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NEW NORMALITY AXIOMS AND DECOMPOSITIONS OF NORMALITY

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Abstract. Generalizations of normality, called (weakly) (functionally) θ-normal spaces, are introduced and studied. This leads to decompositions of normality. It turns out that every paracompact space is θ-normal. Moreover, every Lindelöf space as well as every almost compact space is weakly θ-normal. Preservation of θ-normality and its variants under mappings is studied. This in turn strengthens several known results pertaining to normality.

1. Introduction

In this paper we introduce four generalizations of normality. All four of them coincide with normality in the class of θ-regular spaces (see Definition 3.9) while two of them characterize normality in Hausdorff spaces. Furthermore all four of them serve as a necessary ingredient towards a decomposition of normality.

Throughout the present paper no separation axioms are assumed unless explicitly stated otherwise. For example, we do not assume a paracompact space to be Hausdorff or regular. Thus, in particular, every pseudometrizable space as well as every compact space is paracompact.

2. Basic definitions and preliminaries

Definition 2.1. [10] *Let X be a topological space and let $A \subset X$. A point $x \in X$ is in θ-closure of A if every closed neighbourhood of x intersects A. The θ-closure of A is denoted by $\mathrm{cl}_\theta A$. The set A is called θ-closed if $A = \mathrm{cl}_\theta A$.*

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The complement of a θ-closed set will be referred to as a θ-open set.

PROPOSITION 2.2. *For a topological space X the following statements are equivalent.*

(a) *X is Hausdorff.*
(b) *Every compact subset of X is θ-closed.*
(c) *Every singleton in X is θ-closed.*

REMARK 2.3. The above result is due to Dickman and Porter (see [2, 1.2] and [3, 2.3]).

LEMMA 2.4. [3, 2.4] *A topological space X is regular if and only if every closed set in X is θ-closed.*

Next we quote the following lemma which is utilized in [7] and is fairly immediate in view of Definition 2.1

LEMMA 2.5. [7] *A subset A of a topological space X is θ-open if and only if for each $x \in A$, there is an open set U such that $x \in U \subset \overline{U} \subset A$.*

3. θ-Normal Spaces And Their Variants

DEFINITION 3.1. *A topological space X is said to be*

(i) *θ-normal if every pair of disjoint closed sets one of which is θ-closed are contained in disjoint open sets.*
(ii) *functionally θ-normal if for every pair of disjoint closed sets A and B one of which is θ-closed there exists a continuous function $f : X \longrightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.*
(iii) *Weakly θ-normal if every pair of disjoint θ-closed sets are contained in disjoint open sets; and*
(iv) *Weakly functionally θ-normal if for every pair of disjoint θ-closed sets A and B there exists a continuous function $f : X \longrightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.*

DEFINITION 3.2. [8] *A topological space X is said to be θ-completely regular if for every θ-closed set F in X and a point $x \notin F$ there is a continuous function $f : X \longrightarrow [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$.*

In view of Lemma 2.4 it is immediate that in the class of regular spaces all the four variants of θ-normality in Definition 3.1 coincide with normality. Moreover, the following implications are immediate from the definitions.

$$
\begin{tikzcd}
\textbf{normal} \arrow[r, Rightarrow] & \textbf{functionaly } \boldsymbol{\theta}\textbf{-normal} \arrow[r, Rightarrow] \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \boldsymbol{\theta}\textbf{-normal} \arrow[d, Rightarrow] \\
\boldsymbol{\theta}\textbf{-completely regular} & \textbf{weakly functionaly } \boldsymbol{\theta}\textbf{-normal} \arrow[r, Rightarrow] & \textbf{weakly } \boldsymbol{\theta}\textbf{-normal}
\end{tikzcd}
$$

None of the above implications is reversible (see Examples 3.6, 3.7, 3.8 and [6, Example 3.4]). Moreover, every Hausdorff weakly functionally θ-normal space is θ-completely regular.

THEOREM 3.3. *For a topological space X, the following statements are equivalent.*

(a) *X is θ-normal.*
(b) *For every θ-closed set A and every open set U containing A there exists an open set V such that $A \subset V \subset \overline{V} \subset U$.*
(c) *For every closed set A and every θ-open set U containing A there exists an open set V such that $A \subset V \subset \overline{V} \subset U$.*
(d) *For every pair of disjoint closed sets A and B one of which is θ-closed there exist open sets U and V such that $A \subset U$, $B \subset V$ and $\overline{U} \cap V = \emptyset$.*

PROOF. To prove the assertion (a) $\Rightarrow$ (b), let X be a θ-normal space and let U be an open set containing a θ-closed set A. Now A is a θ-closed set which is disjoint from the closed set $X - U$. By θ-normality of X there are disjoint open sets V and W containing A and $X - U$, respectively. Then $A \subset V \subset X - W \subset U$, since $X - W$ is closed, $A \subset V \subset \overline{V} \subset U$.

To prove the implication (b) $\Rightarrow$ (c), let U be a θ-open set containing a closed set A. Then $X - A$ is an open set containing the θ-closed set $X - U$. So by hypothesis there exists an open set W such that $X - U \subset W \subset \overline{W} \subset X - A$. Let $V = X - \overline{W}$. Then $A \subset V \subset X - W \subset U$. Since $X - W$ is closed, $A \subset V \subset \overline{V} \subset U$.

To prove (c) $\Rightarrow$ (d), let A be a closed set disjoint from a θ-closed set B. Then $A \subset X - B$ and $X - B$ is θ-open. By hypothesis there exists an open set U such that $A \subset U \subset \overline{U} \subset X - B$. Again, by hypothesis there exists an open set W such that $\overline{U} \subset W \subset \overline{W} \subset X - B$. Let $V = X - \overline{W}$. Then U and V are open sets containing A and B, respectively and have disjoint closures. The assertion (d) $\Rightarrow$ (a) is obvious.

The proof of the following characterization of weakly θ-normal spaces is similar to that of Theorem 3.3 and hence is omitted. ∎

THEOREM 3.4. *A topological space X is weakly θ-normal if and only if for every θ-closed set A and a θ-open set U containing A there is an open set V such that $A \subset V \subset \overline{V} \subset U$.*

For a characterization of functionally θ-normal spaces analogous to Urysohn's lemma (see [6]), and for a similar characterization of weakly functionally θ-normal spaces and their relationships with the existence of partition of unity see [7].

The following result shows that in the class of Hausdorff spaces the notions of normality and (functional) θ-normality coincide.

THEOREM 3.5. *For a Hausdorff space X, the folloving statements are equivalent.*

(a) *X is normal.*
(b) *X is functionally θ-normal.*
(c) *X is θ-normal.*

PROOF. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious. To prove (c) $\Rightarrow$ (a), let X be a θ-normal Hausdorff space. By Proposition 2.2 every singleton in X is θ-closed. So by θ-normality of X every closed set in X and a point outside it are contained in disjoint open sets. Thus X is regular and so by Lemma 2.4 every closed set in X is θ-closed. Consequently, every pair of disjoint closed sets in X are separated by disjoint open sets. ∎

EXAMPLE 3.6. A Hausdorff veakly functionally θ-normal space which is not θ-normal. Let X be the real line with every point having neighbourhoods as in the usual topology with the exception of 0. A basic neighbourhood of 0 is of the form $(-\varepsilon, \varepsilon) - K$, where $\varepsilon > O$ and $K = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$. It is easily verified that the space X has the desired properties.

The co-finite topology on an infinite set as well as the co-countable topology on an uncountable set is functionally θ-normal but not normal. Moreover, although every finite topological space is functionally θ-normal, finite spaces need not be normal.

EXAMPLE 3.7. A finite functionally θ-normal space which is not normal. Let $X = \{a, b, c, d\}$. Let $\mathcal{V}$ be the topology on X generated by taking $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{d\}\}$ as a subbase. Then $\{d\}$ and $\{a, b, c\}$ are the only θ-closed sets in X. Define function $f : X \longrightarrow [0, 1]$ by taking $f(d) = 1$ and $f(x) = 0$ for $x \neq d$. Then f is a continuous function and separates every pair of disjoint closed sets if one of them is θ-closed. However x is not normal as the disjoint closed sets $\{a\}$ and $\{c\}$ can not be separated by disjoint open sets.

EXAMPLE 3.8. A weakly θ-normal space which is not weakly functionally θ-normal. Let X denote the interior S^0 of the unit square S in the plane together with the points $(0, 0)$ and $(1, 0)$, i. e. $X = S^0 \cup \{(0, 0), (1, 0)\}$. Every point in S^0 has the usual Euclidean neighourhoods. The points $(0, 0)$ and $(1, 0)$ have neighbourhoods of the form U_n and V_n respectively, where

$$U_n = \{(0, 0)\} \cup \left\{(x, y) : 0 < x < \frac{1}{2},\ 0 < y < \frac{1}{n}\right\}$$

and

$$V_n = \{(1,0)\} \cup \left\{(x,y) : \frac{1}{2} < x < 1,\ 0 < y < \frac{1}{n}\right\}.$$

The space X is weakly θ-normal, since every pair of disjoint θ-closed sets are separated by disjoint open sets. However, the θ-closed sets $\{(0,0)\}$ and $\{(1,0)\}$ do not have disjoint closed neighbourhoods and hence cannot be functionally separated.

The space of Example 3.6 is a Hausdorff weakly functionally θ-normal space which fails to be normal. This motivates the search for an appropriate class of spaces (besides regular spaces) in which the notions of normality and weak (functional) θ-normality coincide. The answer is enfolded in the concept of a θ-regular space.

Definition 3.9. *A topological space X is said to be θ-regular if for each closed set F and each open set U containing F, there exists a θ-open set V such that $F \subset V \subset U$.*

In view of Lemma 2.4 it follows that every regular space is θ-regular. The two-point Sierpinski space [9, p. 44] is a θ-regular space which is not regular. Moreover, by Lemma 2.5 it follows that a T_1-space is regular if and only if it is θ-regular.

In general a normal space need not be regular. However, the following holds.

Proposition 3.10. *Every normal space is θ-regular.*

Proof. Let A be a closed set and U be an open set containing A. Let $B = X - U$. Then A and B are disjoint closed sets in X. By Urysohn's lemma there exists a continuous function $f : X \longrightarrow [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Let $V = f^{-1}[0,1/2)$ and $W = f^{-1}(1/2,1]$. Then $A \subset V \subset X - W \subset U$. We claim that V is a θ-open set. Let $x \in V$. Then $f(x) \in [0,1/2)$. So there is a closed neighbourhood N of $f(x)$ contained in $[0,1/2)$. Let $U_x = \mathrm{int} f^{-1}(N)$. Then $x \in U_x \subset \overline{U}_x \subset f^{-1}(N) \subset V$. By Lemma 2.5, V is θ-open. □

The space of Example 3.7 is a functionally θ-normal space which fails to be θ-regular. The following Theorem is central to the paper, since it provides a decomposition of normality in terms of θ-regularity and variants of θ-normality.

Theorem 3.11. *Let X be a θ-regular space. Then the following statements are equivalent.*

(a) *X is normal.*
(b) *X is functionally θ-normal.*
(c) *X is θ-normal.*

(d) *X is weakly functionally θ-normal.*
(e) *X is weakly θ-normal.*

PROOF. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (e) are immediate. To prove (e) $\Rightarrow$ (c), let A and B be any two disjoint closed subsets of X such that one of them, say B, is θ-closed. Then $B \subset X - A$ and so by θ-regularity of X there is a θ-open set W such that $B \subset W \subset X - A$. Since X is weakly θ-normal, by Theorem 3.4 there exists an open set V such that $B \subset V \subset \overline{V} \subset W \subset X - A$. Clearly $U = (X - \overline{V})$ and V are disjoint open sets containing A and B respectively.

Finally to show that (c) $\Rightarrow$ (a), let E and F be any two disjoint closed subsets of X. Since X is θ-regular, there is a θ-open set W such that $E \subset W \subset X - F$. Then $X - W$ is a θ-closed set containing the closed set F and is disjoint from E. By θ-normality of X there are disjoint open sets U and V containing E and $X - W$, respectively and so E and F, respectively. □

A topological space X is said to be almost compact [1] if for every open cover $\mathcal{U}$ of X there is a finite subcollection $\{U_1, \dots, U_n\}$ of $\mathcal{U}$ such that $X = \bigcup_{i=1}^{n} \overline{U}_i$. A Hausdorff almost compact space is called H-closed.

Dickman and Porter showed that every H-closed space is weakly θ-normal (see [2, 2.4]). This result was significantly improved by Espelie and Joseph [5]. In particular, it is immediate from Theorem 1 of [5] that every almost compact space is weakly θ-normal.

Unlike normality, θ-normality is enjoyed by every paracompact space and hence by every compact space.

THEOREM 3.12. *A paracompact space is θ-normal.*

PROOF. Let A and B be disjoint closed sets such that one of them, say B, is θ-closed. Then $A \subset X - B$ and $X - B$ is θ-open so for each $a \in A$, there is an open set U_a such that $a \in U_a \subset \overline{U}_a \subset X - B$. Then the collection $\mathcal{U} = \{U_a : a \in A\} \cup \{X - A\}$ is an open covering of X. By paracompactness of X, let $\mathcal{V}$ be a locally finite open refinement of $\mathcal{U}$. Let $\mathcal{D}$ denote the subcollection of $\mathcal{V}$ consisting of those members of $\mathcal{V}$ which intersects A. Then $\mathcal{D}$ covers A. Furthermore, if $D \in \mathcal{D}$, then $\overline{D}$ is disjoint from B and as D intersects A, it lies in some $U_{a'}$ whose closure is disjoint from B.

Let $V = \cup\{D : D \in \mathcal{D}\}$. Then V is an open set in X containing A. Since $\mathcal{D}$ is locally finite; $\overline{V} = \cup\{\overline{D} : D \in \mathcal{D}\}$ and $\overline{V}$ is disjoint from B. Thus V and $(X - \overline{V})$ are disjoint open sets containing A and B respectively. Hence X is θ-normal. □

COROLLARY 3.13. *A paracompact θ-regular space is normal.*

PROOF. This is immediate in view of Theorems 3.11 and 3.12. □

COROLLARY 3.14. *A Hausdorff space is compact if and only if it is almost compact and θ-normal.*

PROOF. Necessity is immediate in view of Theorem 3.12 and sufficiency is an easy consequence of Theorem 3.5 and the fact that every regular almost compact space is compact [1]. ◻

REMARK 3.15. In a paracompact space any two sets which are contained in disjoint closed sets one of which is θ-closed are separated by disjoint open sets.

THEOREM 3.16. *A Lindelöf space is weakly θ-normal.*

PROOF. Let X be a Lindelöf space and let A and B be disjoint θ-closed subsets of X. Then A and B are Lindelöf sets in X. Since B is θ-closed, $(X - B)$ is θ-open and $A \subset X - B$. So for each point a of A there is an open set containing a whose closure does not intersect B and consequently the family $\mathcal{U}$ of all open sets whose closure do not intersect B is a cover of A. Similarly, the family $\mathcal{V}$ of all open sets whose closures do not intersect A is a cover of B. Then there is a sequence $\{U_n : n \in \mathbf{Z}^+\}$ of members of $\mathcal{U}$ which covers A and a sequence $\{V_n : n \in \mathbf{Z}^+\}$ of members of $\mathcal{V}$ which covers B. For each n, let $U_n = U_n - \cup\{\overline{V}_k : k \leq n\}$ and $V_{n'} = V_n - \cup\{\overline{U}_k : k \leq n\}$. Each of the set $U_{n'}$ and $V_{n'}$ is open. The collection $\{U_{u'} : n \in \mathbf{Z}^+\}$ covers A, because each $x \in A$ belongs to U_n for some n, x belongs to none of the sets $\overline{V}_k$. Similarly, the collection $\{V_{n'} : n \in \mathbf{Z}^+\}$ covers B. Finally, the open sets $U = \bigcup\limits_{n=1}^{\infty} U_{n'}$ and $V = \bigcup\limits_{n=1}^{\infty} V_{n'}$, are disjoint and contain A and B, respectively. ◻

REMARK 3.17. The above result is false with 'weak θ-normal' replaced by 'θ- normal'. The space X of Example 3.6 is a Hausdorff second countable weakly functionally θ-normal space which is not θ-normal.

COROLLARY 3.18. *A θ-regular Lindelöf space is normal.*

PROOF. This is immediate in view of Theorem 3.11 and 3.16. ◻

REMARK 3.19. In a Lindelöf space any two sets which are contained in disjoint θ-closed sets are separated by disjoint open sets.

THEOREM 3.20. *A θ-completely regular compact space is functionally θ-normal.*

PROOF. Let X be a compact, θ-completely regular space, let A be a closed set disjoint from a θ-closed set B. Since A is closed, it is compact. Since X is θ-completely regular, for every point $x \in A$ there exists a continuous function $f : X \longrightarrow [O, 1]$ such that $f_x(x) = 0$ and $f_x(B) = 1$. Let $U_x = f_x^{-1}[0, 1)$. Now $\mathcal{U} = \{U_x : x \in A\}$ is an open covering of A. Since A is compact, there exists a finite subcollection $\{U_{x_1}, \dots, U_{x_n}\}$ which covers A. Define a function $g : X \longrightarrow [0, 1l$ by $g(x) = 2\max\left[\frac{1}{2}, \min\{f_{x_1}(x), \dots, f_{x_n}(x)\}\right] - 1$. Then it is easily verified that g is continuous, $g(A) = 0$ and $g(B) = 1$. Hence X is functionally θ-normal. ∎

The following diagram summarizes the relationships between compactness and generalized versions of normality discussed is this paper.

$$\begin{tikzcd}
 & \text{compact} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] \arrow[drr, Rightarrow, "+\theta\text{-CR}"] & & \\
\text{almost compact} \arrow[dr, Rightarrow] & \text{Lindelöf} \arrow[d, Rightarrow] & \text{paracompact} \arrow[d, Rightarrow] & \text{functionaly } \theta\text{-normal} \arrow[dl, Rightarrow] \\
 & \text{weakly } \theta\text{-normal} & \theta\text{-normal} \arrow[l, Rightarrow] &
\end{tikzcd}$$

The example of open ordinal space [9, p. 68] a non-Lindelöf, non-paracompact, normal Hausdorff space which is not almost compact, shows that none of the above implications is reversible.

An open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of a space X is said to be shrinkable if there exists an open cover $\mathcal{V} = \{v_\alpha : \alpha \in \Lambda\}$ of X such that $\overline{V}_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$.

THEOREM 3.21. *For a topological space X, consider the folloving statements.*

1. *X is θ-normal.*
2. *Every point-finite θ-open cover of X is shrinkable.*
3. *X is weakly-θ-normal.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

PROOF. To prove (1) $\Rightarrow$ (2), suppose X is a θ-normal space and let $\mathcal{U} = \{U_\alpha : a \in \Lambda\}$ be a point-finite θ-open cover of X. Well order the set Λ. For convenience we may assume that $\Lambda = \{1, 2, \dots, \alpha, \dots\}$. Now construct $\{V_\alpha : \alpha \in \Lambda\}$ by transfinite induction as follows: Let

$$F_1 = X - \bigcup_{\alpha > 1} U_\alpha.$$

Then F_1 is a θ-closed subset of X and $F \subset U_1$. By Theorem 3.3 there exists an open set V_1 such that $F_1 \subset V_1 \subset \overline{V}_1 \subset U_1$. Next suppose that V_β has been defined for each $\beta < \alpha$ and let

$$F_\alpha = X - \left[\left(\bigcup_{\beta<\alpha} V_\beta\right) \bigcup \left(\bigcup_{\gamma>\alpha} U_\gamma\right)\right].$$

Then F_α is closed and contained in the θ-open set U_α. Again by Theorem 3.3 there is an open set V_α such that

$$F_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha.$$

Now, $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ is a shrinking of $\mathcal{U}$ provided it a covers X. Let $x \in X$. Then x belongs to only finitely many members of $\mathcal{U}$, say $U_{\alpha_1}, \ldots, U_{\alpha_k}$. Let $\alpha = \max\{\alpha_1, \ldots, \alpha_k\}$. Now x does not belongs to U_γ for $\gamma > \alpha$ and hence if $x \notin V_\beta$ for $\beta < \alpha$, then $x \in F_\alpha \subset V_\alpha$. Thus in any case $x \in V_\beta$ for some $\beta \leq \alpha$. So $\mathcal{V}$ is a cover of X and hence $\mathcal{V}$ is a shrinking of $\mathcal{U}$.

To prove (2) $\Rightarrow$ (3), let A and B be disjoint θ-closed subsets of X. Then $\{X - A, X - B\}$ is a point-finite θ-open cover of X. Let $\{U, V\}$ be a shrinking of $\{X - A, X - B\}$. Then $X - \overline{U}$ and $X - \overline{V}$ are disjoint open sets containing A and B, respectively. ☐

4. Preservation Under Mappings

Definition 4.1. *A function $f : X \longrightarrow Y$ is said to be θ-continuous if for each $x \in X$ and each open set V containing $f(x)$ there is an open set U containing x such that $f(\overline{U}) \subset V$.*

The following lemma seems to be known and is an easy consequence of the fact that a function $f : X \longrightarrow Y$ is θ-continuous if and only if $\mathrm{cl}_\theta\, f^{-1}(B) \subset f^{-1}(\mathrm{cl}_\theta\, B)$ for each $B \subset Y$.

Lemma 4.2. *Let $f : X \longrightarrow Y$ be a θ-continuous function and let K be a θ-closed set in Y. Then $f^{-1}(K)$ is θ-closed in X.*

Theorem 4.3. *A closed continous image of a θ-normal space is θ-normal.*

Proof. Let $f : X \longrightarrow Y$ be a continuous closed function from a θ-normal space X onto Y. Let A and B be disjoint closed sets in Y such that B is θ-closed. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets in X. In view of Lemma 4.2 $f^{-1}(B)$ is θ-closed. Since X is θ-normal, there exist disjoint open sets U and V containing $f^{-1}(A)$ and $f^{-1}(B)$, respectively. It is easily verfied that $Y - f(X - U)$ and $Y - f(X - V)$ are disjoint open sets containing A and B, respectively. ☐

Corollary 4.4. *A Hausdorff continuous closed image of a θ-normal space is normal.*

Proof. This is immediate in view of Theorems 3.5 and 4.3. ☐

THEOREM 4.5. *A θ-continuous closed image of a weakly θ-normal space is weakly θ-normal.*

The proof of Theorem 4.5 makes use of Lemma 4.2 and is similar to that of Theorem 4.3 and hence omitted.

COROLLARY 4.6. *A θ-regular θ-continuous closed image of a weakly θ-normal space is normal.*

PROOF. This result is immediate in view of Theorems 3.11 and 4.5. □

Recall that a function $f : X \longrightarrow Y$ which is both open and closed is referred to as a clopen function.

THEOREM 4.7. *A continuous clopen image of a (weakly) functionally θ-normal space is (weakly) functionally θ-normal.*

PROOF. Let $f : X \longrightarrow Y$ be a continuous clopen function from a (weakly) functionally θ-normal space X onto Y. Let A and B be disjoint closed subsets of Y such that one of them is θ-closed. Suppose B is θ-closed. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets in X and by Lemma 4.2 $f^{-1}(B)$ is θ-closed. Since X is functionally θ-normal, there exists a continuous function $g : X \longrightarrow [0,1]$ such that $g(f^{-1}(A)) = 0$ and $g(f^{-1}(B)) = 1$. Now, define a mapping $h : Y \longrightarrow [0,1]$ by $h(y) = \sup\{g(x) : x \in f^{-1}(Y)\}$. Since f is a clopen function, by [4, Exercise 16, p. 96] h is continuous and $h(A) = 0$ and $h(B) = 1$. A similar proof holds in case X is weakly functionally θ-normal and in this case f is only required to be 'θ-continuous'. □

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