Tube formulas for relative fractal drums in Euclidean spaces via Lapidus zeta functions

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Abstract

Relative fractal drums generalize the notion of fractal sets in Euclidean spaces of arbitrary dimension. We establish pointwise and distributional fractal tube formulas for a large class of relative fractal drums. These fractal tube formulas are expressed as sums of residues of suitable meromorphic functions over the complex dimensions of the relative fractal drum under consideration (i.e., over the poles of its Lapidus zeta function which generalizes the well-known zeta function for fractal strings). These results generalize to higher dimensions the corresponding ones previously obtained for fractal strings by M. L. Lapidus and M. van Frankenhuijsen. We illustrate our results by several examples and apply them to obtain a new Minkowski measurability criterion.

Definitions and preliminaries

The notion of a relative fractal drum (A,Ω) **(in short RFD):**

- $\emptyset\neq A\subset\mathbb{R}^N$
- δ-neighbourhood of A: $A_\delta=\{x\in\mathbb{R}^N : d(x,A)<\delta\}$
- $d(x,A)$ denotes the Euclidean distance from x to A
- $\Omega\subset\mathbb{R}^N$, $|\Omega|<\infty$, $\exists\delta>0$, such that $\Omega\subseteq A_\delta$, $\quad r\in\mathbb{R}$
- **lower and upper** r**-dimensional Minkowski content of** (A,Ω):

$$\mathcal{M}_*^r(A,\Omega):=\liminf_{\delta\to0^+}\frac{|A_\delta\cap\Omega|}{\delta^{N-r}};\qquad \mathcal{M}^{*r}(A,\Omega):=\limsup_{\delta\to0^+}\frac{|A_\delta\cap\Omega|}{\delta^{N-r}}$$

Relative Minkowski (or box) dimension

- **lower and upper box dimension of** (A,Ω):

$$\underline{\dim}_B(A,\Omega)=\inf\{r\in\mathbb{R}\ :\ \mathcal{M}_*^r(A,\Omega)=0\}$$
$$\overline{\dim}_B(A,\Omega)=\inf\{r\in\mathbb{R}\ :\ \mathcal{M}^{*r}(A,\Omega)=0\}$$
$$\underline{\dim}_B(A,\Omega)=\overline{\dim}_B(A,\Omega)\Rightarrow\exists\dim_B(A,\Omega)$$

- if $\exists D\in\mathbb{R}$ such that

$$0<\mathcal{M}_*^D(A,\Omega)=\mathcal{M}^{*D}(A,\Omega)<\infty,$$

then we define (A,Ω) to be **Minkowski measurable** $\Rightarrow$ $D=\dim_B(A,\Omega)$

The relative distance zeta function [LapRaŽu1]

- generalization of M. L. Lapidus' definition of a zeta function associated to bounded (fractal) sets (Catania 2009)
- (A,Ω) RFD in $\mathbb{R}^N$, $|\Omega|<\infty$, $s\in\mathbb{C}$ and fix $\delta>0$
- the **Lapidus (or distance) zeta function of** (A,Ω):

$$\zeta_A(s,\Omega;\delta):=\int_{A_\delta\cap\Omega}d(x,A)^{s-N}\,dx \tag{1}$$

Theorem 1 (Holomorphicity theorem from [LapRaŽu1])
Let (A,Ω) be an RFD in $\mathbb{R}^N$, then
(a) $\zeta_A(s,\Omega)$ is holomorphic on $\{\mathrm{Re}\,s>\overline{\dim}_B(A,\Omega)\}$, and

$$\zeta_A'(s,\Omega)=\int_{A_\delta\cap\Omega}d(x,A)^{s-N}\log d(x,A)\,dx$$

(b) $\mathbb{R}\ni s<\overline{\dim}_B(A,\Omega)\Rightarrow$ the integral (1) defining $\zeta_A(s,\Omega)$ diverges
(c) $(\exists D=\dim_B(A,\Omega)<N)(\mathcal{M}_*^D(A,\Omega)>0)\Rightarrow$
$\zeta_A(s,\Omega)\to+\infty$ when $\mathbb{R}\ni s\to D^+$

The relative tube zeta function [LapRaŽu1]

Let (A,Ω) an RFD in $\mathbb{R}^N$ and fix $\delta>0$.

- the **tube zeta function of** (A,Ω):

$$\widetilde{\zeta}_A(s,\Omega;\delta):=\int_0^\delta t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t \tag{2}$$

- the exact analog of the the holomorphicity theorem holds for $\widetilde{\zeta}_A(s,\Omega;\delta)$
- the following functional equation connects the two zeta functions:

$$\zeta_A(s,\Omega;\delta)=\delta^{s-N}|A_\delta\cap\Omega|+(N-s)\widetilde{\zeta}_A(s,\Omega;\delta) \tag{3}$$

Fractal tube formulas for RFDs

- **the goal**: derive an asymptotic formula for the relative tube function $t\mapsto|A_t\cap\Omega|$ as $t\to0^+$ directly from the distance zeta function $\zeta_A(\,\cdot\,,\Omega)$ of (A,Ω)
- more precisely, express $|A_t\cap\Omega|$ as a sum of residues over the **complex dimensions** of (A,Ω) (the poles of $\zeta_A(\,\cdot\,,\Omega)$)
- apply this to derive a **Minkowski measurability criterion** for a large class of RFDs
- we observe that the tube zeta function can be expressed as

$$\widetilde{\zeta}_A(s,\Omega;\delta)=\int_0^{+\infty}t^{s-1}\left(\chi_{(0,\delta)}(t)t^{-N}|A_t\cap\Omega|\right)\,\mathrm{d}t \tag{4}$$

- **Mellin inversion theorem** $\Rightarrow$

Theorem 2 (The integral representation [Ra])
Let (A,Ω) be an RFD in $\mathbb{R}^N$ and fix $\delta>0$.
Then, for every $t\in(0,\delta)$ and any $c>\overline{\dim}_B(A,\Omega)$, we have

$$|A_t\cap\Omega|=\frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}t^{N-s}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s. \tag{5}$$

- goal: express (5) as a sum over the residues of $\widetilde{\zeta}_A(\,\cdot\,,\Omega)$ or, by using the functional equation (3), of $\zeta_A(\,\cdot\,,\Omega)$

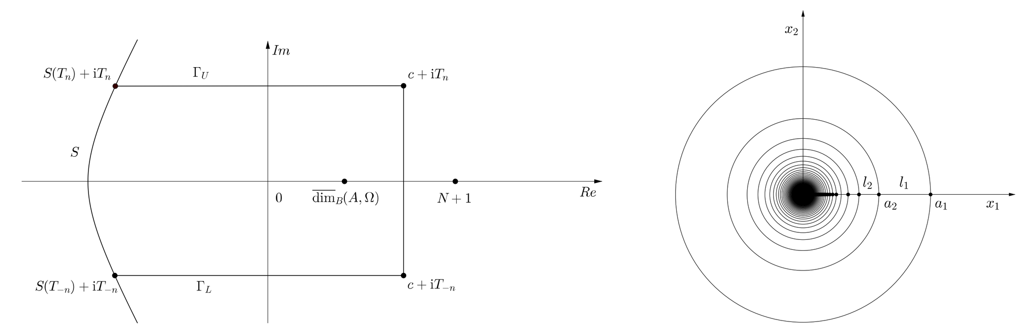

Figure 1. Left: The screen and the window. By using the residue theorem we express $|A_t\cap\Omega|$ as a sum over the complex dimensions of (A,Ω); that is over the poles of $\widetilde{\zeta}_A(s,\Omega;\delta)$ or of $\zeta_A(s,\Omega;\delta)$. Right: The fractal nest generated by the a-string.

Definition 3 (The screen, the window and admissibility of RFDs; adapted from [Lap–vFr])

the screen: $\quad S:=\{S(\tau)+\mathbb{i}\tau\ :\ \tau\in\mathbb{R}\}$

$S(\tau)$ is bounded, real-valued, Lipschitz continuous:

$$|S(x)-S(y)|\leq\|S\|_{\mathrm{Lip}}|x-y|,\qquad\text{for all }x,y,\in\mathbb{R}$$

$$\inf S:=\inf_{\tau\in\mathbb{R}}S(\tau)\qquad\text{and}\qquad\sup S:=\sup_{\tau\in\mathbb{R}}S(\tau)$$

the window: $\quad W:=\{s\in\mathbb{C}:\mathrm{Re}\,s\geq S(\mathrm{Im}\,s)\}$

- (A,Ω) is **admissible** if its relative tube (or distance) zeta function can be meromorphically extended to an open connected neighborhood of some window W

Definition 4 (d-Languidity; adapted from [Lap–vFr])
An admissible (A,Ω) is d-languid if for some $\delta>0$, $\zeta_A(\,\cdot\,,\Omega;\delta)$ satisfies: $(\exists\kappa_d\in\mathbb{R})$, $(\exists C>0)$, $\exists(T_n)_{n\in\mathbb{Z}}$ such that $T_{-n}<0<T_n$ for $n\geq1$ and $\lim_{n\to\pm\infty}|T_n|=+\infty$ satisfying

L1 For all $n\in\mathbb{Z}$ and all $\sigma\in(S(T_n),c)$,

$$|\zeta_A(\sigma+\mathbb{i}T_n,\Omega;\delta)|\leq C(|T_n|+1)^{\kappa_d},$$

where $c>\overline{\dim}_B(A,\Omega)$ is some constant.

L2 For all $\tau\in\mathbb{R}$, $|\tau|\geq1$,

$$|\zeta_A(S(\tau)+\mathbb{i}\tau,\Omega;\delta)|\leq C|\tau|^{\kappa_d}.$$

Definition 5 (Strong d-languidity; adapted from [Lap–vFr])
(A,Ω) is strongly d-languid if **L1** is satisfied for all $\sigma\in(-\infty,c)$ and, additionally, $\exists(S_m)_{m\geq1}$ such that
$\sup S_m\to-\infty$ and $\sup_{m\geq1}\|S_m\|_{\mathrm{Lip}}<\infty$ and

L2' there exist $B,C>0$ such that for all $\tau\in\mathbb{R}$ and $m\geq1$,

$$|\zeta_A(S_m(\tau)+\mathbb{i}\tau,\Omega;\delta)|\leq CB^{|S_m(\tau)|}(|\tau|+1)^{\kappa_d}.$$

Definition 6 (Complex dimensions of an RFD).
Assume that (A,Ω) is admissible for some window W.

- visible complex dimensions of (A,Ω) (with respect to W):

$$\mathcal{P}(\zeta_A(\,\cdot\,,\Omega;\delta),W):=\{\omega\in W:\omega\text{ is a pole of }\zeta_A(\,\cdot\,,\Omega;\delta)\}.$$

- $(W=\mathbb{C})\Rightarrow$ the set of (all) complex dimensions of (A,Ω).
- the set of principal complex dimensions of (A,Ω):

$$\dim_{PC}(A,\Omega):=\{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega;\delta),W):\mathrm{Re}\,\omega=\overline{\dim}_B(A,\Omega)\}.$$

The pointwise fractal tube formula

- let $V^{[k]}_{(A,\Omega)}(t)$ be the k-th primitve function of $|A_t\cap\Omega|$
- for $k\in\mathbb{N}$ we let $(s)_0:=1\quad(s)_k:=s(s+1)\cdots(s+k-1)$
- for $k\in\mathbb{Z}$ we let $(s)_k:=\frac{\Gamma(s+k)}{\Gamma(s)}$

Theorem 7 (Pointwise formula with error term [Ra])

- let (A,Ω) be d-languid for some κ_d and let $\overline{\dim}_B(A,\Omega)<N$
- let $k>\kappa_d$ be a nonnegative integer

Then, for every $t\in(0,\delta)$ we have

$$V^{[k]}_{(A,\Omega)}(t)=\sum_{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)}\operatorname{res}\left(\frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega;\delta),\omega\right)+R^{[k]}(t). \tag{6}$$

The error term $R^{[k]}$ is given by the absolutely convergent integral

$$R^{[k]}(t)=\frac{1}{2\pi\mathbb{i}}\int_S\frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega;\delta)\,\mathrm{d}s.$$

We have the following pointwise error estimate:

$$R^{[k]}(t)=O(t^{N-\sup S+k})\quad\text{as}\quad t\to0^+.v$$

Moreover, if $(\forall\tau\in\mathbb{R})(S(\tau)<\sup S)$, then

$$R^{[k]}(t)=o(t^{N-\sup S+k})\quad\text{as}\quad t\to0^+.$$

Theorem 8 (Exact pointwise tube formula [Ra])

- let (A,Ω) be strongly d-languid for some $\delta>0$ and $\kappa_d\in\mathbb{R}$
- let $k>\kappa_d-1$ be a nonnegative integer and $\overline{\dim}_B(A,\Omega)<N$

Then, for every $t\in(0,\min\{1,\delta,B^{-1}\})$ the fractal tube formula (6) is an exact tube formula, i.e., $R^{[k]}(t)\equiv0$ and $W=\mathbb{C}$. Here, B is the constant appearing in **L2'**.

When can we apply the fractal tube formula at level $k=0$?

- tube formula with error term: **if** $\kappa_d<0$
- exact tube formula: **if** $\kappa_d<1$

The distributional fractal tube formula

- by removing the restriction on κ_d we derive a tube formula only in the sense of **Schwartz's distributions**
- exact analogs of the the tube formula with and without the error term hold distributionally for **any** exponent $\kappa_d\in\mathbb{R}$ and **any** $k\in\mathbb{Z}$; for instance, at level $k=0$ we obtain:

$$|A_t\cap\Omega|=\sum_{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)}\operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_A(s,\Omega),\omega\right)+R^{[0]}(t)$$

The Minkowski measurability criterion

Theorem 9 (The Minkowski measurability criterion [Ra])

- let (A,Ω) be such that $\exists D:=\dim_B(A,\Omega)$ and $D<N$
- assume that (A,Ω) is d-languid for a screen passing strictly between the critical line $\{\mathrm{Re}\,s=D\}$ and all the complex dimensions of (A,Ω) with real part strictly less than D

Then, the following is equivalent:
(a) (A,Ω) is Minkowski measurable.
(b) D is the only pole of $\zeta_A(\,\cdot\,,\Omega)$ located on the critical line $\{\mathrm{Re}\,s=D\}$ and it is simple.
Furthermore, we then have

$$\mathcal{M}^D(A,\Omega)=\frac{\operatorname{res}(\zeta_A(\,\cdot\,,\Omega),D)}{N-D}.$$

- $(a)\Rightarrow(b)$: follows from the distributional tube formula and the **Uniqueness theorem for almost periodic distributions** due to **Schwartz**
- $(b)\Rightarrow(a)$: is a consequence of a **Tauberian theorem** due to **Wiener** and **Pitt** (conditions for this direction to hold can be considerably weakened)
- the assumption $D<N$ can be removed by appropriately embedding the RFD in $\mathbb{R}^{N+1}$

Examples

The Sierpiński gasket

$$\zeta_A(s;\delta)=\frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)}+2\pi\frac{\delta^s}{s}+3\frac{\delta^{s-1}}{s-1}$$

$$\mathcal{P}(\zeta_A)=\{0,1\}\cup\left(\log_2 3+\frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right)$$

By letting $\omega_k:=\log_2 3+\mathbf{p}k\mathbb{i}$ and $\mathbf{p}:=2\pi/\log 2$ we have that

$$|A_t|=\sum_{\omega\in\mathcal{P}(\zeta_A)}\operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s;\delta),\omega\right)$$
$$=t^{2-\log_2 3}\frac{6\sqrt{3}}{\log 2}\sum_{k=-\infty}^{+\infty}\frac{(4\sqrt{3})^{-\omega_k}t^{-\mathbf{p}k\mathbb{i}}}{(2-\omega_k)(\omega_k-1)\omega_k}+\left(\frac{3\sqrt{3}}{2}+\pi\right)t^2,$$

valid pointwise for all $t\in(0,1/2\sqrt{3})$.

The fractal nest generated by the a**-string** (See Figure 1, right.)
$a>0$, $a_j:=j^{-a}$, $l_j:=j^{-a}-(j+1)^{-a}$, $\Omega:=B_{a_1}(0)$

$$\zeta_{A_a}(s;\Omega)=\frac{2^{2-s}\pi}{s-1}\sum_{j=1}^{\infty}l_j^{s-1}(a_j+a_{j+1})$$

$$\mathcal{P}(\zeta_{A_a}(\,\cdot\,,\Omega))\subseteq\{1,2/(a+1),1/(a+1)\}\cup\{-m/(a+1):m\in\mathbb{N}\}$$

$a\neq1$, $D:=2/(1+a)\Rightarrow$

$$|(A_a)_t\cap\Omega|=\frac{2^{2-D}D\pi a^{D-1}t^{2-D}}{(2-D)(D-1)}+\big(4\pi\zeta(a)-2\pi\big)t+O\big(t^{2-\frac{1}{a+1}}\big),$$

$$a=1\Rightarrow|(A_1)_t\cap\Omega|=\operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_{A_1}(s,\Omega),1\right)+o(t)$$
$$=2\pi t\log t^{-1}+\mathrm{const}\cdot t+o(t)\quad\text{as }t\to0^+$$

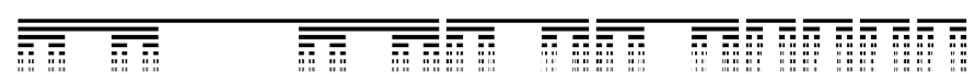

Figure 2. The Cantor set of second order. Only the first three iterations are shown here. More precisely, from left to right we have the middle-third Cantor set C in $[0,1]$, then two copies of C scaled by $1/3$, and then four copies of C scaled by $1/9$.

The second order Cantor set (See Figure 2.)
Let C be the standard middle-third Cantor set in $[0,1]$, $\Omega:=(0,1)$. Define the generator $(G,\Omega):=(\Omega\setminus C,\Omega)$ and let $r_1=r_2=1/3$ be the scaling ratios of the self-similar RFD (C_2,Ω_2).

$$\zeta_{C_2}(s,\Omega_2)=\frac{3^s\,\zeta_C(s,\Omega)}{3^s-2}=\frac{3^s}{2^{s-1}s(3^s-2)^2}$$

$$\mathcal{P}(\zeta_{C_2}(\,\cdot\,\Omega_2))=\{0\}\cup\left(\log_3 2+\frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right)$$

$$|(C_2)_t\cap\Omega_2|=t^{1-\log_3 2}\left(\log t^{-1}G(\log t^{-1})+H(\log t^{-1})\right)+2t$$

where $G,H:\mathbb{R}\to\mathbb{R}$ are nonconstant, periodic with $T=\log 3$.

- in general, a complex dimension ω of order m generates terms of type $t^{N-\omega}(\log t^{-1})^{k-1}$ for $k=1,\dots,m$ in $|A_t\cap\Omega|$

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