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SPECTRAL SEQUENCES OF UNBOUNDED BICOMPLEXES

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Abstract. The two spectral sequences of the right hand half plane bicomplexes are considered and related to the two cohomologies of the bicomplexes, one with the direct sum the other with the direct product as the totalisation operator.

1. Introduction

Bicomplexes (or double complexes) appear in many topological and geometric problems. Usually they have non-zero entries in one quadrant only, but in several cases one has to consider other bicomplexes as well and in this case there are two totalisation operators – the direct sum and the product. The former is the standard one, but in several cases also the latter appears in interesting theories, in particular in cyclic and equivariant cohomology, e.g. [4], [6], [7], [2], §14 of [5]. In [2] spectral sequence arguments for the right hand half plane bicomplexes were used to show that the totalisation with the product gives a Milnor additive cohomology of $\mathbb{Z}$-graded differential sheaves.

Bicomplexes have two natural filtrations giving rise to two spectral sequences [1], [8]. It turns out that for the right hand half plane bicomplexes the first filtration is complete in the sense of [3] for the product totalisation and the second filtration is complete for the direct sum totalisation.

In this short note we show by direct calculation that the second spectral sequence converges to the cohomology with respect to the direct sum and that in some cases the first spectral sequence converges to the cohomology with respect to the direct product. We prove also the corresponding comparison theorems and give some explicit counterexamples.

Some of our results follow from [3] (but these proofs cannot be performed within the category of bicomplexes with their natural filtrations). Both theorems can be proved also indirectly using the the spectral sequence properties of bounded complexes and the direct and inverse limit arguments.

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2. Totalisation: sum vs. product

A bicomplex $C = (C^{p,q}, \delta, d)$ of modules (over a fixed ring) is a family of modules $C^{p,q}$, $p, q \in \mathbb{Z}$ together with two families of differential homomorphisms – the horizontal ones $\delta : C^{p,q} \to C^{p+1,q}$ and the vertical ones $d : C^{p,q} \to C^{p,q+1}$ – which commute, i.e. $d\delta = \delta d$. In this paper we consider only bicomplexes C for which $C^{p,q} = 0$ if $p < 0$. To such a bicomplex we can associate two single complexes (called the total complexes of the bicomplex)

$$(\mathrm{STot}C)^n = \bigoplus_{p+q=n} C^{p,q} \qquad (\mathrm{PTot}C)^n = \prod_{p+q=n} C^{p,q}$$

In both cases the total differential $D : C^{p,q} \longrightarrow C^{p+1,q} \oplus C^{p,q+1}$ is defined by $D(x) = \delta(x) + (-1)^p d(x)$. Denote the respective cohomologies by

$$H^n(C) = \mathrm{H}^n(\mathrm{STot}C), \qquad \hat{H}^n(C) = \mathrm{H}^n(\mathrm{PTot}C).$$

The fundamental observation is the following: if the horizontal differentials δ are exact it follows that $H^*(C) = 0$; if, however, the vertical differentials d are exact it follows that $\hat{H}^*(C) = 0$. As opposed to the case of bounded bicomplexes the vertical differentials d being exact does not imply that $H^* = 0$ as we can show explicitly:
Counterexample 1: Let P be the bicomplex with $P^{i,-i} = \mathbb{R}$ and $P^{i,-i-1} = \mathbb{R}$ for every $i \geqslant 0$, all other $P^{p,q}$ are 0 and the differentials are the identity morphisms where possible. Then the differentials d are exact, but the element $1 \in P^{0,0}$ gives rise to a non-trivial element in $H^0(P)$.

Similarly we can show that the horizontal differentials being exact does not imply $\hat{H}^* = 0$.
Counterexample 2: Let Q be constructed similarly to P except that $Q^{0,0} = 0$. Then the horizontal differentials are exact, but $\sum_{i\geqslant 0} x_i$, where $x_i = 1 \in Q^{i,-i-1}$ gives rise to a non-trivial element in $\hat{H}^{-1}(Q)$.

3. Spectral sequences

There are two natural filtrations of a bicomplex. The first filtration is

$$_IF_pC = \{C^{i,j}; i \geqslant p\}$$

(completed with zeroes to a bicomplex) giving rise to the first spectral sequence $_IE_r(C)$ with the first differential $d_0 = d$; the second filtration is

$$_{II}F_pC = \{C^{i,j}; j \geqslant p\}$$

(completed with zeroes to a bicomplex) giving rise to the second spectral sequence $_{II}E_r(C)$ with the first differential $d_0 = \delta$.

Counterexample 1 shows that in general the first spectral sequence is not related to $H^*(C)$ for an unbounded bicomplex C. The following theorem expresses the relation of the second spectral sequence to H^*.

THEOREM A. *Let A, B and C be bicomplexes such that $A^{p,q} = B^{p,q} = C^{p,q} = 0$ if $p < 0$. Then the following holds:*

1. *The spectral sequence ${}_{II}E_r(C)$ converges to the associated graded module of $H^*(C)$ in the usual sense: let $F_qH^n(C)$ be the image of $H^n({}_{II}F_qC)$ in $H^n(C)$ under the homomorphism induced by the inclusion, this filtration $F_qH^*(C)$ is such that $\bigcup_q F_qH^*(C) = H^*(C)$ and $\bigcap_q F_qH^*(C) = 0$ and there is an isomorphism*
$$ {}_{II}E_\infty^{n-q,q} \longrightarrow F_qH^n(C)/F_{q+1}H^n(C)\,. $$
2. *If a morphism of bicomplexes $f : A \to B$ induces an isomorphism of ${}_{II}E_\infty(-)$, it induces an isomorphism $f^* : H^*(A) \to H^*(B)$.*

Proof. In each non-trivial cohomology class of $H^n(C)$ we can find a cocycle
$$ \gamma = \sum_{i=0}^{n-q} \gamma^{i,n-i}, \qquad \gamma^{i,n-i} \in C^{i,n-i} $$
such that $n - q$ is minimal, i.e. γ is not cohomologous to any cocycle $\sum_{i=0}^{r} \varphi^{i,n-i}$, $\varphi^{i,n-i} \in C^{i,n-i}$ with $r < n - q$. For such a cocycle γ the element $\gamma^{n-q,q}$ determines a non-trivial element $[\gamma^{n-q,q}] \neq 0 \in E_\infty^{n-q,q}$ (we omit the left index and write only E_r where it is clear which spectral sequence is meant). Thus we can define a homomorphism
$$ \begin{array}{rcl} F_qH^n(C)/F_{q+1}H^n(C) & \longrightarrow & E_\infty^{n-q,q} \\ [\gamma] & \longmapsto & [\gamma^{n-q,q}] \end{array} $$
and check that it is an isomorphism.

For the second claim first prove injectivity of f^*. By the above argument every cohomologically non-trivial 0-cocycle of A is cohomologous to one of the form $\alpha = \sum_{i=0}^{p} \alpha_i$, $\alpha_i \in A^{i,-i}$, such that $[\alpha_p]$ is a non-trivial element of $E_\infty(A)$. Then $f\alpha$ is a 0-cocycle of B and since f_∞ is an isomorphism $f\alpha_p$ is a non-trivial element in $E_\infty(B)$ and so $f\alpha$ cannot be a coboundary.

To show that $f^* : H^*(A) \to H^*(B)$ is onto pick a non-trivial element β of $H^n(B)$ and find its representative $\sum_{i=0}^{p} \beta_i$, $\beta_i \in B^{i,-i}$ such that $[\beta_p]$ is a non-trivial element of $E_\infty(B)$. By assumption there is exactly one $[\alpha_p] \in E_\infty(A)$ mapping to $[\beta_p]$ by f_∞. Therefore there exists an n-cocycle of A of the form $\sum_{i=0}^{p} \alpha_i$. For the n-cocycle $\sum_{i=0}^{p} \beta_i - f\alpha_i$ find a cohomologous cocycle $\sum_{i=0}^{q} \gamma_i$, $\gamma_i \in B^{i,-i}$, of B such that $[\gamma_q]$ is a non-trivial element of $E_\infty(B)$ (note that in that case $q < p$) unless it is null-cohomologous. In finitely many steps like this obtain a cocycle of A which maps to β. □

The case of the first spectral sequence is slightly more complicated. Let $F_p\hat{H}^n(C)$ denote the image of $\hat{H}^n({}_IF_pC)$ in $\hat{H}^n(C)$ under the homomorphism induced by the inclusion. We obtain isomorphisms $F_p\hat{H}^{p+q}(C)/F_{p+1}\hat{H}^{p+q}(C) \to {}_IE_\infty^{p,q}(C)$ again, but $F_p\hat{H}^*(C)$ it may happen that $\bigcap_p F_p\hat{H}^*(C)$ (which can be shown directly to be isomorphic to $\lim^1 H^{*-1}(C/ {}_IF_pC))$ is not trivial. Thus ${}_IE_\infty(C) = 0$ need not imply that $\hat{H}^*(C) = 0$ as we show with a counterexample.

Counterexample 3: First define bicomplexes $C_i = {}_{II}F_iQ$ where Q is as in Counterexample 2 and denote the cocycle $1 \in C_i^{i,-i}$ by c_i. Let C be the bicomplex defined by: $C = \oplus_{i \in \mathbb{N}} C_i$. Then we obtain $E_\infty(C) = 0$, but $\hat{H}^0(C) \neq 0$, since

$$\sum_{i \geqslant 1} c_i \in \mathrm{PTot}C$$

is a cocycle of degree 0, but it is not a coboundary.

However, in some cases there is a simple relationship between the first spectral sequence and $\hat{H}^*$ (while Counterexample 2 shows that the second spectral sequence is in general not related to $\hat{H}^*$).

THEOREM B. *Let A, B and C be bicomplexes such that $A^{p,q} = B^{p,q} = C^{p,q} = 0$ if $p < 0$. Then the following holds:*

1. *If for some r we have ${}_IE_r(C) = {}_IE_\infty(C)$ then the first spectral sequence converges to the associated graded module of $\hat{H}^*(C)$.*
2. *If for some r a morphism of bicomplexes $f : A \to B$ induces an isomorphism of ${}_IE_r(-)$, it induces an isomorphism $f^* : \hat{H}^*(A) \to \hat{H}^*(B)$. Thus, in particular, ${}_IE_r(C) = 0$, for some r, implies $\hat{H}^*(C) = 0$.*
3. *If, however, A, B and C are bicomplexes of finitely generated modules then the first spectral sequence converges to the associated graded module of $\hat{H}^*(C)$ sense and if $f : A \to B$ induces an isomorphism of ${}_IE_\infty(-)$, it induces an isomorphism of $\hat{H}^*(-)$.*

Proof. If ${}_IE_r(C) = {}_IE_\infty(C)$ then we can show that every cocycle which can be moved arbitrarily to the right is a coboundary, i. e. $\bigcap_p F_p\hat{H}^n(C) = 0$. Thus every non-trivial cohomology class of degree n has a representative $\gamma = \sum_{i \geqslant p} \gamma^{i,n-i}$, $\gamma^{i,n-i} \in C^{i,n-i}$ with maximal p. Then we can define a homomorphism

$$\begin{array}{rcl} F_p\hat{H}^n(C)/F_{p+1}\hat{H}^n(C) & \longrightarrow & E_\infty^{p,n-p} \\ [\gamma] & \longmapsto & [\gamma^{p,n-p}] \end{array}$$

and check that it is an isomorphism.

Let $f : A \to B$ induce an isomorphism of ${}_IE_r$. First show that the induced homomorphism f^* is one-to-one: In every non-trivial cohomology class of degree 0 there exists a cocycle $\alpha_1 = \sum_{i=k}^\infty \alpha_{1,i}$, $\alpha_{1,i} \in A^{i,-i}$ such that $0 \neq [\alpha_{1,k}]_r \in E_r(A)$. If $[\alpha_{1,k}]_{r+q} \in \mathrm{Im} d_{r+q}$, the cocycle α_1 is cohomologous to a cocycle $\alpha_2 = \sum_{i=l}^\infty \alpha_{2,i}$ such that $0 \neq [\alpha_{2,l}]_{r+q+1} \in E_{r+q+1}(A)$. If $[\alpha_{2,l}]$ does not survive (as a non-trivial element) to $E_\infty(A)$, proceed as before. Either we eventually obtain a cocycle $\alpha = \sum_{i=m}^\infty \alpha_i$ such that $0 \neq [\alpha_m]_\infty$, or we obtain an infinite sequence of elements $[\alpha_i]_{r(i)} \in E_{r(i)}(A)$ for which there exist $[\gamma_i]_{r(i)} \in E_{r(i)}(A)$ such that $[\alpha_i]_{r(i)} = d_{r(i)}[\gamma_i]_{r(i)}$ and in this case (because of non-triviality of the cohomology class) infinitely many elements γ_i are in the same module $A^{p,q}$. In any case by assumption the cohomology class of $f\alpha$ produces the same picture in the spectral sequence of B and thus also is non-trivial.

To show that f^* is onto pick a non-trivial cohomology class $[\beta] \in \hat{H}^0(B)$. There exists a cocycle $\beta = \sum_{i=p}^{\infty} \beta_i$, $\beta_i \in B^{i,-i}$, such that $0 \neq [\beta_p]_r \in E_r(B)$. By assumption there exists a cocycle $\alpha_1 = \sum_{i=p}^{\infty} \alpha_{1,i}$ such that $[f\alpha_{1,p}]_r = [\beta_p]_r$. If the cocycle $\beta - f\alpha_1$ is not a coboundary, it is cohomologous to a cocycle $\gamma = \sum_{i=q}^{\infty} \gamma_i$ with $q > p$ and $[\gamma_q]_r \neq 0$. For $[\gamma_q]_r$ find a cocycle α_2 similar to α_1 above and proceed with $\beta - f(\alpha_1 + \alpha_2)$. In this way obtain a cocycle $\alpha = \sum_{i=1}^{\infty} \alpha_i$ such that $\beta - f\alpha$ is a coboundary.

For the third claim use the fact that since the module $C^{i,-i-1}$ is finitely generated there are only finitely many r such that for some $\gamma \in C^{i,-i-1}$ we have $d_r[\gamma]_r \neq 0$. Thus we show that $\bigcap_p {}_IF_p\hat{H}^0(C) = 0$. Therefore every non-trivial cohomology class is represented by a non-trivial entry in ${}_IE_\infty$. Using this we show that f induces an isomorphism of $\hat{H}^*(-)$. □

Remark. In case C is a bicomplex of vector spaces Theorem A implies that $H^n(C) \cong \oplus_i {}_{II}E_\infty^{i,n-i}(C)$. If C is a bicomplex of vector spaces such that the first spectral sequence of C stabilizes after finitely many steps or all the vector spaces $C^{p,q}$ are finitely generated, then Theorem B implies $\hat{H}^n(C) \cong \prod_i {}_IE_\infty^{i,n-i}(C)$.

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