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# PERIODIC SOLUTION OF A FIRST ORDER NONCONVEX HAMILTONIAN SYSTEM

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Abstract. The first order Hamiltonian system is considered with $T$–periodic Hamiltonian that is sub-quadratic at infinity. Two kinds of sub-quadraticity are considered. The existence of $T$–periodic solution is proved using variational methods.

## 1. Introduction and main result

In this paper we shall consider a Hamiltonian system

$$\dot{z} = JH'_z(t,z) \tag{H}$$

where $J(x,y) = (-y,x)$, for all $z = (x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, and $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}; \mathbb{R})$. The prime denotes the derivative with respect to the variable $z$.

We suppose that the Hamiltonian $H(t,z)$ is $T$-periodic in time, i.e. $H(t+T,z) = H(t,z)$, $t \in \mathbb{R}, z \in \mathbb{R}^{2N}$ and sub-quadratic at infinity in the following sense:

There exist $r, \theta_1, \theta_2, p$ positive real numbers, $0 < \Theta_1 \leq \Theta_2$, $1 < p < 2$, and $\frac{\Theta_1}{p} > \frac{\Theta_2}{2}$, such that for all $|z| \geq r > 0$ we have (uniformly in $t$):

$$|H'(t,z)| \ \leq \Theta_2 |z|^{p-1}, \tag{H1}$$

and

$$\Theta_1 |z|^p \ \leq H'(t,z) \cdot z. \tag{H2}$$

Here we prove the following theorem:

Theorem 1. *Assume* (H1) *and* (H2). *Then* (H) *has a T-periodic solution obtained as a critical point of the functional* $\Psi(z)$ *defined by formula* (2.1).

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The resonant case, when the Hamiltonian $H$ is of the form

$$H(t,z)=\hat{H}(t,z)-\frac{k}{2}|z|^2,\quad k>0,$$

where $\hat{H}$ satisfies conditions (H1) and (H2), is considered as well. Then:

Theorem 2. *Assume that $k<1$ and let there exist a natural number $l\in\mathbb{N}$ such that $l\frac{kT}{2\pi}\in\mathbb{N}$. Then, equation* (H) *has an $lT$-periodic solution.*

Theorems 1 and 2 are also proved under the weaker conditions (H1'), (H2') and (H3) introduced on page 112. For the convenience we rewrite them here:

$$\text{(H1')}\qquad \limsup_{|z|\to\infty}\frac{|H'(t,z)|}{|z|}=0\quad \text{uniformly on}\, t\in\mathbb{R};$$

$$\text{(H2')}\qquad \frac{H'(t,z)z}{|H'(t,z)|\,|z|}\geq\alpha>0,$$

outside a ball of radius $r>0$ uniformly in $t$, and

$$\text{(H3)}\qquad |H'(t,z)||z|\geq\beta>0,$$

for $|z|\ \geq r>0$ uniformly in time.

Let us point out here that in [8] we proved the existence of $T$-periodic without the hypothesis (H3), using the topological degree.

## 2. Some basic Preliminaries

A natural, and widely used approach to solvability of the problem (H) is to consider the functional

$$\Psi(z)=\frac{1}{2}\int_0^T -J\dot{z}\cdot z dt-\int_0^T H(t,z)\,dt \tag{2.1}$$

defined on the Hilbert space $E:=H^{1/2}_{per}(O,T;\mathbb{R}^{2N})$ of $T$-periodic functions from $\mathbb{R}$ to $\mathbb{R}^{2N}$ of class $C^1$, with the norm

$$\|z\|_E=\left(2\pi\sum_{n\in\mathbb{Z}}|z_n|^2\sqrt{1+|n|^2}\right)^{1/2}$$

induced by the scalar product

$$\langle z,v\rangle_E=2\pi\sum_{n\in\mathbb{Z}}z_n\cdot v_n\sqrt{1+|n|^2}.$$

Here $v_n,z_n\in\mathbb{R}^{2N}$ are the Fourier coefficients of the expansion

$$z=\sum_{n\in\mathbb{Z}}z_n e^{i\,n\frac{2\pi}{T}t}. \tag{2.2}$$

and $z_n\cdot v_n$ denotes the scalar product in $\mathbb{R}^{2N}$. The norm on $\mathbb{R}^{2N}$ is denoted by $|\cdot|$. The space $E$ is continuously embedded in $L^p$ for all $p\geq 1$.

Let us introduce the following notation:

$$\begin{aligned} E^n &= \operatorname{span}\{e^{i\,n\frac{2\pi}{T}t}\}, \\ E^- &= \operatorname{span}\{e^{i\,n\frac{2\pi}{T}t} \mid n<0\}, \\ E^0 &= \mathbb{R}^{2N}, \\ E^+ &= \operatorname{span}\{e^{i\,n\frac{2\pi}{T}t} \mid n>0\}. \end{aligned}$$

where $n \in \mathbb{Z}$. Then

$$E = E^- \oplus E^0 \oplus E^+$$

is an orthogonal decomposition of $E$.

The elements from $(E^0)^\perp$ will be denoted by $u$ and these functions coincide with elements of $E$ having zero mean. Each function $z \in E$ can be written in the unique way as

$$\begin{aligned} z &= u + m \\ &= u^- + m + u^+ \end{aligned}$$

where $u^\pm \in E^\pm, u = u^+ + u^-$, and $m \in E^0$.

On the space $E^- \oplus E^+$ we shall introduce the equivalent norm

$$\|u\|_{H^{1/2}} = \left(2\pi \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |n|\,|u_n|^2\right)^{1/2}$$

and the equivalent scalar product

$$\langle u, v\rangle_{H^{1/2}} = 2\pi \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |n| u_n \cdot v_n.$$

Functional $\Psi$ is Fréchet differentiable on $E$ and $\Psi'(z)$ has the form

$$\Psi'(z) = Lz - K(z)$$

where

$$\begin{aligned} \langle Lz, v\rangle_E &= \int_0^T -J\dot z \cdot v\,dt \\ &= 2\pi \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} n z_n \cdot v_n \end{aligned}$$

and

$$\langle K(z), v\rangle_E = \int_0^T H'(t,z) v\,dt, \quad \text{for all } v \in E.$$

Moreover, $L : E \to E$ is self adjoint, $\ker L = E^0 = \mathbb{R}^{2N}$, and the restriction $L/_{E^-\oplus E^+} : E^- \oplus E^+ \to E^- \oplus E^+$ is invertible with continuous inverse. The spectrum of $L$ is

$$\sigma(L) = \left\{ \frac{n}{(1+n^2)^{1/2}} \mid n \in \mathbb{Z} \right\} \cup \{-1, 1\}.$$

The numbers $\lambda_n = \frac{n}{(1+n^2)^{1/2}}$, $n \in \mathbb{Z}$, $n \neq \pm 1$ are eigenvalues of $L$ and the corresponding subspaces $E_n$ have dimension $2N$.

Because of (H1) the operator $K : E \to E$ is uniformly continuous on bounded sets in E in the sense of Krasnosel'ski. Moreover, being derivative of a weakly continuous function it is compact.

## 3. Palais–Smale–Cerami condition

Let us derive some elementary but important facts about $L$ and the behaviour of $H(t,z)$ at infinity.

Lemma 1. *For all $z \in E$ we have*

$$\int_0^T -J\dot{z} \cdot z \, dt = \|u^+\|^2_{H^{1/2}} - \|u^-\|^2_{H^{1/2}}. \tag{3.3}$$

$$\int_0^T -J\dot{z} \cdot u^+ \, dt = \|u^+\|^2_{H^{1/2}} \tag{3a}$$

$$\int_0^T -J\dot{z} \cdot u^- \, dt = -\|u^-\|^2_{H^{1/2}} \tag{3b}$$

Proof. The derivative of function $z$ has expansion

$$\dot{z} \quad = \quad \frac{2\pi}{T} \sum_{n \in \mathbb{Z}} i\, n z_n e^{i\, n \frac{2\pi}{T} t},$$

obtained from (2.2). Then,

$$\begin{aligned} -J\dot{z} \quad &= \quad \frac{2\pi}{T} \sum_{n \in \mathbb{Z}} n z_n e^{i\, n \frac{2\pi}{T} t}, \\ \int_0^T -J\dot{z} \cdot z \, dt \quad &= \quad \frac{2\pi}{T} \sum_{n \in \mathbb{Z}} T n |z_n|^2. \end{aligned}$$

and because of $z = u^- + u^+$, we have

$$\begin{aligned} \int_0^T -J\dot{z} \cdot z \, dt &= \int_0^T -J\dot{u}^- \cdot u^- \, dt + \int_0^T -J\dot{u}^+ \cdot u^+ \, dt \\ &= \quad 2\pi \sum_{\substack{n \in \mathbb{Z} \\ n > 0}} |n| \, |z_n|^2 - 2\pi \sum_{\substack{n \in \mathbb{Z} \\ n > 0}} |n| \, |z_n|^2 \\ &= \quad \|u^+\|^2_{H^{1/2}} - \|u^-\|^2_{H^{1/2}}, \end{aligned}$$

which proves formula (3.3). Formulas (3a) and (3b) follows immediately using the same technique. ∎

LEMMA 2. *Let us assume* (H1) *and* (H2). *Then, there exist real numbers* $C_1, C_2$ *such that for all* $t \in \mathbb{R}$ *and* $z \in \mathbb{R}^n$

$$C_1 + \frac{\Theta_1}{p}|z|^p \leq H(t,z) \leq C_2 + \frac{\Theta_2}{p}|z|^p, \qquad C_1, C_2 \in \mathbb{R} \tag{3.4}$$

PROOF. To prove this, put $s = |z|, z_0 = z/s$. Then

$$\begin{aligned} H(t,sz_0) &= H(t,rz_0) + \int_r^s H'(t,\tau z_0)\cdot z_0 d\tau \\ &\geq H(t,rz_0) + \int_r^s \frac{1}{\tau}\Theta_1|\tau z_0|^p d\tau \\ &\geq \inf_{\substack{|z|\leq r \\ t\in\mathbb{R}}}\{H(t,z)\} - \frac{\Theta_1 r^p}{p} + \frac{\Theta_1 s^p}{p}, \end{aligned}$$

which proves the left-hand inequality. In the same way one can prove the remaining part. ∎

LEMMA 3. *Assume that* $H(t,z)$ *satisfies* (H1) *and* (H2). *Then* $\Psi$ *satisfies the Palais-Smale-Cerami condition in the following sense: If* $(z_n) \subseteq E$, $n \in \mathbb{N}$, *is such that*

$$\begin{cases} \Psi(z_n) \text{ is bounded and} \\ \|\Psi'(z_n)\|_{E'}\|z_n\|_E \to 0 \end{cases} \tag{3.5}$$

*then* $(z_n)$ *has a convergent subsequence.*

The sequence which satisfies (3.5) is called the PSC-sequence.

PROOF. <u>1st step: $(z_n)$ is bounded in $L^p$-norm.</u>

By the assumption $\Psi'(z_n)z_n =: \varepsilon_n \to 0$, which can be written in the form

$$\int_0^T -J\dot{z}_n \cdot z_n\, dt - \int_0^T H'(t,z_n)\cdot z_n\, dt = \varepsilon_n \to 0. \tag{3.6}$$

On the other side, boundedness of $\Psi(z_n)$ can be written in the form

$$-C \leq \frac{1}{2}\int_0^T -J\dot{z}_n\cdot z_n\, dt - \int_0^T H(t,z_n)\, dt \leq C, \qquad C \in \mathbb{R} \tag{3.7}$$

and consequently:

$$-C \leq \varepsilon_n + \frac{1}{2}\int_0^T H'(t,z_n)\cdot z_n\, dt - \int_0^T H(t,z_n)\, dt \leq C$$

where $\varepsilon_n \to 0$ and it can be omitted in the above formula by the appropriate choice of $C$. Using the left-hand inequality and (3.4) we have

$$
\begin{aligned}
-C + \frac{\Theta_1}{p}\|z_n\|^p_{L^p} &\le \frac{1}{2}\int_0^T H'(t, z_n)\cdot z_n\,dt \\
&\le \frac{1}{2}\int_0^T (\Theta_2|z_n|^p + \Theta_2 r^{p-1}|z_n|\,)\,dt \\
&\le \frac{\Theta_2}{2}\|z_n\|^p_{L^p} + C\|z_n\|_{L^p}
\end{aligned}
$$

where $C$ is a generic constant. Finally,

$$
(\frac{\Theta_1}{p} - \frac{\Theta_2}{2})\|z_n\|^p_{L^p} \le C + C\|z_n\|_{L^p}.
$$

Hence, we have proved the first step since $\frac{\Theta_1}{p} - \frac{\Theta_2}{2} > 0$.

2nd step: $\|u_n^+\|_{H^{1/2}}$ is bounded.

Using the assumption that $z_n$ is PSC-sequence, it is easy to prove that $\Psi'(z_n)u_n^+$ converges to zero too, which can be written in the form

$$
\int_0^T -J\dot{z}_n\cdot u_n^+\,dt - \int_0^T H'(t, z_n)\cdot u_n^+\,dt =: \varepsilon_n \to 0.
$$

But, using the definition of $H^{1/2}$-scalar product we have

$$
\int_0^T -J\dot{z}_n\cdot u_n^+\,dt = \|u_n^+\|^2_{H^{1/2}},
$$

which implies that

$$
\begin{aligned}
\|u_n^+\|^2_{H^{1/2}} &\le \varepsilon_n + \int_0^T (\Theta_2|z_n|^{p-1} + C)|u_n^+|\,dt \\
&\le \varepsilon_n + \int_0^T \Theta_2|z_n|^{p-1}|u_n^+|\,dt + \int_0^T C|u_n^+|\,dt \\
&\le \varepsilon_n + \Theta_2\left(\int_0^T (|z_n|^{p-1})^{\frac{p}{p-1}}\,dt\right)^{\frac{p-1}{p}}\left(\int_0^T |u_n^+|^p\,dt\right)^{\frac{1}{p}} + \\
&\qquad\qquad + \int_0^T C|u_n^+|\,dt \\
&\le C + \Theta_2\,\|z_n\|^{p-1}_{L^p}\|u_n^+\|_{L^p} + C\,\|u_n^+\|_{L^p}
\end{aligned}
$$

where $C$ is a generic constant. The previous inequality and the embedding $H^{1/2} \hookrightarrow L^p$ imply that there exists a constant $K > 0$ such that $\|u_n^+\|_{L^p} \le \|u_n^+\|_{H^1/2}$ and

$$
\|u_n^+\|^2_{H^{1/2}} \le C + \Theta_2 K\,\|z_n\|^{p-1}_{L^p}\|u_n^+\|_{H^{1/2}} + KC\,\|u_n^+\|_{H^{1/2}},
$$

The boundedness of $\|u_n^+\|_{H^{1/2}}$ now follows from the boundedness of $||z_n||_{L^p}$ proved in the first step.

3rd step: $\|u_n^-\|_{H^{1/2}}$ is bounded.

Because of lemmas 1 and 2, inequality (3.7) implies that

$$-C \leq \frac{1}{2}\|u_n^+\|^2_{H^{1/2}} - \frac{1}{2}\|u_n^-\|^2_{H^{1/2}} - \int_0^T (C + \frac{\Theta_1}{p}|z_n|^p)\, dt$$

and

$$\frac{1}{2}\|u_n^-\|^2_{H^{1/2}} \leq C(1+T) + \frac{1}{2}\|u_n^+\|^2_{H^{1/2}} - \frac{\Theta_1}{p}\|z_n\|^p_{L^p}.$$

The right-hand side of this inequality is bounded by steps one and two.

4th step: Conclusion

From orthogonal decomposition $z_n = u_n^- + m_n + u_n^+$ we have

$$\|z_n\|^2_E = \|u_n^-\|^2_E + T|m_n|^2 + \ \|u_n^+\|^2_E.$$

The norms $\|\cdot\|_E$ and $\|\cdot\|_{H^{1/2}}$ are equivalent on $E^- \oplus E^+$ and $|m_n|$ is bounded because $\|z_n\|_{L^p}$ is bounded and $(m_n)$ is contained in a finite-dimensional space. This implies that $\|z_n\|_E$ is bounded. If $z_n \to 0$ we have nothing to prove. Otherwise, $\Psi'(z_n) \to 0$. Then using the fact that $Lz_n = Lu_n$ and since the restriction $L_0 = L/_{E^- \oplus E^+}$ is invertible we obtain

$$\begin{aligned} L_0 u_n - K(z_n) &= \Psi'(z_n) \to 0, \\ u_n &= L_0^{-1}\Psi'(z_n) + L_0^{-1}K(z_n). \end{aligned}$$

The right-hand side has a convergent subsequence because $K$ is compact and $(z_n)$ is bounded in $E$. Hence $(u_n)$ has a convergent subsequence. The sequence $(m_n)$ is also bounded and, passing eventualy to a subsequence, we have the conclusion because $z_n = u_n + m_n$. □

## 4. Proof of theorem 1

The decomposition of the space $E$ below allows us to use the theorem of Benci–Rabinowitz in the Appendix. The functional $\Psi$ is of the form $\Psi(z) = \frac{1}{2}\langle Lz, z\rangle - K(z)$ where $L\colon E \to E$ is a self adjoint linear operator and $K(z)$ has compact derivative. Let us consider the orthogonal decomposition

$$E = E_1 \oplus E_2$$

where $E_1 = \mathbb{R}^{2N} \oplus E^-$ and $E_2 = E^+$.

On $E_1$ we have $\Psi(z) \to -\infty$ when $\|z\|_E \to +\infty$.

To see this take $z \in E_1$. Then,

$$\begin{aligned}\Psi(z) &= -\frac{1}{2}\|u^-\|^2_{H^{1/2}} - \int_0^T H(t,z)\,dt\\ &\leq -\frac{1}{2}\|u^-\|^2_{H^{1/2}} - \frac{\Theta_1}{p}\|z\|^p_{L^p} - C_1T.\end{aligned}$$

If $z \in E^-$ this conclusion is immediate, while if $u \in \mathbb{R}^N$ this follows from the fact that all norms on a finite dimensional space are equivalent.

$\Psi$ is coercive on $E_2$, i.e. $\Psi(u^+) \to +\infty$ when $\|u^+\|_{H^{1/2}} \to +\infty$.

To prove that we use lemmas 1 and 2 to conclude that

$$\begin{aligned}\Psi(u^+) &= \frac{1}{2}\int_0^T -Ju^+ \cdot u^+\,dt - \int_0^T H(t,u^+)\,dt\\ &= \frac{1}{2}\|u^+\|^2_{H^{1/2}} - \int_0^T H(t,u^+)\,dt\\ &\geq \frac{1}{2}\|u^+\|^2_{H^{1/2}} - C_2T - \frac{\Theta_2}{p}\|u^+\|^p_{L^p}\end{aligned}$$

Now let us recall the fact that the space $H^{1/2}$ is embedded into $L^p$, $p > 1$, and that $\|u^+\|_{L^p} \leq K\|u^+\|_{H^{1/2}}$ for some $K > 0$. The consequence of this is

$$\frac{1}{2}\|u^+\|^2_{H^{1/2}} - \frac{\Theta_1 K}{p}\|u^+\|^p_{H^{1/2}} - C_1T \leq \Psi(u^+)$$

which proves the coercivity of $\Psi/_{E^+}$ because $1 < p < 2$. Now the conclusion follows from the theorem of Benci–Rabinowitz.

## 5. The resonant case

In this part we shall consider the Hamiltonian of the form

$$H(t,z) = \hat{H}(t,z) - \frac{k}{2}|z|^2, \quad k > 0 \tag{5.8}$$

where $\hat{H}(t,z)$ satisfies conditions (H1) and (H2). Equation (H) can be written in the form

$$-J\dot{z} + kz = \hat{H}'(t,z),$$

and the associated functional $\Psi_k : E \to \mathbb{R}$ is

$$\Psi_k(z) = \frac{1}{2}\int_0^T -J\dot{z}\cdot z\,dt + \frac{1}{2}\int_0^T k|z|^2\,dt - \int_0^T \hat{H}(t,z)\,dt. \tag{5.9}$$

Evidently,

$$\Psi_k(z) = \frac{1}{2}\|u^+\|_{H^{1/2}} - \frac{1}{2}\|u^-\|_{H^{1/2}} + \frac{k}{2}\|z\|^2_{L^2} - \int_0^T \hat{H}(t,z)\,dt. \tag{5.10}$$

which follows from lemma 1.

Let us investigate some basic properties of the linear operator $L_k : E \to E$ defined by

$$\langle L_k z, v\rangle_E = \int_0^T (-J\dot{z} + kz)v, \qquad z, v \in E. \tag{5.11}$$

We shall assume that

$$k\frac{T}{2\pi} = K \in \mathbb{N}. \tag{5.12}$$

Obviously, $L_k : E \to E$ is self adjoint, $\ker L_k = E^{-K} = \operatorname{span}\{e^{-iK\frac{2\pi}{T}t}\}$, the restriction $L_k/_{(E^{-K})^\perp}$ is invertible and the spectrum of $L_k$ is

$$\sigma(L_k) = \left\{ \frac{n+K}{\sqrt{1+|n|^2}} \mid n \in \mathbb{Z} \right\} \cup \{-1, 1\}.$$

Lemma 1. *If we assume* (5.12)*, then* $\Psi_k$ *satisfies the Palais-Smale-Cerami condition.*

Proof. The idea of the proof is the same as that of lemma 3. Let $(z_n) \subset E$ be a sequence such that

$$\begin{cases} |\Psi_k(z_n)| \ \le C \text{ and} \\ \|\Psi_k'(z_n)\|_{E'} \|z_n\|_E \to 0 \end{cases} \tag{5.13}$$

<u>1<sup>st</sup> step: $(z_n)$ is bounded in $L^p$-norm and in $L^2$-norm.</u>

The same calculation as in lemma 3 gives

$$-C \le \varepsilon_n + \frac{1}{2}\int_0^T \hat{H}'(t, z_n)\cdot z_n\, dt - \int_0^T \hat{H}(t, z_n)\, dt \le C,$$

where $\varepsilon_n \to 0$, and the boundedness in $L^p$-norm follows. Boundedness in $L^2$-norm follows from the embedding $L^p \hookrightarrow L^2$, for $p < 2$.

<u>2<sup>nd</sup> step: $\|u_n^+\|_{H^{1/2}}$ is a bounded sequence.</u>

$\Psi_k'(z_n)u_n^+$ is convergent because

$$0 \le |\Psi_k'(z_n)u_n^+| \le \|\Psi_k'(z_n)\|_{E'}\, \|z_n\|_E \to 0.$$

This means that

$$\int_0^T L_k z_n \cdot u_n^+\, dt - \int_0^T \hat{H}'(t, z_n)\cdot u_n^+\, dt =: \varepsilon_n \to 0.$$

On the other hand we have the following expression for $\langle L_k z, v\rangle_E$:

$$\begin{aligned}\langle L_k z, v\rangle_E &= \int_0^T L_k z \cdot v\,dt\\ &= \int_0^T L_k \hat{z}\cdot\hat{v}\,dt\\ &= 2\pi\sum_{n\neq -K}(n+K)z_n\cdot v_n.\end{aligned}$$

If we use it in the above formula we obtain

$$2\pi\sum_{\substack{n\neq -K\\ n>0}}(n+K)|z_n|^2 = \varepsilon_n + \int_0^T \hat{H}'(t,z_n)\cdot u_n^+\,dt,$$

and

$$\begin{aligned}\|u_n^+\|^2_{H^{1/2}} + \frac{2\pi}{T}K\sum_{n>0}T|z_n|^2 &= \varepsilon_n + \int_0^T \hat{H}'(t,z_n)\cdot u_n^+\,dt;\\ \|u_n^+\|^2_{H^{1/2}} + k\|u_n^+\|^2_{L^2} &= \varepsilon_n + \int_0^T \hat{H}'(t,z_n)\cdot u_n^+\,dt.\end{aligned}$$

Using (H1) and repeating the same arguments as in the $2^{\text{nd}}$ step on page 106 we finally have

$$\|u_n^+\|^2_{H^{1/2}} + k\|u_n^+\|^2_{L^2} \le C + C\|u_n^+\|_{H^{1/2}} + C\Theta_2\|z_n\|^{p-1}_{L^p}\|u_n^+\|_{H^{1/2}},$$

where $C$ is a constant such that $\sup\varepsilon_n < C$. Using the imbedding $H^{1/2}\hookrightarrow L^2$ the conclusion follows.

$3^{\text{rd}}$ step: $\|u_n^-\|_{H^{1/2}}$ is bounded.

$\Psi_k(z_n)$ is a bounded sequence which implies that

$$\begin{aligned}C &\le \frac{1}{2}\int_0^T L_k z_n\cdot z_n\,dt - \int_0^T \hat{H}(t,z_n)\,dt\\ &\le \frac{1}{2}\|u_n^+\|^2_{H^{1/2}} - \frac{1}{2}\|u_n^-\|^2_{H^{1/2}} + \frac{k}{2}\|z_n\|^2_{L^2} - C_1T - \frac{\Theta_1}{p}\|z_n\|^p_{L^p},\end{aligned}$$

where $\Theta_1, C_1$ are the constants from lemma 2. Since $\|u_n^+\|_{H^{1/2}}, \|z_n\|_{L^2}, \|z_n\|_{L^p}$ are already bounded we conclude that $\|u_n^-\|_{H^{1/2}}$ is bounded.

$4^{\text{th}}$ step: Conclusion

We can write $z_n = u_n^+ + m_n + u_n^- = u_n^+ + m_n + z_n^{-K} + \hat{u}_n^-$ where $u_n^\pm\in E^\pm$, $m_n\in E^0$, $z_n^{-K}\in E^{-K}$, $\hat{u}_n^-\in E^-\setminus E^{-K}$ and $u_n^- = z_n^{-K} + \hat{u}_n^-$ . The spaces $E^{-K}$ and $E^0$ are finite dimensional and all the norms on finite dimensional spaces are equivalent. As $(z_n^{-K})$ and $(m_n)$ are bounded sequences in $L^p$-norm, they are bounded also in $E$. Finally, $(z_n)$ is bounded in $E$. On the other hand $\Psi_k'(z_n) \to 0$,

$$\Psi_k'(z_n) = \hat{L}_k \hat{z}_n - K(z_n) \in E'$$

where $K$ is a compact operator and the restriction $\hat{L}_k := L_k/_{(E^{-K})^\perp}$ is invertible. Then

$$\hat{z}_n = \hat{L}_k^{-1} \Psi_k'(z_n) + \hat{L}_k^{-1} K(z_n),$$

which implies that $(\hat{z}_n)$ has a convergent subsequence. The sequence $(z_n^{-K})$ also has a convergent subsequence because it is a bounded sequence in a finite dimensional space. Therefore $(z_n)$ has a convergent subsequence. ∎

## 6. Proof of theorem 2

Let us denote $T_1 = lT$. Then $H$ and $\hat{H}$ are $T_1$-periodic and we can apply previous considerations to $T_1$-periodic functions. Put $K = \frac{kT_1}{2\pi}$. Then $K \in \mathbb{N}$ and

$$\Psi_k(z) = \frac{1}{2}\|u^+\|_{H^{1/2}} - \frac{1}{2}\|u^-\|^2_{H^{1/2}} + \frac{k}{2}\|z\|^2_{L^2} - \int_0^{T_1} \hat{H}(t,z)\,dt.$$

where we have used the same notation i.e. $H^{1/2}$ in the context of $T_1$-periodic functions. From lemmas 1 and 2 we have inequality

$$\begin{aligned} &\tfrac{1}{2}\|u^+\|_{H^{1/2}} - \tfrac{1}{2}\|u^-\|^2_{H^{1/2}} + \tfrac{k}{2}\|z\|^2_{L^2} - C_2T - \tfrac{\Theta_2}{p}\|z\|^p_{L^p} \\ &\qquad \le \Psi_k(z) \le \\ &\tfrac{1}{2}\|u^+\|_{H^{1/2}} - \tfrac{1}{2}\|u^-\|^2_{H^{1/2}} + \tfrac{k}{2}\|z\|^2_{L^2} - C_1T - \tfrac{\Theta_1}{p}\|z\|^p_{L^p}. \end{aligned}$$

This inequality suggests to decompose the space into the orthogonal sum $E = E_1 \oplus E_2$ where $E_1 = E^-$ and $E_2 = E^0 + E^+$. Now, the following two claims are easy to prove:

1. $\Psi_k$ is coercive on $E_2$, i.e. $\Psi_k(z) \to +\infty$ when $\|z\| \to +\infty, z \in E_2$.
2. $\Psi_k(z) \to -\infty$ when $\|z\|_E \to +\infty,\ z \in E_1$

The first one is evident because $p < 2$, and the second requires some technical calculations. Recall that for $u \in E^- \oplus E^+$ we always have $\|u\|_{L^2} \le \|u\|_{H^{1/2}}$. So, for any $u \in E_1$ there holds

$$\begin{aligned} \Psi_k(u^-) &\le -\frac{1}{2}\|u^-\|^2_{H^{1/2}} + \frac{k}{2}\|u^-\|^2_{L^2} - C_1T - \frac{\Theta_1}{p}\|u^-\|^p_{L^p} \\ &\le \frac{1}{2}(k-1)\|u^-\|^2_{H^{1/2}} - C_1T - \frac{\Theta_1}{p}\|u^-\|^p_{L^p}. \end{aligned}$$

Here $k < 1$ and the last expression evidently tends to $-\infty$ when $\|u^-\| \to +\infty$. If $u \in \ker L_K = E^{-K}$ then

$$\Psi_k(u) \leq -C_1 T - \frac{\Theta_1}{p}\|u\|^p_{L^p}$$

which implies that $\Psi_k(u) \to -\infty$ when $\|u\|_{L^p} \to +\infty$. It should be noticed that $E^{-K}$ is finite dimensional and all its norms are equivalent. So, $\Psi_k(u) \to -\infty$ when $\|u\|_{L^p} \to +\infty$, what we wanted to prove.

To finish the proof of the theorem let us point out that we are exactly in the situation described in the saddle point theorem (theorem A2 in the Appendix). □

Corollary 1. *Assume that Hamiltonian system* (H) *is autonomous and* $\hat{H}$ *satisfies* (H1) *and* (H2) *for some* $0 < k < 1$. *Then,* (H) *has a T-periodic solution for any* $T > 0$ *of the form* $T = \frac{2\pi}{k} n$, $n \in \mathbb{N}$.

Proof. We can choose $\tilde{T} > 0$ such that $1 = \frac{k}{2\pi}\tilde{T}$. Theorem 2 gives the existence of $n\tilde{T}$-periodic solution for all $n \in \mathbb{N}$. □

## 7. Relaxed sublinearity

The conditions (H1) and (H2) seems to be quite restrictive and that there is still enough room for relaxation. A weaker form of sub-linearity is that

(H1') $$\limsup_{|z|\to\infty} \frac{|H'(t,z)|}{|z|} = 0 \quad \text{uniformly on}\, t \in \mathbb{R};$$

From dynamic point of view one expects periodic solutions to appear as a consequence of attractive force, attractive at least outside a ball centered at the origin. If $F(q)$ is a force at position $q \in \mathbb{R}^n$ then the force is attractive at $q$ if it satisfies

$$-\alpha \geq \frac{q \cdot F(q)}{|q|\,|F|}$$

for some $\alpha > 0$. A slight generalisation of this condition is

(H2') $$\frac{H'(t,z)\cdot z}{|H'(t,z)|\,|z|} \geq \alpha > 0,$$

outside a ball of radius $r > 0$ uniformly in $t$.

Another reasonable dynamic condition is that the force should not vanish at infinity. The condition $|F| \geq \beta > 0$ outside some ball seems to be too strong. A weaker condition is $|F|\,|q| \geq \beta > 0$ outside a ball. Still weaker condition is

(H3) $$|H'(t,z)||z| \geq \beta > 0,$$

for $|z| \;\geq r > 0$ uniformly in time. A Hamiltonian that satisfies conditions (H1) and (H2) satisfies also (H1)', (H2)' and (H3) with $\beta = \Theta_1/\Theta_2$.

We want to prove the existence of $T$-periodic solution of hamiltonian system (H) with new, relaxed conditions. The first step in that direction is to prove that functional $\Psi : E \to \mathbb{R}$ defined by (2.1) satisfies the Palais–Smale–Cerami condition. If we have inequality of the form

$$\|u_n\|_{L^\infty} \leq \delta |m_n| + \gamma, \qquad 0 < \delta < 1 \tag{7.14}$$

where $z_n = u_n + m_n$, $\int_0^T u_n dt = 0$, $m_n \in \mathbb{R}^{2N}$ for a PSC-sequence $u_n$, then we could prove the Palais-Smale condition for $\Psi$ using the same technique as in [7].

Lemma 1. *Let $(z_n) \subset E$ be a PS-sequence that satisfies (3.5). Then*

$$\|u_n\|_{H^{1/2}} \leq \delta |m_n| + \gamma, \tag{7.15}$$

*where $\delta = \dfrac{2\varepsilon T^{1/2}}{1-2\varepsilon}$, $\varepsilon > 0$ and $\gamma \in \mathbb{R}$. Moreover, $\varepsilon$ can be taken such that $\delta < 1$ is arbitrarily small.*

The passage from inequality (7.15) to inequality (7.14) is 'not possible' because the space $E$ is not included in $L^\infty$. At the moment it seems that the best we can prove is the inequality

$$\|u_n\|_{L^p} \leq \delta |m_n| + \gamma, \qquad 0 < \delta < 1, \tag{7.16}$$

for $1 < p < +\infty$, but this inequality is still not sufficient to prove PSC–condition.

Proof of lemma 1. Because of (3.5) we have

$$\langle \Psi'(z_n), u_n^+ \rangle_{E',E} =: \varepsilon_n \to 0 \quad n \longrightarrow +\infty$$

This means that

$$\int_0^T -J\dot{z}_n \cdot u_n^+ \, dt = \int_0^T H'(t, z_n) \cdot u_n^+ \, dt + \varepsilon_n.$$

Condition (H1') is equivalent to the following one

$$\big(\forall \varepsilon > 0\big)\big(\exists C_\varepsilon \in \mathbb{R}\big) \quad \text{such that} \quad |H'(t,z)| \leq \varepsilon |z| + C_\varepsilon \text{ uniformly in } t.$$

Using this on the right-hand side of above equality and formula (3a), Lemma 2, on the left-hand side we obtain

$$\begin{aligned}
\|u_n^+\|^2_{H^{1/2}} &\leq \textstyle\int_0^T (\varepsilon |z_n| + C_\varepsilon) |u_n^+| \, dt + \varepsilon_n \\
&\leq \varepsilon \, \|z_n\|_{L^2} \|u_n^+\|_{L^2} + C_\varepsilon T^{1/2} \|u_n^+\|_{L^2} + \varepsilon_n \\
&\leq \varepsilon \, \big( \|u_n\|_{L^2} + T^{1/2} |m_n| \big) \, \|u_n^+\|_{L^2} + C_\varepsilon T^{1/2} \|u_n^+\|_{L^2} + \varepsilon_n \\
&\leq \varepsilon \, \big( \|u_n\|_{H^{1/2}} + T^{1/2} |m_n| \big) \, \|u_n^+\|_{H^{1/2}} + C_\varepsilon T^{1/2} \|u_n^+\|_{H^{1/2}} + \varepsilon_n,
\end{aligned}$$

where we used that for $u \in E$,

$$\begin{aligned} \|u\|_{L^2}^2 &= 2\pi \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |u_n|^2 \\ &\leq 2\pi \sum_{n\neq 0} |n|\,|u_n|^2 \\ &= \|u\|_{H^{1/2}}^2. \end{aligned}$$

Using the triangle inequality $\|u_n\|_{H^{1/2}} \leq \|u_n^+\|_{H^{1/2}} + \|u_n^-\|_{H^{1/2}}$, after rearranging we obtain

$$(1-\varepsilon)\|u_n^+\|_{H^{1/2}}^2 \leq \varepsilon\|u_n^-\|_{H^{1/2}}\|u_n^+\|_{H^{1/2}} + T^{1/2}(\varepsilon|m_n| + C_\varepsilon)\|u_n^+\|_{H^{1/2}} + \varepsilon_n,$$

and in the same way

$$(1-\varepsilon)\|u_n^-\|_{H^{1/2}}^2 \leq \varepsilon\|u_n^-\|_{H^{1/2}}\|u_n^+\|_{H^{1/2}} + T^{1/2}(\varepsilon|m_n| + C_\varepsilon\|u_n^-\|_{H^{1/2}} + \varepsilon_n.$$

The second inequality follows from

$$\langle \Psi'(z_n), -u_n^- \rangle = \varepsilon_n \to 0, \qquad n \to +\infty.$$

Summing up both of them we obtain

$$(1-\varepsilon)(\|u_n^-\|_{H^{1/2}}^2 + \|u_n^+\|_{H^{1/2}}^2) \leq$$
$$2\varepsilon\|u_n^-\|_{H^{1/2}}\|u_n^+\|_{H^{1/2}} + T^{1/2}\big(\varepsilon|m_n| + \ C_\varepsilon\big)\big(\|u_n^+\|_{H^{1/2}} + \|u_n^-\|_{H^{1/2}}\big) + 2\varepsilon_n.$$

Using the notation $\alpha = \|u_n^-\|_{H^{1/2}}, \beta = \|u_n^+\|_{H^{1/2}}, \xi = \alpha+\beta$, and the inequality $(\alpha+\beta)^2 \leq 2(\alpha^2+\beta^2)$ we have

$$\begin{aligned} (\frac{1}{2}-\varepsilon)(\alpha+\beta)^2 &\leq \frac{1}{2}(\alpha+\beta)^2 - \varepsilon(\alpha+\beta)^2 \\ &\leq \alpha^2+\beta^2-\varepsilon(\alpha+\beta)^2 \\ &\leq T^{1/2}(\varepsilon|m_n| + C_\varepsilon)(\alpha+\beta) + 2\varepsilon_n. \end{aligned}$$

If we take $\varepsilon < \frac{1}{2}$ and use the inequality $\xi \leq \frac{B+\sqrt{B^2+4AC}}{2A} \leq \frac{B}{A} + \sqrt{\frac{C}{A}}$ whenever $A\xi^2 \leq B\xi + C$ where $A, B, C$ are positive, then,

$$0 \leq \xi \leq \frac{2(\varepsilon|m_n| + C_\varepsilon T^{1/2}}{1-2\varepsilon} + (\frac{4\varepsilon_n}{1-2\varepsilon})^{1/2} \tag{7.17}$$

or

$$\|u_n\|_{H^{1/2}} \leq \xi \leq \frac{2\varepsilon T^{1/2}}{1-2\varepsilon}|m_n| + \frac{2C_\varepsilon T^{1/2} + 2\sqrt{\varepsilon_n}(1-2\varepsilon)^{1/2}}{1-2\varepsilon}$$

which proves the lemma. ∎

To prove inequality (7.14) one needs a relation between $\|\cdot\|_{L^\infty}$ norm and $\|\cdot\|_{L^2}$ norm (or $\|\cdot\|_{H^{1/2}}$ norm). The continuity of $L^{-1}$ in this pair of norms is just what we are looking for. Here is a slight generalisation of required inequality.

Lemma 2. *Let us assume* (5.12). *Then, for each* $\hat{z} \in (E^{-K})^{\perp}$ *we have*

$$\|\hat{z}\|_{L^\infty} \leq \sqrt{\frac{T}{12}}\|L_k\hat{z}\|_{L^2}. \tag{7.18}$$

Proof. Let $\hat{z} = \sum_{n\neq -K} z_n e^{i\,n\frac{2\pi}{T}t}$ be the Fourier expansion for $\hat{z}$. Then

$$L_k\hat{z} = \frac{2\pi}{T}\sum_{\substack{n\in\mathbb{Z}\\ n\neq -K}}(n+K)z_n e^{i\,n\frac{2\pi}{T}t},$$

en hence

$$\|L_k\hat{z}\|_{L^2} = \frac{2\pi}{T^{1/2}}\Big(\sum_{n\neq -K}(n+K)^2|z_n|^2\Big)^{1/2}.$$

$$\begin{aligned}
|\hat{z}(t)| &\leq \sum_{n\neq -K}|z_n| \\
&= \sum_{n\neq -K}|n+K|\,|z_n|\frac{1}{|n+K|} \\
&= \Big(\sum_{n\neq -K}|n+K|^2|z_n|^2\Big)^{1/2}\Big(\sum_{n\neq -K}\frac{1}{(n+K)^2}\Big)^{1/2} \\
&= \frac{\pi}{\sqrt{3}}\frac{T^{1/2}}{2\pi}\|L_k\hat{z}\|_{L^2} \\
&= \frac{T^{1/2}}{2\sqrt{3}}\|L_k\hat{z}\|_{L^2}. \qquad \square
\end{aligned}$$

In the special case when $k = 0$, we have $L_k = L$ and inequality (7.18) can be rewritten in the form $\|u\|_{L^\infty} \leq \sqrt{\frac{T}{12}}\|Lu\|_{L^2}$ for all $u \in E$ such that $\int_0^T u(t)\,dt = 0$. Inequality (7.14) is now a consequence of this inequality, lemma 1 and the fact that the constants $\delta$ and $\gamma$ in (7.14) do not depend upon $n$. $\square$

Lemma 3. *Assume that conditions* (H1'), (H2') and (H3) *ere satisfied. Then* $\Psi$ *satisfies the PSC-condition.*

Proof. We repeat the proof given in [7] without going to the details. First, we prove that

$$H'(t,z)\cdot m \geq const > 0, \quad \text{for } |m| \text{ big enough,} \tag{$*$}$$

where $m = \frac{1}{T}\int z(t)\,dt$ is the constant part of $z$. Because of (H2') and (H3) we have

$$\begin{aligned} H'\bigl(t,z(t)\bigr)\cdot m &= H'\bigl(t,z(t)\bigr)\cdot\bigl(z(t)-u(t)\bigr)\\ &= \bigl|H'\bigl(t,z(t)\bigr)\bigr|\bigl|z(t)\bigr|\left(\frac{H'\bigl(t,z(t)\bigr)\cdot z(t)}{|H'\bigl(t,z(t)\bigr)||z(t)|}-\frac{H'\bigl(t,z(t)\bigr)\cdot u(t)}{|H'\bigl(t,z(t)\bigr)||z(t)|}\right)\\ &\geq \beta\left(\alpha-\frac{|u(t)|}{|z(t)|}\right). \end{aligned}$$

Now, using (7.14) we can write

$$|z(t)| \geq (1-\delta)|m| - \gamma,$$

for $|m| \geq R > 0$ and $R$ big enough (recall that $0 < \delta < 1$). Using this a priori bound we get

$$\begin{aligned} \frac{|u(t)|}{|z(t)|} &\leq \frac{\delta|m|+\gamma}{|m|(1-\delta)-\gamma}\\ &= \frac{\delta+\gamma/|m|}{(1-\delta)-\gamma/|m|}, \end{aligned}$$

and the expression on the right hand side can be made less than $\frac{\alpha}{2}$ when $|m| \to +\infty$. This proves $(*)$.

Let us suppose now that that $(z_n)$ is PSC-sequence. Then,

$$\|\Psi'(z_n)\|\cdot\|m_n\| \to 0. \tag{7.19}$$

On the other side we always have inequality

$$\|\Psi'(z_n)\|\cdot\|m_n\| \geq \Bigl|\int H'(t,z_n)\cdot m_n\,dt\Bigr|.$$

We want to conclude that $\|m_n\|$ is bounded. If not, then, for $R$ big enough and $|m_n| \geq R > 0$ the right hand side of above inequality is bounded bellow by a positive constant, because of $(*)$, which contradicts (7.19).

Till now we have proved boundedness of $|m_n|$ and boundedness of $|z_n(t)|$. To finish the proof we should follow the same arguments as in the $4^{\text{th}}$ step of the proof of lemma 3. □

Now, it is easy to see that theorems 1 and 2 remain true also under relaxed conditions (H1'), (H2') and (H3).

## 8. Appendix

Let $\Psi \in C^1(E,\mathbb{R})$. We say that $\Psi$ satisfies the Palais–Smale–Cerami (PSC) condition in an open interval $(\alpha,\beta) \subset \mathbb{R}$, if for any $c \in (\alpha,\beta)$ and any sequence $(u_n) \subset E$ such that $\Psi(u_n) \to c$ and $\|\Psi'(u_n)\|\,\|u_n\| \to 0$ there exists convergent subsequence of $u_n$ .

This definition is a slightly generalized version of (PS)–condition where the precompactness is required for a wider class of sequences $(u_n)$ for which $\Psi(u_n) \to c$ and $\Psi'(u_n) \to 0$.

The following result, known as Deformation theorem, plays an important role in the study of critical points. We shall state a simplified version that is still sufficient to prove the existence of critical points.

THEOREM A1 (Deformation theorem). *Suppose that $\Psi \in C^1(E, \mathbb{R})$ satisfies the following assumptions:*

($\Psi$1) $\Psi(u) = \frac{1}{2}\langle Lu, u\rangle - b(u)$, *where*
 (i) *$L$ is continuous self-adjoint operator on $E$;*
 (ii) *$b \in C^1(E, \mathbb{R})$ and $b'$ is a compact operator.*

($\Psi$2) *$E = \oplus E_\lambda$ where $E_\lambda$'s are eigenspaces of $L$ of finite dimension.*

($\Psi$3) *$\Psi$ satisfies $(PSC)$-condition or $(PS)$-condition in $\mathbb{R}$.*

*Given $c > 0$, if $c$ is not a critical value of $\Psi$, then there exist constants $\bar{\varepsilon} > \varepsilon > 0$ and a homeomorphism $\eta : E \to E$ such that*

(a) $\eta(\Psi^{c+\varepsilon}) \subset \Psi^{c-\varepsilon}$,
(b) $\eta(u) = u \qquad \forall u \notin \Psi^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}])$.

*Moreover, $\eta$ can be chosen to be of the form*

(c) $\eta(u) = e^{\alpha(u)L}u + K(u)$,

*where $\alpha : E \to \mathbb{R}$ is a continuous linear functional and $K$ compact, and such that all $\eta$'s of the form (c) form a group, which we denote by $G$.*

The proof of this theorem is technically quite complicated and it is given in [3].

THEOREM A2 (Benci–Rabinowitz). *Let us suppose that $\Psi \in C^1(E, \mathbb{R})$ satisfies $(\Psi 1), (\Psi 2)$ and $(\Psi 3)$ and*

$$E = W \oplus V \qquad \text{(orthogonal sum)}$$

*is a decomposition of $E$ into the sum of two $L$-invariant subspaces. Furthermore, assume that*

(i) *$\Psi(w) \to -\infty$ when $\|w\| \to +\infty, w \in W$;*
(ii) *The restriction $\Psi/V$ is bounded from below.*

*Then $\Psi$ has a critical point $u \in E$.*

PROOF. Let $R > 0$ and $Q = \{w \in W \mid \|u\| \leq R\}$ be such that

$$\sup \Psi(\partial Q) < \inf \Psi(V).$$

This is possible because of (i) and (ii). Then $Q$ and $V$ intersect with respect to $G_{\partial Q}$ (the subgroup of $G$ that leaves $\partial Q$ fixed). A number

$$c = \inf_{g \in G_{\partial Q}} \sup \Psi(g(Q))$$

belongs to interval $[\inf \Psi(V), \sup \Psi(Q)]$. We claim that $c$ is a critical value of $\Psi$. Assuming the contrary, according to Deformation theorem, there exists a homeomorphism $\eta : E \to E$ and $\bar{\varepsilon} > \varepsilon > 0$ such that

$$\eta(\Psi^{c+\varepsilon}) \subset \Psi^{c-\varepsilon}$$

$$\eta(u) = u \text{ for } u \notin \Psi^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]).$$

By the definition of $c$ there exists $h \in G_{\partial Q}$ such that

$$\sup \Psi(h(Q)) < c + \varepsilon.$$

Then $\eta \circ h \in G_{\partial Q}$ because $u \in \partial Q \Rightarrow \eta(h(u)) = \eta(u) = u$ and

$$\sup \Psi(\eta(h(Q))) < c - \varepsilon,$$

which contradicts the definition of $c$. □

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