Quasi-particles in the principal picture of $\widehat{\mathfrak{sl}}_2$ and Rogers-Ramanujan-type identities

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Rogers-Ramanujan identities are two analytic identities, for $a = 0, 1$,

$$\prod_{m\geq 0} \frac{1}{(1-q^{5m+1+a})(1-q^{5m+4-a})} = \sum_{m\geq 0} \frac{q^{m^2+am}}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Let $a = 0$.

- The coefficient of q^n obtained from the product side is a number of partitions of n with parts congruent $\pm 1 \bmod 5$.
- The nth coefficient of the summand

$$\frac{q^{m^2}}{(q)_m}, \qquad (q)_m = (1-q)(1-q^2)\cdots(1-q^m),$$

is a number of partitions of $n = j_1 + \cdots + j_s$ with parts j_r at most m such that a difference between two consecutive parts is at least two.

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Factor q^{m^2} "represents" the partition

$$(2m-1)+\ldots+3+1=m^2,$$

on which we "add" partitions $\lambda_1+\ldots+\lambda_s$ with parts at most m "counted" by $1/(q)_m$.

For example, q^{16} "represents" the smallest partition

$$7+5+3+1=16$$

satisfying difference condition, on which we "add" partitions with parts at most 4, such as $14=4+3+3+3+1$, to obtain a partition satisfying difference condition

Similar argument applies for a difference $d>2$.

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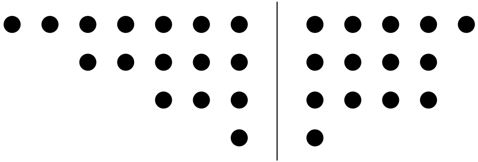

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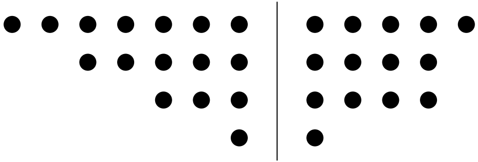

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Analytic Rogers-Ramanujan identities have Gordon-Andrews-Bressoud's generalization ([Go], [A2], [Br1], [Br2])

$$\prod_{\substack{m\geq 1\\ m\not\equiv 0,\pm r \pmod{2l+1}}} (1-q^m)^{-1} = \sum_{n_1,n_2,\ldots,n_{l-1}\geq 0} \frac{q^{N_1^2+N_2^2+\ldots+N_{l-1}^2+N_r+N_{r+1}+\ldots+N_{l-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{l-1}}}$$

where $N_j = n_j + n_{j+1} + \ldots + n_{l-1}$ and $l \geq 2$, $1 \leq r \leq l$.
(We omitted even moduli.)

- Combinatorial Gordon identities: The nth coefficient of the right hand side is is a number of partitions of $n = j_1 + \cdots + j_s$ such that

$$j_1 \geq j_2 \cdots \geq j_s, \qquad j_p \geq 2 + j_{p+l-1}, \qquad j_{s-r+1} > 1.$$

- Combinatorial RR identities for $l = 2$, $a = 0$ for $r = 2$.
- In general we have difference conditions and initial conditions.

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From the Weyl-Kac formula for $\widehat{\mathfrak{sl}}_2$ and $\Lambda = k_0\Lambda_0 + k_1\Lambda_1$ we have ([LM]),

$$\operatorname{ch}_q L(\Lambda) = \begin{cases} F \cdot \prod_{\substack{n\geq 1 \\ n\not\equiv 0, \pm(k_0+1)}} (1-q^n)^{-1} & \text{if } k_0 \neq k_1, \\ F \cdot \prod_{\substack{n\geq 1 \\ n\not\equiv 0, (k_0+1)}} (1-q^n)^{-1} \prod_{\substack{n\geq 1 \\ n\equiv k_0+1}} (1-q^n) & \text{if } k_0 = k_1, \end{cases}$$

where all congruences are modulo $k_0 + k_1 + 2$ and

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Let $\mathfrak{g} = \mathfrak{sl}_2$ with the invariant form $\langle x, y \rangle = \operatorname{tr} xy$ and the standard basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The affine Lie algebra $\widehat{\mathfrak{sl}}_2$ **"in the homogeneous picture"** is

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^1, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is a nonzero central element in $\hat{\mathfrak{g}}$, and for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$,

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle \, m\delta_{m+n,0} c,$$
$$[d, x \otimes t^m] = mx \otimes t^m.$$

The affine Lie algebra $\widehat{\mathfrak{sl}}_2$ **"in the principal picture"** is the subalgebra

$$\hat{\mathfrak{g}} \cong h \otimes \mathbb{C}[t^2, t^{-2}] \oplus \operatorname{span}\{e, f\} \otimes t\mathbb{C}[t^2, t^{-2}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

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Set

$$B(n) = (e + f) \otimes t^n \quad \text{for } n \in 2\mathbb{Z} + 1,$$

$$X(n) = \begin{cases} (f - e) \otimes t^n & \text{if } n \in 2\mathbb{Z} + 1, \\ h \otimes t^n & \text{if } n \in 2\mathbb{Z}. \end{cases}$$

The set

$$\{B(m), X(n), c, d : m \in 2\mathbb{Z} + 1, n \in \mathbb{Z}\}$$

is a basis of $\hat{\mathfrak{g}}$. We have

$$[B(m), B(n)] = m\delta_{m+n,0}c \quad \text{for } m, n \in 2\mathbb{Z} + 1;$$

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span $\{B(m), c : m \in 2\mathbb{Z} + 1\}$ is the principal Heisenberg subalgebra of $\widehat{\mathfrak{sl}}_2$.

The principal Heisenberg subalgebra normalizes $\{X(n) : n \in \mathbb{Z}\}$.

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Let $\Lambda = k_0\Lambda_0$ for $k_0 = 2l + 1$ odd.

Theorem ([LW4])

The set of vectors

$$Z_{j_1}(\beta)\cdots Z_{j_s}(\beta)v_\Lambda, \quad s \geq 0,$$

such that

- $j_1 \leq j_2 \leq \ldots \leq j_s < 0,$
- $j_{m+l} - j_m \geq 2 \quad$ *for all* $m,\ 1 \leq m \leq s - l\,,$

is a basis of $\Omega_{L(\Lambda)}$.

Remark

This basis corresponds with the basis of the standard module $L(\Lambda)$ ([MP]),

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Set

$$X(\zeta) = \sum_{n \in \mathbb{Z}} X(n)\zeta^n,$$

$$B(\zeta) = \sum_{n \in 2\mathbb{Z}+1} B(n)\zeta^n.$$

We have

$$[B(\zeta), B(\xi)] = c \sum_{n \in 2\mathbb{Z}+1} n(\zeta/\xi)^n,$$

$$[B(\zeta), X(\xi)] = 2X(\xi) \sum_{n \in 2\mathbb{Z}+1} (\zeta/\xi)^n,$$

$$[X(\zeta), X(\xi)] = -2B(\xi)\delta(-\zeta/\xi) + c(D\delta)(-\zeta/\xi).$$

Note that $(1+\zeta/\xi)\delta(-\zeta/\xi) = (1+\zeta/\xi)\sum_{n\in\mathbb{Z}}(-\zeta/\xi)^n = 0.$

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Since

$$(\zeta\xi)^{-1}(\zeta+\xi)^2 X(\zeta)X(\xi) = (\zeta\xi)^{-1}(\zeta+\xi)^2 X(\xi)X(\zeta),$$

we can define for a positive integer p

$$X^{(p)}(\zeta) := \lim_{\zeta_i\to\zeta} \prod_{1\leq i<j\leq p} (\zeta_i\zeta_j)^{-1}(\zeta_i+\zeta_j)^2 X(\zeta_1)\cdots X(\zeta_p).$$

The **quasi-particle** $X^{(p)}(n)$ of degree n and charge p is the coefficient in

$$X^{(p)}(\zeta) = \sum_{n\in\mathbb{Z}} X^{(p)}(n)\zeta^n,$$

Quasi-particles in homogeneous picture [FS], [Ge] are coefficients in

$$x_{p\alpha}(z) = x_\alpha(z)^p = \left(\sum_{n\in\mathbb{Z}} x_\alpha(n)z^{-n-1}\right)^p.$$

$x_\theta(z)^{k+1} = 0$ are defining relations for level k standard modules.

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The **quasi-particle** $X^{(p)}(n)$ of degree n and charge p is the coefficient in

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Quasi-particles in homogeneous picture [FS], [Ge] are coefficients in

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Theorem

The set $\mathcal{B}_{M(\Lambda)}$ of vectors

$$B(i_1)\cdots B(i_r)X^{(p_1)}(j_1)\cdots X^{(p_s)}(j_s)v_\Lambda \tag{V}$$

such that

$$r \geq 0 \quad \text{and odd} \quad i_1 \leq \ldots \leq i_r \leq -1, \tag{V1}$$
$$s \geq 0 \quad \text{and} \quad 1 \leq p_1 \leq \ldots \leq p_s, \tag{V2}$$
$$j_s \leq -p_s, \tag{V3}$$
$$p_l < p_{l+1} \quad \text{implies} \quad j_l \leq -p_l - 2p_l(s-l), \tag{V4}$$
$$p_l = p_{l+1} \quad \text{implies} \quad j_l \leq -2p_l + j_{l+1} \tag{V5}$$

is a basis of the Verma module $M(\Lambda)$.

For spanning we use relations such as

$$X^{(p)}(\zeta)X^{(q)}(\zeta) \sim X^{(p+q)}(\zeta)$$

$$\left(\sum_{i\in\mathbb{Z}} a(i)\zeta^i\right)\left(\sum_{j\in\mathbb{Z}} b(j)\zeta^j\right) \sim \left(\sum_{n\in\mathbb{Z}} c(n)\zeta^n\right)$$

$$c(n) \sim \cdots + a(n-j_0+1)b(j_0-1) + a(n-j_0)b(j_0) + a(n-j_0-1)b(j_0+1) + \ldots$$

to express monomial $a(n-j_0)b(j_0)$, for example when $a=b$ and $n=2j_0$.

- for $p=q=1$ we have difference 2 (V5)
- for $p=q=2$ we have difference 4 (V5)
- for $p=1$, $q=2$ we have “interaction” (V4)

For linear independence we “check” the character.

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Relations for standard modules $L(\Lambda)$

$$E^{\pm}(\zeta) = \sum_{i>0} E^{\pm}(\pm i)\zeta^{\pm i} = \exp\left(2 \sum_{n\in\pm(2\mathbb{N}+1)} B(n)\zeta^{n}/n\right).$$

On level k standard $\hat{\mathfrak{g}}$-module $L(k_0\Lambda_0 + k_1\Lambda_1)$ we have ([LW4], [MP]):

- For $p \geq k+1$ $\qquad X^{(p)}(\zeta) = 0.$
- For $p, q \geq 0$, $p+q=k$,

$$a_p X^{(p)}(\zeta) - (-1)^{k_0} a_q E^{-}(-\zeta)X^{(q)}(-\zeta)E^{+}(-\zeta) = 0,$$

  where $a_r = 2^{-r(r-2)}/r!$.
- There are some "initial" relations $I_p^{\Lambda}(n)v_{\Lambda} = 0$ for $n > -n_{\Lambda}(p)$.

Note that for $k=3$ we can express $X^{(2)}(\zeta)$ in terms of $X^{(1)}(-\zeta)$.
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$$1 \leq p_1 \leq \ldots \leq p_s \leq k/2 \qquad \& \qquad \text{initial conditions hold}$$

give a spanning set of $L(\Lambda)$. By using analytic identities we see this is a basis.

For $L((l+1)\Lambda_0 + l\Lambda_1)$ we can prove linear indepenence of monomial vectors (V) directly. Hence

Corollary

$$\prod_{\substack{n\geq 1 \\ n\not\equiv 0,\pm(l+2) (mod\ 2l+3)}} (1-q^n)^{-1} = \sum_{n_1,n_2,\ldots,n_l\geq 0} \frac{q^{N_1^2+N_2^2+\ldots+N_l^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_l}}.$$

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Questions:

- Can one prove "directly" all Gordon-Andrews-Bressoud identites?
- Can one extend the construction to $\widehat{\mathfrak{sl}}_n$?
- What is the proper VOA setting for this construction? We have

$$X^{(p)}(z^{-1/2}) = 2^{2(p-1)} z^p \, x(z)_{-1} \ldots x(z)_{-1} x(z)$$

where $x(z)$ is a field for twisted representaion of VOA for $\widehat{\mathfrak{sl}}_2$. But then, where "lives"

$$a_p X^{(p)}(\zeta) - (-1)^{k_0} a_q E^-(-\zeta) X^{(q)}(-\zeta) E^+(-\zeta) \, ?$$

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Thank you for your attention.

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