GLASNIK MATEMATIČKI
Vol. 38(58)(2003), 45 – 55

LOCAL STABILITY OF THE ADDITIVE FUNCTIONAL EQUATION

Soon-Mo Jung
Hong-Ik University, Korea

Abstract. In this paper, we prove the Hyers-Ulam stability of the additive functional equation for a number of unbounded domains. We moreover prove the stability of Jensen's functional equation for a large class of restricted domains.

1. Introduction

The starting point of studying the stability of functional equations seems to be the famous talk of S. M. Ulam [15] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by D. H. Hyers [3] under the assumption that G_1 and G_2 are Banach spaces. Later, the result of Hyers was significantly generalized by Th. M. Rassias [12]. It should be remarked that we can find in the books [4, 8] a lot of references concerning the stability of functional equations (or see [2, 5, 6]).

In [13, 14], F. Skof investigated the Hyers-Ulam stability of the additive functional equation for many cases of restricted domains in $\mathbb{R}$. Later,

2000 *Mathematics Subject Classification.* 39B82, 39B62.
Key words and phrases. Hyers-Ulam stability, local stability, additive functional equation, Jensen's functional equation.
The author was supported by Korea Research Foundation Grant. (KRF–DP0031).

L. Losonczi [10] proved the local stability of the additive equation for more general cases and applied the result to the proof of stability of the Hosszú's functional equation.

In Section 2, the Hyers-Ulam stability of the additive equation will be investigated for a large class of unbounded domains. Moreover, in Section 3, we will apply the previous result to the proof of the local stability of the Jensen's functional equation on unbounded domains.

Throughout this paper, let E_1 and E_2 be a real (or complex) normed space and a Banach space, respectively.

2. Stability of additive equation on restricted domains

Assume that $\varphi : (0, \infty) \to [0, \infty)$ is a decreasing mapping for which there exists a $d > 0$ such that

$$\varphi(s) \le s \tag{2.1}$$

for any $s \ge d$.

We may now define

$$B_1 = \{(x, y) \in E_1 \backslash \{0\} \times E_1 \,:\, \|y\| < \varphi(\|x\|)\} \cup \{(0, y) \in E_1^2 \,:\, y \in E_1\}$$

and

$$B_2 = \{(x, y) \in E_1^2 \,:\, \|x + y\| < d\}.$$

In the following theorem, we generalize the theorems of Skof [13, 14] and of Losonczi [10] concerning the stability of the additive equation on restricted domains.

Theorem 2.1. *If a mapping $f : E_1 \to E_2$ with $\|f(0)\| \le \varepsilon$ satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon \tag{2.2}$$

for some $\varepsilon \ge 0$ and all $(x, y) \in E_1^2 \backslash (B_1 \cup B_2)$, then there exists a unique additive mapping $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \le 39\varepsilon \tag{2.3}$$

for all $x \in E_1$.

Proof. First, we assume that $(x, y) \in B_2$ satisfies $x \ne 0$, $y \ne 0$ and $x + y \ne 0$. For this case, we can choose a $z_1 \in E_1$ with

$$\|z_1\| \ge \varphi(\|x + y\|), \quad \|z_1\| \ge \varphi(\|x\|), \quad \|x + z_1\| \ge \varphi(\|y\|),$$

$$\|x + y + z_1\| \ge d, \quad \|x + z_1\| \ge d.$$

Thus, the pairs $(x+y, z_1)$, (x, z_1) and $(y, x+z_1)$ do not belong to $B_1 \cup B_2$. Hence, it follows from (2.2) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \| - f(x+y+z_1) + f(x+y) + f(z_1)\| \\ &\quad + \|f(x+z_1) - f(x) - f(z_1)\| \\ &\quad + \|f(x+y+z_1) - f(y) - f(x+z_1)\| \\ &\leq 3\varepsilon \end{aligned} \tag{2.4}$$

for any $(x, y) \in B_2$ with $x \neq 0$, $y \neq 0$ and $x + y \neq 0$.

When $x = 0$ or $y = 0$, we have

$$\|f(x+y) - f(x) - f(y)\| = \|f(0)\| \leq \varepsilon.$$

Taking this fact into account, we see that the inequality (2.4) is valid for all $(x, y) \in B_2$ with $x + y \neq 0$.

We now assume that $(x, y) \in B_2$ satisfies $x + y = 0$ and $\|x\| \geq d$. (In this case, $\|y\| = \| - x\| \geq d$.) In view of (2.1), both the pairs $(-x, -x)$ and $(x, -2x)$ do not belong to $B_1 \cup B_2$. Hence, it follows from (2.2) that

$$\|f(-2x) - 2f(-x)\| \leq \varepsilon$$

and

$$\|f(-x) - f(x) - f(-2x)\| \leq \varepsilon.$$

From the last two inequalities we get

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &= \|f(0) - f(x) - f(-x)\| \\ &\leq \|f(0)\| + \|f(-2x) - 2f(-x)\| \\ &\quad + \|f(-x) - f(x) - f(-2x)\| \\ &\leq 3\varepsilon. \end{aligned}$$

Considering all the previous inequalities including (2.2), we may conclude that f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq 3\varepsilon \tag{2.5}$$

for all $(x, y) \in E_1^2 \backslash (B_1 \cup B_2) \cup \{(u, v) \in B_2 \ : \ \|u\| \geq d\}$.

Now, let $(x, y) \in E_1^2$ be arbitrarily given with $\|x\| \geq d$ and $\|y\| \geq d$. Since φ is decreasing, we see by (2.1) that

$$\varphi(\|x\|) \leq \varphi(d) \leq d \leq \|y\|,$$

and this implies that $(x, y) \notin B_1$. If, moreover, the given pair (x, y) belongs to B_2, then $(x, y) \in \{(u, v) \in B_2 \ : \ \|u\| \geq d\}$. Otherwise, $(x, y) \in E_1^2 \backslash (B_1 \cup B_2)$. Hence, it follows from (2.5) that

$$\|f(x+y) - f(x) - f(y)\| \leq 3\varepsilon \tag{2.6}$$

for all $(x, y) \in E_1^2$ with $\|x\| \geq d$ and $\|y\| \geq d$.

Assume that $(x,y) \in E_1^2$ with $\|x\| < d$ and $\|y\| \geq 4d$. In this case, we may choose a $z_2 \in E_1$ with $2d \leq \|z_2\| < 3d$. Then, it holds that

$$\begin{array}{lll} \|x - z_2\| \geq d & \text{and} & \|y + z_2\| \geq d, \\ \|x - z_2\| \geq d & \text{and} & \|z_2\| \geq 2d, \\ \|-z_2\| \geq 2d & \text{and} & \|y + z_2\| \geq d, \\ \|z_2\| \geq 2d & \text{and} & \|-z_2\| \geq 2d. \end{array} \tag{2.7}$$

It then follows from (2.6) and (2.7) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \|f(x+y) - f(x-z_2) - f(y+z_2)\| \\ &\quad + \|-f(x) + f(x-z_2) + f(z_2)\| \\ &\quad + \|-f(y) + f(-z_2) + f(y+z_2)\| \\ &\quad + \|f(0) - f(z_2) - f(-z_2)\| + \|-f(0)\| \\ &\leq 13\varepsilon \end{aligned} \tag{2.8}$$

for $(x,y) \in E_1^2$ with $\|x\| < d$ and $\|y\| \geq 4d$.

Combining (2.6) and (2.8), we have

$$\|f(x+y) - f(x) - f(y)\| \leq 13\varepsilon \tag{2.9}$$

for all $(x,y) \in E_1^2$ with $\|y\| \geq 4d$. Since the Cauchy difference $f(x+y) - f(x) - f(y)$ is symmetric with respect to x and y, we may conclude that the inequality (2.9) is true for all $(x,y) \in E_1^2$ with $\|x\| \geq 4d$ or $\|y\| \geq 4d$.

If $(x,y) \in E_1^2$ satisfies $\|x\| < 4d$ and $\|y\| < 4d$, then we can choose a $z_3 \in E_1$ with $\|z_3\| \geq 8d$. Then, we have $\|x + z_3\| \geq 4d$. Since the inequality (2.9) holds true for all $(x,y) \in E_1^2$ with $\|x\| \geq 4d$ or $\|y\| \geq 4d$, we get

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \|-f(x+y+z_3) + f(x+y) + f(z_3)\| \\ &\quad + \|f(x+z_3) - f(x) - f(z_3)\| \\ &\quad + \|f(x+y+z_3) - f(y) - f(x+z_3)\| \\ &\leq 39\varepsilon \end{aligned}$$

for any $(x,y) \in E_1^2$ with $\|x\| < 4d$ and $\|y\| < 4d$.

The last inequality, together with (2.9), yields

$$\|f(x+y) - f(x) - f(y)\| \leq 39\varepsilon$$

for all $x, y \in E_1$.

According to [1], there exists a unique additive mapping $A : E_1 \to E_2$ that satisfies the inequality (2.3) for each x in E_1. $\square$

Corollary 2.2. *Let $d > 0$ and $\varepsilon \geq 0$ be given. If a mapping $f : E_1 \to E_2$ with $\|f(0)\| \leq \varepsilon$ satisfies the inequality* (2.2) *for all $x, y \in E_1$ with $\max\{\|x\|, \|y\|\} \geq d$ and $\|x + y\| \geq d$, then there exists a unique additive mapping $A : E_1 \to E_2$ that satisfies the inequality* (2.3) *for each $x \in E_1$.*

PROOF. Because of the symmetry property of the Cauchy difference with respect to x and y, we can without loss of generality assume that f satisfies the inequality (2.2) for all $x, y \in E_1$ with $\|y\| \geq d$ and $\|x+y\| \geq d$.

For a constant mapping $\varphi(s) = d$ ($s > 0$), let us define

$$B_1 = \{(x,y) \in E_1 \backslash \{0\} \times E_1 \,:\, \|y\| < d\} \cup \{(0,y) \in E_1^2 \,:\, y \in E_1\}$$

and

$$B_2 = \{(x,y) \in E_1^2 \,:\, \|x+y\| < d\}.$$

Since

$$E_1^2 \backslash B_1 = \{(x,y) \in E_1 \backslash \{0\} \times E_1 \,:\, \|y\| \geq d\}$$

and

$$E_1^2 \backslash B_2 = \{(x,y) \in E_1^2 \,:\, \|x+y\| \geq d\},$$

we have

$$E_1^2 \backslash (B_1 \cup B_2) = \{(x,y) \in E_1 \backslash \{0\} \times E_1 \,:\, \|y\| \geq d \text{ and } \|x+y\| \geq d\}.$$

Thus, it follows from our hypothesis that f satisfies the inequality (2.2) for all $(x,y) \in E_1^2 \backslash (B_1 \cup B_2)$.

According to Theorem 2.1, there exists a unique additive mapping $A : E_1 \to E_2$ that satisfies the inequality (2.3) for all $x \in E_1$. □

In 1983, Skof [14] presented an interesting asymptotic behavior of the additive mappings:

> *A mapping $f : \mathbb{R} \to \mathbb{R}$ is additive if and only if $|f(x+y) - f(x) - f(y)| \to 0$ as $|x| + |y| \to \infty$.*

Without difficulty, the above theorem can be extended to mappings from a real normed space to a Banach space. We will now apply the previous corollary to a generalization of the above theorem of Skof:

COROLLARY 2.3. *A mapping $f : E_1 \to E_2$ is additive if and only if*

$$\|f(x+y) - f(x) - f(y)\| \to 0$$

as $\|x+y\| \to \infty$.

PROOF. On account of the hypothesis, there exists a decreasing sequence (ε_n) with $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon_n$$

for all $(x,y) \in E_1^2$ with $\|x+y\| \geq n$. With $y = 0$ and $\|x\| \to \infty$, our hypothesis implies $f(0) = 0$.

By Corollary 2.2, there exists a unique additive mapping $A_n : E_1 \to E_2$ such that

$$\|f(x) - A_n(x)\| \leq 39\varepsilon_n \tag{2.10}$$

for all $x \in E_1$.

Now, let l and m be integers with $m > l > 0$. Then, the inequality (2.10) implies that

$$\|f(x) - A_m(x)\| \leq 39\varepsilon_m \leq 39\varepsilon_l,$$

for $x \in E_1$, and further the uniqueness of A_n implies that $A_m = A_l$ for all integers $l, m > 0$, i.e., $A_n = A_1$ for any $n \in \mathbb{N}$. By letting $m \to \infty$ in the last inequality, we get

$$\|f(x) - A_1(x)\| = 0,$$

for any $x \in E_1$, which means that f is additive. The reverse assertion is trivial. $\square$

3. Stability of Jensen's equation on restricted domains

Z. Kominek investigated in [9] the Hyers-Ulam stability of the Jensen's functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$$

for the class of mappings defined on a bounded subset of $\mathbb{R}^N$. On the other hand, the author proved in [7] the Hyers-Ulam stability of that equation on unbounded domains.

In this section, we will use Theorem 2.1 to generalize the theorems of the author and of Kominek.

Let $\varphi_1 : [0, \infty) \to [0, \infty)$ be a decreasing mapping that satisfies $\varphi_1(0) = d_0 > 0$. Let us define

$$\begin{aligned} B_1 &= \{(x, y) \in E_1 \backslash \{0\} \times E_1 : \|y\| < \varphi_1(\|x\|)\} \cup \{(0, y) \in E_1^2 : y \in E_1\}, \\ B_2 &= \{(x, y) \in E_1^2 : \|x + y\| < d_0\}, \\ D &= \{(0, y) \in E_1^2 : \|y\| \geq d_0\}. \end{aligned}$$

Theorem 3.1. *If a mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \varepsilon \tag{3.11}$$

for some $\varepsilon \geq 0$ and all $(x, y) \in E_1^2 \backslash (B_1 \cup B_2) \cup D$, then there exists a unique additive mapping $A : E_1 \to E_2$ such that

$$\|f(x) - A(x) - f(0)\| \leq 78\varepsilon \tag{3.12}$$

for any $x \in E_1$.

Proof. If we substitute $g(x)$ for $f(x) - f(0)$ in (3.11), then we have

$$\left\|2g\left(\frac{x+y}{2}\right) - g(x) - g(y)\right\| \leq \varepsilon \tag{3.13}$$

for any $(x, y) \in E_1^2 \backslash (B_1 \cup B_2) \cup D$. With $x = 0$ and $\|y\| \geq d_0$, the inequality (3.13) yields

$$\left\|2g\left(\frac{y}{2}\right) - g(y)\right\| \leq \varepsilon$$

for each $y \in E_1$ with $\|y\| \geq d_0$. Let us replace y by $x+y$ ($\|x+y\| \geq d_0$) in the above inequality to get

$$\left\| 2g\left(\frac{x+y}{2}\right) - g(x+y) \right\| \leq \varepsilon \tag{3.14}$$

for all $x, y \in E_1$ with $\|x+y\| \geq d_0$.

It follows from (3.13) and (3.14) that

$$\begin{aligned} \|g(x+y) - g(x) - g(y)\| &\leq \left\| g(x+y) - 2g\left(\frac{x+y}{2}\right) \right\| \\ &\quad + \left\| 2g\left(\frac{x+y}{2}\right) - g(x) - g(y) \right\| \\ &\leq 2\varepsilon \end{aligned}$$

for every $(x,y) \in E_1^2 \backslash (B_1 \cup B_2) \cup D$ with $\|x+y\| \geq d_0$. Since $(x,y) \in E_1^2 \backslash (B_1 \cup B_2)$ implies $\|x+y\| \geq d_0$, the mapping g surely satisfies

$$\|g(x+y) - g(x) - g(y)\| \leq 2\varepsilon$$

for all $(x,y) \in E_1^2 \backslash (B_1 \cup B_2)$.

It trivially holds that $\varphi_1(s) \leq s$ for all $s \geq d_0$. On account of Theorem 2.1, there exists a unique additive mapping $A : E_1 \to E_2$ such that

$$\|g(x) - A(x)\| \leq 78\varepsilon$$

for each x in E_1. □

Let $\varphi_2 : (0, \infty) \to [0, \infty)$ be a continuous and decreasing mapping that satisfies

$$0 < d = \inf\{s > 0 \,:\, \varphi_2(s) = 0\} < \infty.$$

Furthermore, let us assume that the restriction $\varphi_2|_{(0,d]}$ is strictly decreasing.

Now, we define

$$\begin{aligned} B_1 &= \{(x,y) \in E_1 \backslash \{0\} \times E_1 \,:\, \|y\| < \varphi_2(\|x\|)\} \cup \{(0,y) \in E_1^2 \,:\, y \in E_1\}, \\ B_2 &= \{(x,y) \in E_1^2 \,:\, \|x+y\| < d_0\}, \\ D &= \{(0,y) \in E_1^2 \,:\, \|y\| \geq d_0\}, \end{aligned}$$

where we set $d_0 = \inf\{d, \lim_{s \to 0+} \varphi_2(s)\}$.

Corollary 3.2. *If a mapping $f : E_1 \to E_2$ satisfies the inequality* (3.11) *for some $\varepsilon \geq 0$ and all $(x,y) \in E_1^2 \backslash (B_1 \cup B_2) \cup D$, then there exists a unique additive mapping $A : E_1 \to E_2$ satisfying the inequality* (3.12) *for all $x \in E_1$.*

Proof. First, we may define a mapping $\varphi_0 : [0, \infty) \to [0, \infty)$ by

$$\varphi_0(s) = \begin{cases} d_0, & \text{for } s = 0, \\ \inf\{\varphi_2(s), \inf \varphi_2^{-1}(s)\}, & \text{for } s > 0, \end{cases}$$

where we set $\varphi_2^{-1}(t) = \{s > 0 \,:\, \varphi_2(s) = t\}$ and $\inf \emptyset = \infty$. (We cannot exclude the case $\varphi_2^{-1}(s) = \emptyset$ from the above definition). Let us define

$$\begin{aligned} \tilde{B}_1 &= \{(x,y) \in E_1 \backslash \{0\} \times E_1 \,:\, \|y\| < \varphi_0(\|x\|)\} \cup \{(0,y) \in E_1^2 \,:\, y \in E_1\}, \\ \tilde{B}_2 &= \{(x,y) \in E_1^2 \,:\, \|x+y\| < d_0\}, \\ \tilde{D} &= \{(0,y) \in E_1^2 \,:\, \|y\| \geq d_0\}. \end{aligned}$$

The fact that $\varphi_0(s) \leq \varphi_2(s)$ for all $s > 0$ implies $\tilde{B}_1 \subset B_1$. Since $B_2 = \tilde{B}_2$ and $D = \tilde{D}$, we get

$$E_1^2 \backslash (B_1 \cup B_2) \cup D \subset E_1^2 \backslash (\tilde{B}_1 \cup \tilde{B}_2) \cup \tilde{D}.$$

Now, assume that $(x,y) \in E_1^2 \backslash (\tilde{B}_1 \cup \tilde{B}_2) \cup \tilde{D}$ but $(x,y) \notin E_1^2 \backslash (B_1 \cup B_2) \cup D$. Because $(x,y) \notin D$ and $(x,y) \notin B_2$, we have

$$x \neq 0 \text{ and } \|x+y\| \geq d_0. \tag{3.15}$$

Moreover, (x,y) should belong to $B_1 \backslash \tilde{B}_1$, i.e.,

$$0 < \inf \varphi_2^{-1}(\|x\|) \leq \|y\| < \varphi_2(\|x\|). \tag{3.16}$$

(Since $\|x\| > 0$ and $\varphi_2|_{(0,d]}$ is strictly decreasing, we have $\inf \varphi_2^{-1}(\|x\|) > 0$.) If we assume that $(y,x) \in B_1$, then we get $\|x\| < \varphi_2(\|y\|)$. This fact implies $\|y\| < \inf \varphi_2^{-1}(\|x\|)$, which is contrary to (3.16). Hence, by (3.15), we conclude that $(y,x) \notin B_1 \cup B_2$. This fact, together with (3.11), yields

$$\left\| 2f\left(\frac{y+x}{2}\right) - f(y) - f(x) \right\| \leq \varepsilon$$

for all $(x,y) \in E_1^2 \backslash (\tilde{B}_1 \cup \tilde{B}_2) \cup \tilde{D}$.

We now define another mapping $\varphi : [0,\infty) \to [0,\infty)$ by

$$\varphi(s) = \begin{cases} d_0, & \text{for } s = 0, \\ \inf\{\varphi_2(s), \inf \varphi_2^{-1}(s)\}, & \text{for } 0 < s \leq d_1, \\ \sup\{\varphi_2(s), \sup \varphi_2^{-1}(s)\}, & \text{for } s > d_1, \end{cases}$$

where $d_1 > 0$ is the unique fixed point of φ_2, i.e., $d_1 = \varphi_2(d_1)$, and we set $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

Let $s_i > 0$ $(i = 1,2,3,4)$ be arbitrarily given with $0 < s_1 < s_2 \leq d_1 < s_3 < s_4$. Since φ_2 is decreasing, we have

$$\lim_{s \to 0+} \varphi_2(s) \geq \varphi_2(s_1) \geq \varphi_2(s_2) \geq d_1 \geq \varphi_2(s_3) \geq \varphi_2(s_4)$$

and

$$d \geq \inf \varphi_2^{-1}(s_1) \geq \inf \varphi_2^{-1}(s_2) \geq d_1 \geq \sup \varphi_2^{-1}(s_3) \geq \sup \varphi_2^{-1}(s_4).$$

Hence, we get

$$\varphi(0) \geq \varphi(s_1) \geq \varphi(s_2) \geq \varphi(s_3) \geq \varphi(s_4),$$

which implies that φ is decreasing.

Similarly as before, we define

$$\begin{aligned}
\hat{B}_1 &= \{(x,y) \in E_1\backslash\{0\} \times E_1 \,:\, \|y\| < \varphi(\|x\|)\} \cup \{(0,y) \in E_1^2 \,:\, y \in E_1\}, \\
\hat{B}_2 &= \{(x,y) \in E_1^2 \,:\, \|x+y\| < d_0\}, \\
\hat{D} &= \{(0,y) \in E_1^2 \,:\, \|y\| \geq d_0\}.
\end{aligned}$$

Since $\hat{B}_1 \supset \tilde{B}_1$, $\hat{B}_2 = \tilde{B}_2$ and $\hat{D} = \tilde{D}$, we may conclude that the inequality (3.11) holds true for all $(x,y) \in E_1^2\backslash(\hat{B}_1 \cup \hat{B}_2) \cup \hat{D}$.

According to Theorem 3.1, there exists a unique additive mapping $A : E_1 \to E_2$ such that the inequality (3.12) is true for any $x \in E_1$. □

The author [7] proved that it needs only to show an asymptotic property of the Jensen difference to identify a given mapping with an additive one:

> *Let X and Y be a real normed space and a real Banach space, respectively. A mapping $f : X \to Y$ with $f(0) = 0$ is additive if and only if*
>
> $$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \to 0$$
>
> *as $\|x\| + \|y\| \to \infty$.*

By using Theorem 3.1, we will now prove an asymptotic behavior of additive mappings which generalizes the above result:

Corollary 3.3. *A mapping $f : E_1 \to E_2$ with $f(0) = 0$ is additive if and only if*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \to 0$$

as $\|x+y\| \to \infty$.

Proof. According to our hypothesis, there exists a decreasing sequence (ε_n) with $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \varepsilon_n \tag{3.17}$$

for all $(x,y) \in E_1^2$ with $\|x+y\| \geq n$.

The mapping $\varphi_1 : [0,\infty) \to [0,\infty)$ defined by $\varphi_1(s) = -s + n$ $(s \geq 0)$ is decreasing. Moreover, it holds that $\varphi_1(0) = n$. Let us define

$$\begin{aligned}
B_1 &= \{(x,y) \in E_1\backslash\{0\} \times E_1 \,:\, \|y\| < -\|x\| + n\} \cup \{(0,y) \in E_1^2 \,:\, y \in E_1\}, \\
B_2 &= \{(x,y) \in E_1^2 \,:\, \|x+y\| < n\}, \\
D &= \{(0,y) \in E_1^2 \,:\, \|y\| \geq n\}.
\end{aligned}$$

Since $B_1 \cup B_2 = \{(x,y) \in E_1^2 \,:\, x = 0 \text{ or } \|x+y\| < n\}$ and $D = \{(x,y) \in E_1^2 \,:\, x = 0 \text{ and } \|x+y\| \geq n\}$, we have

$$E_1^2\backslash(B_1 \cup B_2) = \{(x,y) \in E_1^2 \,:\, x \neq 0 \text{ and } \|x+y\| \geq n\},$$

and hence

$$E_1^2 \backslash (B_1 \cup B_2) \cup D = \{(x, y) \in E_1^2 : \|x + y\| \geq n\}.$$

Therefore, the inequality (3.17) holds true for all $(x, y) \in E_1^2 \backslash (B_1 \cup B_2) \cup D$.

According to Theorem 3.1, there exists a unique additive mapping $A_n : E_1 \to E_2$ such that

$$\|f(x) - A_n(x)\| \leq 78\varepsilon_n \tag{3.18}$$

for all $x \in E_1$. Now, let l and m be positive integers with $m > l$. Then, it follows from (3.18) that

$$\|f(x) - A_m(x)\| \leq 78\varepsilon_m \leq 78\varepsilon_l$$

for $x \in E_1$. However, the uniqueness of A_n implies that $A_m = A_l$ for all positive integers l and m, i.e., $A_n = A_1$ for any $n \in \mathbb{N}$. By letting $m \to \infty$ in the last inequality, we get

$$\|f(x) - A_1(x)\| = 0,$$

for each $x \in E_1$, which implies that f is an additive mapping.

The reverse assertion is trivial because every additive mapping $f : E_1 \to E_2$ is a solution of the Jensen functional equation (see [11]). □

References

[1] G. L. Forti, *The stability of homomorphisms and amenability with applications to functional equations*, Abh. Math. Sem. Univ. Hamburg **57** (1987), 215–226.

[2] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), 143–190.

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[4] D. H. Hyers, G. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Boston/Basel/Berlin, 1998.

[5] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), 125–153.

[6] S.-M. Jung, *Hyers-Ulam-Rassias stability of functional equations*, Dynam. Systems Appl. **6** (1997), 541-566.

[7] S.-M. Jung, *Hyers-Ulam-Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc. **126** (1998), 3137–3143.

[8] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.

[9] Z. Kominek, *On a local stability of the Jensen functional equation*, Demonstratio Math. **22** (1989), 499–507.

[10] L. Losonczi, *On the stability of Hosszú's functional equation*, Result. Math. **29** (1996), 305–310.

[11] J. C. Parnami and H. L. Vasudeva, *On Jensen's functional equation*, Aequationes Math. **43** (1992), 211–218.

[12] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[13] F. Skof, *Sull'approssimazione delle applicazioni localmente δ-additive*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **117** (1983), 377–389.

[14] F. Skof, *Proprieta' locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[15] S. M. Ulam, Problems in Modern Mathematics, Chap. VI, Wiley, 1964.

S.-M. Jung
Hong-Ik University, Mathematics Section,
College of Science and Technology,
339–701 Chochiwon, Korea
E-mail: smjung@wow.hongik.ac.kr

Received: 19.06.2002.