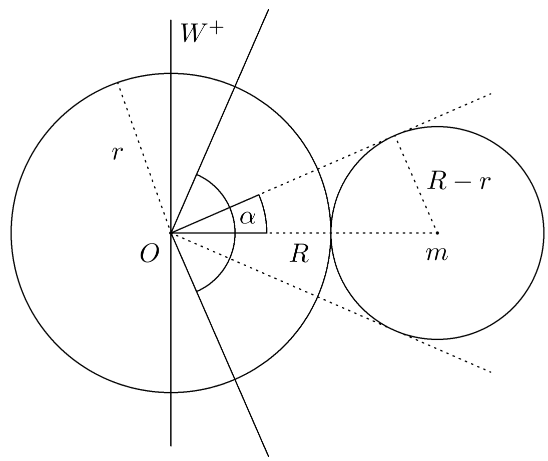
Corollary 5.4. *Same conclusion as in Corollary 5.3 with hypothesis $H(t,z) = \frac{1}{p} f(t)|z|^p + g(t)z$ where $g : [0,T] \to \mathbb{R}^N$.*

## 6. Radial-like Hamiltonians

*Proof of Theorem 1.4.* It is sufficient to prove that for $|m|$ large enough, there exists $\varepsilon > 0$ in ($\varepsilon$SL), such that

$$\sin\alpha := \frac{R_\varepsilon(r) - r}{R_\varepsilon(r)} < \mu. \tag{6.1}$$

In this case $H'(t,z)$ is an element of the positive dual cone generated by the ball $B(m, R_\varepsilon(r) - r)$, hence an element of the half space $\{x \in \mathbb{R}^{2N} \,|\, x \cdot m > 0\}$




and obviously $⨍ JH'(t,z)\cdot Jm = ⨍ H'(t,z)\cdot m \neq 0$ which proves half space localization property. To prove (6.1) we have

$$\frac{R_\varepsilon(r)-r}{R_\varepsilon(r)} = \frac{\frac{r+\gamma}{1-\delta}-r}{\frac{r+\gamma}{1-\delta}} = \frac{r\delta+\gamma}{r+\gamma}.$$

Then (6.1) is equivalent to

$$r(\mu-\delta) > \gamma(1-\mu).$$

Now, $\varepsilon$ can be chosen such that $\mu-\delta>0$ and $r$ can be taken such that

$$r > \frac{\gamma(1-\mu)}{\mu-\delta}$$

which proves the theorem. □

*Proof of Theorem 1.5.* Let $F(t,z) = JH'(t,z)$. Evidently $F$ is sub-linear at infinity. To prove that $H$ is radial-like Hamiltonian let us consider $z$ such that $|z|\geq r$. Then

$$\frac{H'(t,z)\cdot z}{|H'(t,z)||z|} \geq \frac{\Theta_1|z|^p}{\Theta_2|z|^p} = \frac{\Theta_1}{\Theta_2} =: \mu > 0$$

which proves the theorem. □

## 7. Almost convex Hamiltonians

*Proof of Theorem 1.6.* The idea is to prove inequality (1.1) and to get a priori bound on the solution from this inequality. We shall write (H) in the form

$$L_k z := \dot z + kJz = J\hat H'(t,z)$$

or, to simplify the notation

$$L_k z = \hat F(t,z) \tag{7.1}$$

with $\hat{F}(t,z) = J\hat{H}'(t,z)$.

Let us perform decomposition of the equation (7.1) like in ($F_1$), i.e.

$$\begin{array}{rcl} L_k u &=& \hat{F}(t,u+m) - ⨍ \hat{F}(t,u+m) \\ 0 &=& ⨍ F(t,u+m). \end{array} \tag{7.2}$$

The proof will be divided into several steps:

1st step: $\limsup_{|z|\to+\infty} \frac{|\hat{F}(t,z)|}{|z|} \le 2k$, uniformly on $t$.

2nd step: For any $u \in E$ such that $⨍ u = 0$ we have

$$\|u\|_{L^\infty} \le \frac{\sqrt{T}}{2\sqrt{3} - kT} \|L_k u\|_{L^2}.$$

3rd step: If $u \in E, ⨍ u = 0$, is a solution of the first equation (7.2) for given $m \in \mathbb{R}^{2N}$, then

$$\|u\|_{L^\infty} \le \delta|m| + \gamma, \qquad 0 < \delta < 1$$

where $\delta = \frac{2(2k+\varepsilon)T}{\sqrt{3} - 5kT - \varepsilon T}, \gamma = \frac{2C_\varepsilon T}{\sqrt{3} - 5kT - \varepsilon T}$.

4th step: (Conclusion) Using (Rad) and the 3rd step we obtain a priori bound on the solution, because (1.3) is satisfied with $w(m) = Jm$.

Proof of the 1st step, Let us denote by $\hat{G}$ the Legendre transform of $\hat{H}$, i.e.

$$\hat{G}(t,v) = -\hat{H}(t,z) + vz$$

where $v = \hat{H}'(t,z)$ and $z = \hat{G}'(t,v)$. Because of (WS) and the properties of the Legendre transform for each $\varepsilon, 0 < \varepsilon < k$, there exists $C_\varepsilon \in \mathbb{R}$ such that

$$\begin{array}{c} \frac{k-\varepsilon}{2}|z|^2 - C_\varepsilon \le \hat{H}(t,z) \le \frac{k+\varepsilon}{2}|z|^2 + C_\varepsilon \\ \frac{1}{2(k+\varepsilon)}|v|^2 - C_\varepsilon \le \hat{G}(t,v) \le \frac{1}{2(k-\varepsilon)}|v|^2 + C_\varepsilon. \end{array}$$

Functions $\hat{H}$ and $\hat{G}$ are bounded from bellow by a constant $-C_\varepsilon$ and consequently

$$\frac{1}{2(k+\varepsilon)}|v|^2 - C_\varepsilon \le \hat{G}(t,v) \le vz - \hat{H}(t,z) \le vz + C_\varepsilon, \tag{7.3}$$

$$\frac{k-\varepsilon}{2}|z|^2 - C_\varepsilon \le \hat{H}(t,z) \le vz - \hat{G}(t,z) \le vz + C_\varepsilon. \tag{7.4}$$

Dividing (7.3) by $|v|\,|z|$ we have

$$\frac{1}{2(k+\varepsilon)} \frac{|v|}{|z|} \le 1 + \frac{2C_\varepsilon}{|v|\,|z|}, \quad \forall \varepsilon > 0$$

which proves the 1st step.

Proof of the 2nd step.

$$\begin{aligned}\|L_k u\|_{L^2} &= \|\dot{u} + kJu\|_{L^2} \geq \|\dot{u}\|_{L^2} - k\|u\|_{L^2} \\ &\geq \frac{2\sqrt{3}}{\sqrt{T}}\|u\|_{L^\infty} - k\sqrt{T}\|u\|_{L^\infty} \\ &= \frac{2\sqrt{3} - kT}{\sqrt{T}}\|u\|_{L^\infty}\end{aligned}$$

where we have used inequality (2.1) from Lemma 2.1 and inequality $\|u\|_{L^2} \leq \sqrt{T}\|u\|_{L^\infty}$.

Proof of the 3rd step. Using the 1st and 2nd step in the first equation of (7.2) we obtain

$$\begin{aligned}\frac{2\sqrt{3} - kT}{\sqrt{T}}\|u\|_{L^\infty} &\leq \|L_k u\|_{L^2} \\ &\leq 2(2k + \varepsilon)T^{1/2}(\|u\|_{L^\infty} + |m|) + 2C_\varepsilon T^{1/2}\end{aligned}$$

which proves the claim. ∎

Corollary 7.1. *Suppose that $H(t, z) = h(z) + g(t)z$, where $h(z)$ is a convex, radial-like and weakly sub-quadratic. If $g : \mathbb{R} \to \mathbb{R}^N$ is $T$-periodic and non-constant. Then, Hamiltonian system $H$ has a non-constant $T$-periodic solution.*

## 8. Appendix

The following inequalities are useful in the theory of convex Hamiltonian systems. Let $G(z) = \gamma|z|^q + \alpha$ where $\gamma > 0$, $q > 1$ and $\alpha \in R$. Then if $\frac{1}{p} + \frac{1}{q} = 1$,

$$G^*(v) = \left(\frac{1}{\gamma q}\right)^{p/q} \frac{|v|^p}{p} - \alpha.$$

If $H$ is a Legendre function, then:

$$H(z) \leq \gamma|z|^q + \alpha \Longleftrightarrow H^*(v) \geq \left(\frac{1}{\gamma q}\right)^{p/q} \frac{|v|^p}{p} - \alpha, \tag{8.1}$$

$$H(z) \geq \gamma|z|^q + \alpha \Longleftrightarrow H^*(v) \leq \left(\frac{1}{\gamma q}\right)^{p/q} \frac{|v|^p}{p} - \alpha. \tag{8.2}$$

Proposition 8.1. *Let $H$ be a Legendre function such that for some $\gamma > 0$, $q > 1, \alpha, \eta \in R$,*

$$\eta \leq H(z) \leq \gamma|z|^q + \alpha. \tag{8.3}$$

*If* $\frac{1}{p}+\frac{1}{q}=1$ *then*

$$\left(\frac{1}{\gamma q}\right)^{p/q}\frac{|H'(z)|^p}{p}\leq H'(z)z+\alpha-\eta$$

PROOF. Let $v=H'(z)$. From (8.1) and (8.3) we obtain

$$\left(\frac{1}{\gamma q}\right)^{p/q}\frac{|v|^p}{p}-\alpha\leq H^*(v)=vz-H(z)\leq vz-\eta.$$

$\square$

## References


[1] A. Bahri, H. Berestycki, *Existence of forced oscillations for some nonlinear differential equations*, Comm. Pure Appl. Math. **37** (1983), 403–442.

[2] H. Berestycki, *Solutions périodiques de systèmes hamiltoniens, Séminaire Bourbaki*, Soc. Math. France, Paris, 1983, 105–128.

[3] H. Brezis, Analyse fonctionelle, théorie et applications, Masson, Paris, 1983.

[4] F. Clarke, I. Ekeland, *Hamiltonian trajectories having prescribed minimal period*, Comm. Pure, Appl. Math. **33** (1980), 103–116.

[5] L. Čaklović, *Periodic solutions of a first order nonconvex Hamiltonian system*, Glas. Mat. **37** (2002), 101–118.

[6] M. A. Krasnosel'ski, The operator of translation along trajectories of ordinary differential equations, Mir, Moskow, 1966.

[7] M. A. Krasnosel'ski, Topological methods in the theory of nonlinear integral equations, Macmillan, New York, 1964.

[8] I. Ekeland, J. M. Lasry, *On the number of Periodic trajectories for a Hamiltonian flow on a convex energy surface*, Ann. Math. **112**(1980), 283–319

[9] J. Mawhin, Topological degree methods in nonlinear boundary value problems, CBMS **40,** Amer. Math. Soc., Providence, R.I., 1979.

[10] L. Nirenberg, Topics in nonlinear functional analysis, Lecture notes, Courant Institute of Math. Sc., New York Univ., 1974.

[11] P. H. Rabinowitz, *Subharmonic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. **33** (1980), 609–633.

[12] P. H. Rabinowitz, Méthodes topologiques et problémes aux limits non linéaires, Lecture notes Univ. Paris VI, (1975), Redigées par H. Berestycki.

[13] M. Willem, *Subharmonic oscillations of convex Hamiltonian systems*, Nonlinear Anal. **9** (1985), 1303–1311.



L. Čaklović
Department of Mathematics
University of Zagreb
P.O.Box 335, 10002 Zagreb
Croatia
*E-mail*: `caklovic@math.hr`



*Received*: 20.06.2000.
*Revised*: 29.01.2003.