GLASNIK MATEMATIČKI
Vol. 37(57)(2002), 141 – 145

2–ISOMETRIC OPERATORS

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Abstract. An operator T on a complex Hilbert space is called a 2–isometry if $T^{*2}T^2 - 2T^*T + I = 0$. Our underlying purpose in this article is to investigate some algebraic and spectral properties of 2–isometries.

1. Introduction

Let H be a complex Hilbert space. By an operator on H, we shall mean a bounded linear transformation from H to H. Let $\sigma(T)$, $\pi(T)$, $\pi_0(T)$, $\pi_{00}(T)$ and $w(T)$, respectively denote the spectrum, the approximate point spectrum, the point spectrum, the set of eigenvalues with finite multiplicity and the Weyl spectrum of an operator T. We use the symbol $\partial\sigma(T)$ for the boundary of $\sigma(T)$. If for an operator T, $w(T) = \sigma(T) \sim \pi_{00}(T)$, then we say that the Weyl's theorem holds for T.The spectral radius and the numerical radius of T will be denoted by $r(T)$ and $|W(T)|$ respectively. If $r(T) = |W(T)|$, then T is called a spectraloid operator. By saying that an operator T is power bounded, we mean that there exists some $M > 0$ such that $\|T^n\| \leq M$ for each positive integer n. According to [1], an operator T is defined to be a 2–isometry if $T^{*2}T^2 - 2T^*T + I = 0$. In the present note, we explore some properties of 2–isometries.

Clearly every isometry is a 2–isometry. According to [1, Proposition 1.23], an invertible 2–isometry turns out to be a unitary operator. It is obvious from the definition that every 2–isometry is left invertible. In particular if both T and T^* are 2–isometries then T is invertible and so must be unitary.

2. Results

Theorem 2.1. *A power of a 2–isometry is again a 2–isometry.*

2010 *Mathematics Subject Classification.* 47A62, 47A63.
Key words and phrases. Hilbert space, 2–isometry, spectrum.

Proof. Let T be a 2–isometry. We prove the assertion by using the mathematical induction. Since T is a 2–isometry, the result is true for $n = 1$. Now assume that the result is true for $n = k$, i.e.,

$$T^{*2k}T^{2k} - 2T^{*k}T^k + I = 0. \tag{2.1}$$

Then

$$\begin{aligned}
& T^{*2(k+1)}T^{2(k+1)} - 2T^{*k+1}T^{k+1} + I \\
= {} & T^{*2}(T^{*2k}T^{2k})T^2 - 2T^{*k+1}T^{k+1} + I \\
= {} & T^{*2}(2T^{*k}T^k - I)T^2 - 2T^{*k+1}T^{k+1} + I \qquad \text{(by (2.1))} \\
= {} & 2T^{*k+2}T^{k+2} - T^{*2}T^2 - 2T^{*k+1}T^{k+1} + I \\
= {} & 2T^{*k}(T^{*2}T^2 - T^*T)T^k - T^{*2}T^2 + I \\
= {} & 2T^{*k}(T^*T - I)T^k - T^{*2}T^2 + I \qquad (T \text{ is a 2–isometry}) \\
= {} & (2T^{*k+1}T^{k+1} - 2T^{*k}T^k) - T^{*2}T^2 + I \\
= {} & 2(T^{*2}T^2 - T^*T) - T^{*2}T^2 + I \qquad \text{(by (2.1))} \\
= {} & T^{*2}T^2 - 2T^*T + I \\
= {} & 0.
\end{aligned}$$

This shows that the result is true for $n = k + 1$: thus T^n is a 2–isometry for each n. □

It is well known and obvious that a unilateral weighted shift is an isometry iff all its weights lie on the unit circle. In the next result, we obtain a necessary and sufficient condition under which a non–isometric unilateral weighted shift is a 2–isometry.

Theorem 2.2. *A non–isometric unilateral weighted shift T with weights $\{\alpha_n\}$ is a 2–isometry if and only if*

(i) $|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0$ *for each n;*
(ii) $|\alpha_n| \neq 1$ *for each n.*

Proof. Suppose T is a 2–isometry. If $\{e_n\}$ is an orthonormal base for H, then $Te_n = \alpha_n e_{n+1}$ and hence (i) follows. Suppose (ii) is false. Select the least positive integer k such that $|\alpha_k| = 1$. If $k > 1$, then (i) gives $|\alpha_{k-1} = 1$ which is contrary to the selection of k. Therefore $|\alpha_1 = 1$. Using the induction argument and (i), one can show that $|\alpha_n| = 1$ for each positive integer n. But this will contradict our assumption that T is non–isometric. Hence we conclude that (ii) is true. The converse assertion is obvious. □

Corollary 2.3. *Let T be a non–isometric unilateral weighted shift with weights $\{\alpha_n\}$. If T is a 2–isometry, then the following assertions hold.*

(i) $\{|\alpha_n|\}$ *is a strictly decreasing sequence of real numbers converging to 1.*
(ii) $\sqrt{2} > |\alpha_n| > 1$ *for each $n > 1$.*

PROOF. (i) Suppose $|\alpha_{n+1}| \geq |\alpha_n|$ for some n. Then by Theorem 2.2 (i), we find $0 \geq (1-|\alpha_n|^2)^2$ or $|\alpha_n| = 1$. But this contradicts Theorem 2.2 (ii). Thus $\{|\alpha_n|\}$ is a strictly decreasing sequence of real numbers and so must be convergent. By Theorem 2.2 (i), we infer that $|\alpha_n| \to 1$.

(ii) Rewriting equality (i) of Theorem 2.2 as

$$|\alpha_{n+1}|^2 - 2 + 1/|\alpha_n|^2 = 0 \tag{2.2}$$

we get $\sqrt{2} > |\alpha_n|$ for each $n > 1$. By (i) and Theorem 2.2 (ii), $|\alpha_n| > 1$. This finishes the proof of (ii). □

THEOREM 2.4. *A power bounded 2–isometry is an isometry.*

PROOF. Let T be a power bounded 2–isometry. Then there exists a positive real number M such that

$$\|T^n\| \leq M \tag{2.3}$$

for $n = 1, 2, 3, \ldots$. The definition of a 2–isometry yields

$$\|T^2\|^2 + 1 = 2\|T\|^2. \tag{2.4}$$

Since T^n is also a 2–isometry by Theorem 2.1, an induction argument shows that

$$\|T^{2^n}\|^2 = 2^n\|T\|^2 - (2^n - 1) \tag{2.5}$$

for every positive integer n. Now (2.3) and (2.5) will give

$$M^2/2^n \geq \|T\|^2 - 1 + 1/2^n \geq 0.$$

Letting $n \to \infty$, we find $\|T\| = 1$. In particular, $I \geq T^*T$. Since $T^*T \geq I$ [1, Proposition 1.5], we conclude $T^*T = I$. □

REMARK 2.5. Above theorem can be used to show that unlike isometries, the class of 2–isometries is not bounded. To see this, use Theorem 2.2 to construct a 2–isometry T, which is not an isometry. Then by Theorem 2.4, we see that for each $M > 0$, there corresponds a positive integer n such that $\|T^n\| > M$. Since Theorem 2.1 says that T^n is also a 2–isometry, we conclude that the class of 2–isometries contains operators with arbitrarily large norm.

COROLLARY 2.6. *A 2–isometry similar to a spectraloid operator is an isometry.*

PROOF. Let T be a 2–isometry. Suppose it is similar to a spectraloid operator A. Then $r(T^n) = r(A^n) = |W(A^n)|$ for $n = 1, 2, 3, \ldots$. Since $r(T) = 1$, [1], we find $1 = |W(A^n)|$ and hence $\|A^n\| \leq 2$ for each n. Now the similarity of T and A shows that T is power bounded; thus the result follows from the preceding theorem. □

REMARK 2.7. Above corollary shows that unlike the class of isometries, the class of 2–isometries fails to be a subclass of spectraloid operators.

COROLLARY 2.8. *If T is a 2–isometry, then* $1 \in \sigma(T^*T)$.

PROOF. Suppose to the contrary that $1 \notin (T^*T)$. Then the operator $A = T^*T - I$ is invertible. Moreover $A \geq 0$ [1, Proposition 1.5]. From the definition of a 2–isometry it follows that $\sigma T^*AT = A$ or $(A^{1/2}TA^{-1/2})^*(A^{1/2}TA^{-1/2}) = I$ where $A^{1/2}$ denotes the positive square root of A. Thus T is similar to an isometry and so must be an isometry by virtue of Corollary 2.6. This contradicts our supposition that $1 \notin \sigma(T^*T)$. □

In the rest of the article, we shall obtain some spectral properties of 2-isometries.

THEOREM 2.9. *Let T be a 2–isometry. Then*

(i) $z \in \pi(T)$ *implies* $z^* \in \pi(T^*)$.

(ii) $z \in \pi_0(T)$ *implies* $z^* \in \pi_0(T^*)$.

(iii) *Eigenvectors of T corresponding to distinct eigen-values are orthogonal.*

PROOF. (i) Let $z \in \pi(T)$. Choose a sequence $\{x_n\}$ of unit vectors such that $(T - zI)x_n \to 0$. Then $(T^{*2}T^2 - z^2T^{*2})x_n \to 0$ and $T^*Tx_n - zT^*x_n \to 0$. The hypothesis that T is a 2–isometry yields $0 = T^{*2}T^2 - 2T^*T + I = T^{*2}T^2 - z^2T^{*2} - 2T^*T + 2zT^* + z^2T^{*2} - 2zT^* + I$. This will imply $z^2T^{*2}x_n - 2zT^*x_n + x_n \to 0$. Since $\pi(T)$ is a subset of the unit circle [1], we find $(T^* - z^*I)^2x_n \to 0$. From this it follows that $z^* \in \pi(T^*)$.

(ii) The argument is similar to one given in (i).

(iii) Let λ and μ be distinct eigen-values of T. Suppose $Tx = \lambda x$ and $Ty = \mu y$. Then $0 = \langle (T^{*2}T^2 - 2T^*T + I)x, y\rangle = \langle T^2x, T^2y\rangle - 2\langle Tx, Ty\rangle + \langle x, y\rangle = (\lambda^2\mu^{*2} - 2\lambda\mu^* + 1)\langle x, y\rangle$. Since $\lambda \neq \mu$ with $|\lambda| = 1 = |\mu|$, $\lambda^2\mu^{*2} - 2\lambda\mu^* + 1 = (\lambda/\mu - 1)^2 \neq 0$. This leads to $\langle x, y\rangle = 0$ which proves the assertion. □

THEOREM 2.10. *The spectrum of a 2–isometry is the closed unit disc provided it is non–unitary.*

PROOF. Let T be a non–unitary 2–isometry. Then $0 \in \sigma(T) \sim \pi(T)$. Since $\partial\sigma(T) \subseteq \pi(T)$, 0 turns out to be an interior point of $\sigma(T)$. Therefore we can find the largest positive number r such that $\{z : |z| \leq r\}$ is contained in $\sigma(T)$. It is possible to select a complex number z in $\partial\sigma(T)$ such that $r = |z|$. Since $\partial\sigma(T) \subseteq \pi(T) \subseteq \{z : |z| = 1\}$[1], $r = 1$. Consequently we find $\sigma(T) = \{z : |z| \leq 1\}$. □

COROLLARY 2.11. *If T is a 2–isometry, then each isolated point in its spectrum is an eigen–value.*

PROOF. If $\sigma(T)$ has an isolated point, then it is clear from the above theorem that T is unitary and hence the result follows. □

Corollary 2.12. *Let T be a 2–isometry. If the Lebesgue planar measure of $\sigma(T)$ is zero, then T is unitary.*

Corollary 2.13. *The Weyl's theorem holds for 2–isometries.*

Proof. The result holds if T is unitary. Assume that T is non-unitary. Then Theorem 2.10 shows that $\pi_{00}(T) = \emptyset$. Also by Theorem 2.9 (ii) and Lemma 3 of [2], $\sigma(T) \sim \pi_{00}(T) \subseteq w(T)$ and hence $\sigma(T) \subseteq w(T)$. This completes the argument. $\square$

Acknowledgements.

The author is thankful to the referee for suggesting some improvement over the original version of the present article. Also his special thanks are due to Professor Scott McCullough for reading the whole manuscript and making some useful comments and suggestions.

References

[1] J. Agler and M. Stankus, *m–isometric transformations of Hilbert space I*, Integral Equations and Operator Theory, **21** (1995), 383–429.

[2] J. V. Baxley, *Some general conditions implying Weyl's theorem*, Rev. Roum. Math. Pures Appl. **16** (1971), 1163–1166.

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Received: 25.04.2001