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ON A-STATISTICAL CLUSTER POINTS

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Abstract. In this paper we study the concepts of statistical cluster points and statistical core of a sequence for C_λ methods defined by deleting a set of rows from the Cesáro matrix C_1. Also we get necessary conditions on the matrices A and B so that A and B are equivalent in the statistical convergence sense and, study the equality $\Gamma_A(x) = \Gamma_B(x)$,where $\Gamma_A(x)$ is the set of A-statistical cluster points of the real number sequence x.

1. Introduction and notations

In [5] Fridy introduced the concepts of statistical limit points and statistical cluster points of a number sequence. These concepts are compared to the usual concept of limit point of a sequence. In [6] Fridy and Orhan introduced the concepts of statistical limit superior and inferior. They have also given the definition of the statistical core of a real number sequence which is based on the idea of the statistical cluster points of the sequence, and proved the statistical core theorem. Those results have also been extended [7] to the complex case by them, too. In [2] Connor and Kline extended the concept of a statistical limit (cluster) point of a number sequence to a A-statistical limit (cluster) point where A is a nonnegative regular summability matrix. In [3] the present author extended the concepts of statistical limit superior and inferior (as introduced by Fridy and Orhan) to A-statistical limit superior and inferior and given some A-statistical analogue of properties of statistical limit superior and inferior for a sequence of real numbers. Also in [3] the concept of statistical core is extended to A-statistical core.

In this paper we study the concepts of statistical cluster points and statistical core of a sequence for C_λ methods, defined by deleting a set of rows from

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the Cesàro matrix C_1. Also we get necessary conditions on the matrices A and B so that A and B are equivalent in the statistical convergence sense and, study the equality $\Gamma_A(x) = \Gamma_B(x)$,where $\Gamma_A(x)$ is the set of $A-$ statistical cluster points of the real number sequence x.

First we introduce some notation. Let $A = (a_{nk})$ denote a summability matrix which transforms a number sequence $x = (x_k)$ into the sequence Ax whose n-th term is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$. As usual, $\mathbb{N}$ and $\mathbb{C}$ denote the sets of positive integers and complex numbers, respectively.

If K is a set of positive integers, $|K|$ will denote the cardinality of K. The natural density of K [11] is given by

$$\delta(K) := \lim_n (C_1\chi_K)_n = \lim_n \frac{1}{n} \left|\{k \le n : k \in K\}\right|,$$

if it exists, where C_1 is the Cesàro mean of order one and χ_K is the characteristic function of the set K.

We recall the following elementary result concerning natural density (See [11, page 222]):

Let E be an infinite subset of $\mathbb{N}$ and consider E as strictly increasing sequence of positive integers, say $E = \{\lambda(n)\}_{n=1}^{\infty}$. Then

$$\delta(E) = \lim_n \frac{n}{\lambda(n)}$$

provided this limit exists. Because $\delta(E)$ does not exists for all subsets of $\mathbb{N}$, it is convenient to use the upper asymptotic density $\delta^*(E)$, which is defined by

$$\delta^*(E) = \limsup_n \frac{1}{n} \left|\{k \le n : k \in E\}\right|$$

(See [9, p.xvii]). For convenience we state here some properties of δ^*. For arbitrary subsets E and F of $\mathbb{N}$ we have

(i) if $\delta(E)$ exists then $\delta(E) = \delta^*(E)$;
(ii) $\delta(E) \neq 0$ if and only if $\delta^*(E) > 0$;
(iii) if $E \subseteq F$, then $\delta^*(E) \le \delta^*(F)$.

Natural density can be generalized by using a nonnegative regular summability matrix A in place of C_1.

Following Freedman and Sember [4] we say that a set $K \subseteq \mathbb{N}$ has $A-$density if

$$\delta_A(K) = \lim_n \sum_{k\in K} a_{nk} = \lim_n \sum_{k=1}^{\infty} a_{nk}\chi_K(k) = \lim_n (A\chi_K)_n$$

exists where A is a nonnegative regular summability matrix.

The number sequence $x = (x_k)$ is $A-$statistically convergent to L provided that for every $\epsilon > 0$ the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \ge \varepsilon\}$ has $A-$density zero [2, 10]. In this case we write $st_A - \lim x = L$.

By st_A we denote the set of all $A-$statistically convergent sequences.

The number γ is a $A-$statistical cluster point of the number sequence $x = (x_k)$ provided that for every $\epsilon > 0, \delta_A(K_\varepsilon) \neq 0$ where $K_\varepsilon := \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$ [2]. Note that the statement $\delta_A(K) \neq 0$ means that either $\delta_A(K) > 0$ or K fails to have $A-$density.

By $\Gamma_A(x)$ we denote the set of all $A-$statistical cluster points of x. When $A = C_1$ we shall simply write δ instead of δ_{C_1} and Γ instead of Γ_{C_1}.

The sequence $x = (x_k)$ is the $A-$statistical bounded if it has a bounded subsequence $\{x_k\}_{k\in E}$ such that $\delta_A(E) = 1$; $st_A - \limsup x$ and $st_A - \liminf x$ are the greatest and least $A-$statistical cluster point of such an x [3]. Also $A-$statistically bounded sequence x is $A-$statistically convergent if and only if $st_A - \liminf\, x = st_A - \limsup x$ [3].

Note that $A-$statistical boundedness implies that $st_A - \limsup$ and $st_A - \limsup$ are finite [3]. Some results on statistical limit points may be found in [2, 5, 6, 13].

For any complex number sequence $x = (x_k)$ the $A-$statistical core of x is given by

$$st_A - core\,\{x\} = \bigcap_{H\in \mathbf{H}(x)} H,$$

where $\mathbf{H}(x)$ is the collection of all closed half-planes H that satisfy $\delta_A\,\{k \in \mathbb{N} : x_k \in H\} = 1$ (see [3]).

In [3, Theorem 6] it is shown that for every $A-$ statistically bounded complex number sequence $x = (x_k)$

$$st_A - core\,\{x\} = \bigcap_{z\in\mathbb{C}} B_x(z),$$

where

$$B_x(z) := \left\{ w \in \mathbb{C} : |w - z| \leq st_A - \limsup_k |x_k - z| \right\}.$$

When $A = C_1$ we shall simply write st-core instead of $st_{C_1} - core$ (see [6, 7]).

2. C_λ-STATISTICAL CLUSTER POINTS

In [1] Armitage and Maddox introduced the summability method C_λ defined by deleting a set of rows from the Cesàro matrix. They gave some inclusion theorems for C_λ methods. This method has also been studied in [12].

Let E be an infinite subset of $\mathbb{N}$ and consider E as strictly increasing sequence of positive integers, say $E = \{\lambda(n)\}_{n=1}^{\infty}$. The summability method C_λ, as introduced in [1], is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k,$$

where $x = (x_k)$ is a sequence of real or complex numbers and $n = 1, 2, \ldots$. It is clear that C_λ is regular for any λ.

Note that if $A = C_\lambda$, then $\gamma \in \Gamma_{C_\lambda}(x)$ if, for every $\varepsilon > 0$,

$$\delta_{C_\lambda}(K_\varepsilon) = \lim_n (C_\lambda \chi_{K_\varepsilon})_n = \lim_n \frac{1}{\lambda(n)} \left|\{k \leq k(n) : |x_k - \gamma| < \varepsilon\}\right| \neq 0.$$

In the particular case when $\lambda(n) = n$ we see that $(C_\lambda \chi_{K_\varepsilon})_n$ is the C_1 mean of χ_{K_ε}.

In this section we establish inclusion relations between $\Gamma_{C_\lambda}(x)$ and $\Gamma_{C_\mu}(x)$ and between $\Gamma(C_\lambda x)$ and $\Gamma(C_\mu x)$ for C_λ methods. Also we study C_λ-statistical core for a bounded complex sequence.

THEOREM 2.1. *Let $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ be infinite subsets of $\mathbb{N}$. If $E \setminus F$ is finite and $\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then*

$$\delta_{C_\lambda}(K) \neq 0 \textit{ implies } \delta_{C_\mu}(K) \neq 0 \textit{ for every } K \subseteq \mathbb{N}.$$

PROOF. If $E \setminus F$ is finite, then there exists N such that $\{\mu(n) : n \geq N\} \subset F$. For $n \geq N$ let $j(n)$ be such that $\mu(n) = \lambda_{j(n)}$. Then $(j(n))$ increases and $j(n) \to \infty$, (as $n \to \infty$). If $\delta_{C_\lambda}(K) \neq 0$, then

$$\delta^*_{C_\lambda}(K) = \limsup_n \frac{|\{i \leq \lambda(n) : i \in K\}|}{\lambda(n)} > 0.$$

Since $\limsup_n(x_n y_n) \leq (\lim_n x_n)(\limsup_n y_n)$ provided that the right hand side exists, and

$$\frac{\lambda(n)}{\lambda_{j(n)}} \frac{|\{i \leq \lambda(n) : i \in K\}|}{\lambda(n)} \leq \frac{\left|\{i \leq \lambda_{j(n)} : i \in K\}\right|}{\lambda_{j(n)}},$$

we get

$$\delta^*_{C_\mu}(K) = \limsup_n \frac{|\{i \leq \mu(n) : i \in K\}|}{\mu(n)} > 0.$$

Hence $\delta_{C_\mu}(K) \neq 0$. ∎

Since $E \Delta F = (E \setminus F) \cup (F \setminus E), (C_\mu x)_n = (C_1 x)_{\mu(n)}$ and $(C_\lambda x)_n = (C_1 x)_{\lambda(n)}$, we immediately get the following from Theorem 2.1.

THEOREM 2.2. *Let $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ be infinite subsets of $\mathbb{N}$.*

(i) *If $E \setminus F$ is finite and $\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma_{C_\lambda}(x) \subseteq \Gamma_{C_\mu}(x)$.*

(ii) *If $E \Delta F$ is finite and $\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma_{C_\lambda}(x) = \Gamma_{C_\mu}(x)$.*

(iii) *If $E \setminus F$ is finite and $\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma(C_\mu x) \subseteq \Gamma(C_\lambda x)$.*

(iv) *If $E \Delta F$ is finite and $\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0$, then $\Gamma(C_\mu x) = \Gamma(C_\lambda x)$.*

When $\lambda(n) = n$ the following may be deduced from (i) and (iii) of Theorem 2.2.

THEOREM 2.3. *Let $E = \{\mu(n)\}$ be infinite subset of* $\mathbb{N}$.

(i) *If* $\lim_n \frac{n}{\mu(n)} = d \neq 0$, *then* $\Gamma(x) \subseteq \Gamma_{C_\mu}(x)$.

(ii) *If* $\lim_n \frac{n}{\mu(n)} = d \neq 0$, *then* $\Gamma(C_\mu x) \subseteq \Gamma(C_1 x)$.

It is clear from (i) of Theorem 2.2 that for every bounded complex sequence $x = (x_k)$

$$st_{C_\lambda} - \limsup |x| \leq st_{C_\mu} - \limsup |x| \, .$$

So it follows that, for any $z \in \mathbb{C}$,

$$\left\{ w \in \mathbb{C} : |w - z| \leq st_{C_\lambda} - \limsup_k |x_k - z| \right\} \subseteq$$
$$\subseteq \left\{ w \in \mathbb{C} : |w - z| \leq st_{C_\mu} - \limsup_k |x_k - z| \right\}.$$

Now Theorem 6 of [3] implies that

$$\bigcap_{z \in \mathbb{C}} \left\{ w \in \mathbb{C} : |w - z| \leq st_{C_\lambda} - \limsup_k |x_k - z| \right\} \subseteq$$
$$\subseteq \bigcap_{z \in \mathbb{C}} \left\{ w \in \mathbb{C} : |w - z| \leq st_{C_\mu} - \limsup_k |x_k - z| \right\},$$

i.e.,

$$st_{C_\lambda} - core\, \{x\} \subseteq st_{C_\mu} - core\, \{x\} \, .$$

Thus we have

COROLLARY 2.4. *Let $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ be infinite subsets of* $\mathbb{N}$. *If $E \setminus F$ is finite and* $\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0$, *then* $st_{C_\lambda} - core\,\{x\} \subseteq st_{C_\mu} - core\,\{x\}$ *for every bounded complex sequence x.*

We immediately get the next corollary from (ii),(iii) and (iv) of Theorem 2.2 while the latter from Theorem 2.3 for every bounded complex sequence x.

COROLLARY 2.5. *Let $F = \{\lambda(n)\}$ and $E = \{\mu(n)\}$ be infinite subsets of* $\mathbb{N}$. *Then, for every bounded complex sequence x,*

(i) *if $E \Delta F$ is finite and* $\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0$, *then* $st_{C_\lambda} - core\,\{x\} = st_{C_\mu} - core\,\{x\}$;

(ii) *if $E \setminus F$ is finite and* $\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0$, *then* $st - core\,\{C_\mu x\} \subseteq st - core\,\{C_\lambda x\}$;

(iii) *if $E \Delta F$ is finite and* $\lim_n \frac{\lambda(n)}{\mu(n)} = d \neq 0$, *then* $st - core\,\{C_\mu x\} = st - core\,\{C_\lambda x\}$.

Corollary 2.6. *Let* $E = \{\mu(n)\}$ *be infinite subset of* $\mathbb{N}$. *Then, for every bounded complex sequence* x,

(i) *if* $\lim_n \frac{n}{\mu(n)} = d \neq 0$, *then* $st - core\,\{x\} \subseteq st_{C_\mu} - core\,\{x\}$;

(ii) *if* $\lim_n \frac{n}{\mu(n)} = d \neq 0$, *then* $st - core\,\{C_\mu x\} \subseteq st - core\,\{C_1 x\}$.

3. Consistency of A-statistical convergence

In this section we consider the concept of $A-$statistical convergence and recall definitions of inclusion and consistency in the statistical convergence sense as introduced by Fridy and Khan [8]. Also we get necessary conditions on the matrices A and B so that A and B are equivalent in the statistical convergence sense and $\Gamma_A(x) = \Gamma_B(x)$ for a real number sequence x where A and B are nonnegative regular summability matrices.

We begin by giving two definitions.

Definition 3.1. *If* $st_A \supset st_B$, *A is said to be stronger than B in the statistical convergence sense.*

Definition 3.2. *Matrices A and B are called consistent in the statistical convergence sense if* $st_A - \lim x = st_B - \lim x$ *whenever* $x \in st_A \cap st_B$. *If A is stronger than B in the statistical convergence sense and consistent with B in the statistical convergence sense we then write* $A \overset{st}{\supset} B$ [8]. *If* $A \overset{st}{\supset} B$ *and* $B \overset{st}{\supset} A$, *$A$ and B are called equivalent in the statistical convergence sense (denoted by* $A \overset{st}{\sim} B$ *).*

Throughout this section $A = (a_{nk})$ and $B = (b_{nk})$ will denote nonnegative regular summability matrices.

Theorem 3.3. *If the condition*

$$\limsup_n \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| = 0 \tag{$*$}$$

holds, then $\delta_A(K) = 0$ *if and only if* $\delta_B(K) = 0$ *for every* $K \subseteq \mathbb{N}$.

Proof. (Necessity). If $\delta_A(K) = 0$, then $\lim_n \sum_{k \in K} a_{nk} = 0$. Since

$$|(A\chi_K)_n - (B\chi_K)_n| \leq \sum_{k \in K} |a_{nk} - b_{nk}| \leq \sum_{k=1}^{\infty} |a_{nk} - b_{nk}|,$$

we have $\limsup_n |(A\chi_K)_n - (B\chi_K)_n| = 0$ by $(*)$, which implies

$$\delta_B(K) = \lim_n \sum_{k \in K} b_{nk} = 0.$$

Sufficiency follows from the symmetry. □

Hence we can get the following results from Theorem 3.3.

THEOREM 3.4. *If A and B satisfy the condition $(*)$, then*

(i) $st_A = st_B$
(ii) $\Gamma_A(x) = \Gamma_B(x)$

for a real number sequence x.

The statistical limits in (i) of Theorem 3.4 agree (i.e., $st_B - \lim x = L$ implies $st_A - \lim x = L$). Therefore, if A and B satisfy condition $(*)$ of Theorem 3.3, then A and B are consistent in the statistical convergence sense.

Note that the support sets generated by nonnegative summability methods A and B can be used to determine when, if a sequence x is both A- and B-statistically convergent, the A-statistical and B-statistical limits of x agree. In [2] Connor and Kline, using the "$\beta\mathbb{N}$ program" have shown that A and B assign the same statistical limit to x if $K_A \cap K_B \neq \phi$ where the sets K_A and K_B are the support sets of the nonnegative regular summability matrices A and B.

The next corollary shows that we have the same result under different conditions.

COROLLARY 3.5. *If A and B satisfy the conditions $(*)$ of Theorem 3.3, then $A \overset{st}{\sim} B$.*

Recall that A-statistical boundedness implies that $st_A - \limsup$ and $st_A - \liminf$ are finite and $st_A - \limsup x$ and $st_A - \liminf x$ are the greatest and least A-statistical cluster points of such an x [3]. Also

$$st_A - core\{x\} = [st_A - \liminf x,\ st_A - \limsup x]$$

for any $A-$statistically bounded real number sequence x [3].

Hence we can get the following from (ii) of Theorem 3.4.

COROLLARY 3.6. *If A and B satisfy the condition $(*)$, then $st_A - core\{x\} = st_B - core\{x\}$ for every bounded real sequence x.*

Note that the converse of Corollary 3.6 does not hold. This is seen by the following example.

EXAMPLE 3.7. Consider the matrices $A = (a_{nk})$ and $B = (b_{nk})$ defined by

$$a_{nk} = \begin{cases} \dfrac{n}{3(n+1)}, & k = n^2 \\ 1 - \dfrac{n}{3(n+1)}, & k = n^2 + 1 \\ 0, & \text{otherwise;} \end{cases}$$

and

$$b_{nk} = \begin{cases} \dfrac{n}{5(n+1)}, & k = n^2 \\ 1 - \dfrac{n}{5(n+1)}, & k = n^2 + 1 \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that A and B are nonnegative regular matrix summability methods.

Let us define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1, & k = n^2 \\ 0, & \text{otherwise.} \end{cases}$$

If we write $E_1 := \{k = n^2 : n = 1, 2, ...\}$ and $E_2 := \{k \neq n^2 : n = 1, 2, ...\}$, then we have $\delta_A(E_1) = \frac{1}{3}$, $\delta_A(E_2) = \frac{2}{3}$, $\delta_B(E_1) = \frac{1}{5}$, $\delta_B(E_2) = \frac{4}{5}$. Thus $\Gamma_A(x) = \Gamma_B(x) = \{0, 1\}$. Also, $st_A - core\{x\} = st_B - core\{x\} = [0, 1]$. Observe that

$$\limsup_n \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| = \frac{4}{15}.$$

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