Lapidus zeta functions of fractal sets and their residues

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Aims

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- If $\underline{\dim}_B A = \overline{\dim}_B A$ we write $\dim_B A$, box dimension of A.

Minkowski measurable and nondegenerate sets

- If there is $D \geq 0$ with

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The Lapidus zeta function of A (or distance z.f.) is defined by

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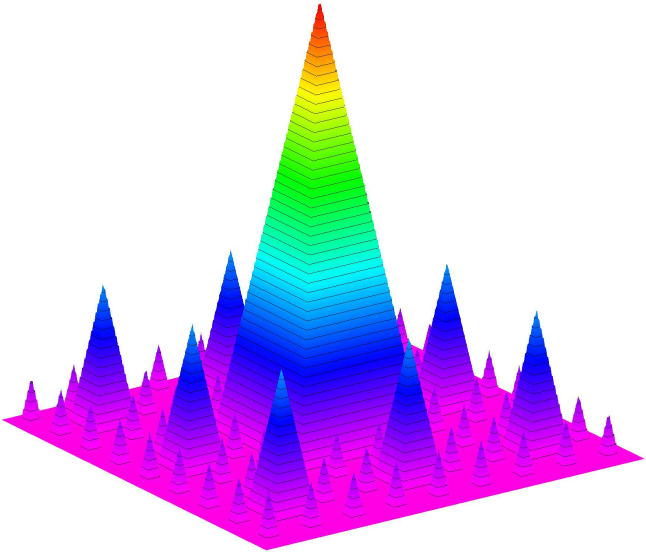
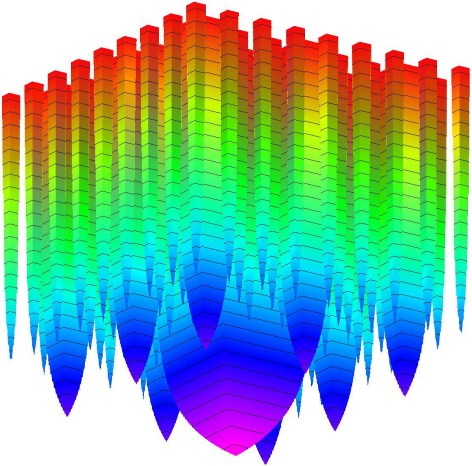
- $\delta < \delta_1$ implies that $\zeta_A(s;\delta_1) - \zeta_A(s) = \int_{A_{\delta,\delta_1}} d(x,A)^{s-N}\,\mathrm{d}x$ is entire function

Graph of Sierpiński carpet distance function $x \mapsto d(x, A)$



Graph of the function $x \mapsto d(x, A)^{s-N}$ for $s < N$



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- (*scaling property*) *If $\lambda > 0$, then*

$$\zeta_{\lambda A}(s; \lambda\delta) = \lambda^s \cdot \zeta_A(s; \delta)$$

  for all $s \in \mathbb{C}$ such that $\mathrm{Re}\, s > \overline{\dim}_B A$.

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Theorem (Harvey & Polking 1970)

Assume that A is a bounded set in $\mathbb{R}^N$ and $\delta > 0$ is given. Then

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- If $D := \dim_B A$ exists, and $\mathcal{M}_*^D(A) > 0$, then the converse also holds (D.Ž., ISAAC Proc. 2009). The Minkowski content condition is essential (D.Ž., RAE 2005)

Complex dimensions of a fractal set A

- Let A be such that ζ_A can be meromorphically extended to an open right half-plane W (window) containing the *critical line* $\{\operatorname{Re} s = D(\zeta_A)\}$

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Definition

The multiset of poles of ζ_A contained in W, is denoted by

$$\mathcal{P}(\zeta_A) = \mathcal{P}(\zeta_A, W).$$

The poles are called complex dimensions of A (depend on W). Complex dimensions contained on the critical line are called principal complex dimensions, and the corresponding multiset is denoted by

$$\dim_{PC} A := \{s \in \mathcal{P}(\zeta_A) : \operatorname{Re} s = D(\zeta_A)\}.$$

It does not depend on W.

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If A is Minkowski nonegenerate, then $s = D$ is a simple pole, and

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Corollary (LRŽ)

If A is Minkowski measurable, i.e., $\mathcal{M}^D(A) \in (0, \infty)$, then

$$\operatorname{res}(\zeta_A, D) = (N - D)\mathcal{M}^D(A).$$

Residue of tube zeta functions at $D := \dim_B A$ ($D \leq N$)

- Tube zeta function associated with the *tube fct.* $t \mapsto |A_t|$:

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In particular, if A *is Minkowski measurable, then*

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- The proof rests on the following identity on $\{\operatorname{Re} s > \overline{D}\}$:

$$\zeta_A(s) = \delta^{s-N}|A_\delta| + (N-s)\tilde{\zeta}_A(s)$$

Minkowski measurable sets

Theorem (LRŽ)

Assume $A \subset \mathbb{R}^N$ and there exist $\alpha > 0$, $\mathcal{M} \in (0, \infty)$ and $D \geq 0$ s.t.

$$|A_t| = t^{N-D}\left(\mathcal{M} + O(t^{\alpha})\right) \quad \textit{as } t \to 0.$$

Then A is Minkowski measurable, $\dim_B A = D$, $\mathcal{M}^D(A) = \mathcal{M}$, $D(\tilde{\zeta}_A) = D$, $\exists!$ meromorphic extension of $\tilde{\zeta}_A(s)$ (at least) to

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The pole $s = D$ is unique, simple, $\operatorname{res}(\tilde{\zeta}_A, D) = \mathcal{M}$.

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Minkowski nonmeasurable sets

Theorem (LRŽ)

Assume $A \subset \mathbb{R}^N$ *and there exist* $D \geq 0$, *a nonconstant periodic fct.* $G : \mathbb{R} \to \mathbb{R}$ *with the minimal period* $T > 0$, *and* $\alpha > 0$, *s.t.*

$$|A_t| = t^{N-D}\left(G(\log t^{-1}) + O(t^{\alpha})\right) \quad \text{as } t \to 0.$$

Then $\dim_B A = D$, $\mathcal{M}_*^D(A) = \min G$, $\mathcal{M}^{*D}(A) = \max G$, $D(\tilde{\zeta}_A) = D$, *and* $\exists!$ *meromorphic extension (at least) to* $\{\mathrm{Re}\, s > D - \alpha\}$. *The set of all of poles is*

$$\mathcal{P}(\tilde{\zeta}_A) = \left\{ s_k = D + \frac{2\pi}{T}\mathrm{i}k : \hat{G}_0\left(\frac{k}{T}\right) \neq 0,\ k \in \mathbb{Z} \right\}$$

they are all simple. Here $\hat{G}_0(s) := \int_0^T e^{-2\pi \mathrm{i} s \cdot t} G(t)\, dt$.

Minkowski nonmeasurable measurable sets

Theorem (...continued)

For all $s_k \in \mathcal{P}(\tilde{\zeta}_A)$, $\operatorname{res}(\tilde{\zeta}_A, s_k) = \frac{1}{T}\hat{G}_0(\frac{k}{T})$. *We have*

$$|\operatorname{res}(\tilde{\zeta}_A, s_k)| \leq \frac{1}{T}\int_0^T G(\tau)\,d\tau, \qquad \lim_{k\to\infty} \operatorname{res}(\tilde{\zeta}_A, s_k) = 0$$

We have $D \in \mathcal{P}(\tilde{\zeta}_A)$,

$$\operatorname{res}(\tilde{\zeta}_A, D) = \frac{1}{T}\int_0^T G(\tau)\,d\tau$$

$$\mathcal{M}_*^D(A) < \operatorname{res}(\tilde{\zeta}_A, D) < \mathcal{M}^{*D}(A).$$

Examples: ternary Cantor set $C^{(2,1/3)}$, generalized Cantor sets $C^{(m,a)}$ $(ma < 1)$

Relative fractal drums

- Relative fractal drum (RFD) is a pair (A, Ω) of nonempty subsets A and Ω (open) of $\mathbb{R}^N$, s.t. $|\Omega| < \infty$ and $\exists \delta > 0$ s.t. $\Omega \subset A_\delta$. (A and Ω may be unbdd.)

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  (here, $-\infty$ can be achieved)
- each bdd set A can be identified with an RFD (A, A_δ), for any $\delta > 0$

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Definition (Relative distance zeta function, LRŽ)

Distance zeta function of the RFD (A,Ω) (or relative distance z.f.) is defined by

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- flatness condition on an RFD: $\overline{\dim}_B(A,\Omega) < 0$
- let (A,Ω) be a fixed RFD

Definition (Relative distance zeta function, LRŽ)

Distance zeta function of the RFD (A,Ω) (or relative distance z.f.) is defined by

$$\zeta_{A,\Omega}(s) := \int_\Omega d(x,A)^{s-N}\,\mathrm{d}x$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ large enough.

- If A is bdd, then $\zeta_A = \zeta_{A,A_\delta}$

Analyticity of relative zeta functions

Theorem (LRŽ)

- *the abscissa of (absolute) convergence is* $D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega)$*; in particular,* $\zeta_{A,\Omega}$ *is holomorphic on* $\{\operatorname{Re} s > \overline{\dim}_B(A,\Omega)\}$*;*

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- *assuming that* $D = \dim_B(A,\Omega)$ *exists and* $\mathcal{M}_*^D(A,\Omega) > 0$, *if* $s \in \mathbb{R}$ *and* $s \to D^+$, *then* $\zeta_{A,\Omega}(s) \to \infty$;

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- *assuming that* $D = \dim_B(A,\Omega)$ *exists and* $\mathcal{M}_*^D(A,\Omega) > 0$*, if* $s \in \mathbb{R}$ *and* $s \to D^+$*, then* $\zeta_{A,\Omega}(s) \to \infty$*;*
- (*scaling property*) *for any* $\lambda > 0$*,*

$$\zeta_{\lambda A,\lambda\Omega}(s) = \lambda^s \zeta_{A,\Omega}(s)$$

for all $s \in \mathbb{C}$ *with* $\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)$*.*

Meromorphic extensions of relative zeta functions

Theorem (LRŽ; gauge functions; Mink. meas. case)

Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ s.t.

$$|A_t \cap \Omega| = t^{N-D}(\log t^{-1})^{m-1}(\mathcal{M} + O(t^\alpha)) \quad \text{as } t \to 0,$$

where $m \in \mathbb{N}$, $D \in (-\infty, N]$. Then $D(\tilde{\zeta}_{A,\Omega}) = D$, and $\tilde{\zeta}_{A,\Omega}$ has a unique meromorphic extension to $\{\operatorname{Re} s > D - \alpha\}$.
$s = D$ is the unique pole, of order m. If $m = 1$, then
$\operatorname{res}(\tilde{\zeta}_{A,\Omega}, D) = \mathcal{M}$.

Converse of the previous theorem

Theorem (LRŽ; gauge functions; Mink. meas. case; converse)

Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ s.t. $\tilde{\zeta}_{A,\Omega}$ is languid, $m \in \mathbb{N}$, $D \in (-\infty, N]$. Let $D(\tilde{\zeta}_{A,\Omega}) = D$, and $\tilde{\zeta}_{A,\Omega}$ has a meromorphic extension to $\{\operatorname{Re} s > D - \alpha\}$, and $s = D$ is the unique pole, of order m. Then

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Furthermore, the supremum of all α satisfying $(*)$ is

$$\sup \alpha = D - \sup\left\{ \operatorname{Re} s : s \in \mathcal{P}(\tilde{\zeta}_{A,\Omega}) \setminus \{D\} \right\}.$$
$(*)$

Meromorphic extensions of relative zeta functions

Theorem (LRŽ; gauge functions; Mink. nonmeas. case)

Let (A,Ω) *be a relative fractal drum in* $\mathbb{R}^N$, *s.t.* $\exists$ $D \geq 0$, *a nonconstant periodic fct.* $G:\mathbb{R}\to\mathbb{R}$ *with the min. period* $T>0$, $m\in\mathbb{N}$, $D\in(-\infty,N]$, $\alpha>0$, *satisfying*

$$|A_t\cap\Omega| = t^{N-D}(\log t^{-1})^{m-1}\left(G(\log t^{-1})+O(t^{\alpha})\right) \quad \text{as } t\to 0$$

Then $\dim_B(A,\Omega)=D$, $D(\tilde{\zeta}_{A,\Omega})=D$, *and* $\tilde{\zeta}_{A,\Omega}$ *has a unique meromorphic extension* (*at least*) *to* $\{\operatorname{Re} s > D-\alpha\}$. *All of its poles are of order* m, *and*

$$\mathcal{P}(\tilde{\zeta}_{A,\Omega}) = \left\{ s_k = D + \frac{2\pi}{T}\mathrm{i}k \in \mathbb{C} : \hat{G}_0\Big(\frac{k}{T}\Big) \neq 0,\ k\in\mathbb{Z} \right\}$$

Also, $s_0 = D \in \mathcal{P}(\tilde{\zeta}_{A,\Omega})$.

Meromorphic extensions of relative zeta functions (continued)

Theorem (LRŽ; gauge functions; Mink. meas. case; cntnd.)

For $s_0 = D$ (i.e., $k = 0$) we have

$$c_{-m}^{(0)} = \frac{(m-1)!}{T} \int_0^T G(\tau)\, d\tau$$

Defining the h-Minkowski content by $\mathcal{M}^{*r}(A, \Omega, h) := \overline{\lim}_{t \to 0^+} \frac{|A_t \cap \Omega|}{h(t) t^{N-r}}$, where $h(t) := (\log t^{-1})^{m-1}$ is the gauge fct., and similarly $\mathcal{M}_*^r(A, \Omega, h)$, we have

$$(m-1)!\mathcal{M}_*^D(A, \Omega, h) < c_{-m}^{(0)} < (m-1)!\mathcal{M}^{*D}(A, \Omega, h).$$

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For $m = 1$ we have $\operatorname{res}(\zeta_{A,\Omega}, D) = \frac{1}{T}\int_0^T G(\tau)\, d\tau$ *and*

$$\mathcal{M}_*^D(A, \Omega) < \operatorname{res}(\zeta_{A,\Omega}, D) < \mathcal{M}^{*D}(A, \Omega).$$

Tensor products of fractal strings

A bounded fractal string $\mathcal{L}$ is any nondecreasing sequence $(\ell_j)_{j\geq 1}$ of positive numbers, such that $\sum_{j=1}^{\infty} \ell_j < \infty$. It can be identified with the set

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Let $\mathcal{L} = (\ell_j)_{j\geq 1}$ and $\mathcal{M} = (m_k)_{k\geq 1}$ be two bdd fractal strings.

Their tensor product is the multiset

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Using iterated tensor products of fractal strings, it is possible to

construct a subset $A \subset [0,1]$ with arbitrarily high multiplicities of complex dimensions, and even with essential singularities of ζ_A.

m-Cantor string

It is easy to see that

$$\zeta_{\mathcal{L}\otimes\mathcal{M}}(s) = \zeta_{\mathcal{L}}(s) \cdot \zeta_{\mathcal{M}}(s)$$

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Let $\mathcal{L}_{CS}$ be the Cantor string, $m \geq 2$, and define its m-fold tensor product (or m-Cantor string) by

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All of its complex dimensions (i.e., poles of $\zeta_{\mathcal{L}_{CS}^{m\otimes}}$),

$$\dim_{PC} \mathcal{L}_{CS}^{m\otimes} = \log_3 2 + \frac{2\pi}{\log 3}\mathrm{i}\mathbb{Z},$$

are of multiplicity m, since

$$\zeta_{\mathcal{L}_{CS}^{m\otimes}}(s) = \big(\zeta_{\mathcal{L}_{CS}}(s)\big)^m = (1 - 2\cdot 3^{-s})^{-m}$$

∞-Cantor string

Taking a disjoint union of scaled copies of $\mathcal{L}_{CS}^{m\otimes}$,

$$\mathcal{L}_{CS}^{\infty} := \bigsqcup_{m=2}^{\infty} \frac{3^{-m}}{m!} \mathcal{L}_{CS}^{m\otimes},$$

$\log_3 2 + \frac{2\pi}{\log 3}\mathrm{i}\mathbb{Z}$ becomes the set of essential singularities of $\zeta_{\mathcal{L}_{CS}^{\infty}}$.

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The proof is based on

$$\begin{aligned}\zeta_{\mathcal{L}_{CS}^{\infty}}(s) &= \sum_{m=2}^{\infty} \frac{3^{-ms}}{(m!)^s} \left(\zeta_{\mathcal{L}_{CS}}(s)\right)^m \\ &= \sum_{m=2}^{\infty} \frac{3^{-ms}}{(m!)^s} \cdot \frac{1}{(1-2\cdot 3^{-s})^m},\end{aligned}$$

which is holomrphic on $\{\operatorname{Re} s > 0\} \setminus \left(\log_3 2 + \frac{2\pi}{\log 3}\mathrm{i}\mathbb{Z}\right)$.

n-Quasiperiodic sets

Definition

$A \subset \mathbb{R}^N$ is said to be 2-quasiperiodic set if $|A_t| = t^{N-D}(G(\log 1/t) + O(t^\alpha))$ as $t \to 0$, for some $D \geq 0$, $\alpha > 0$, and $G(\tau)$ is a 2-quasiperiodic function, that is, $G(\tau) = G_1(\tau) + G_2(\tau)$ and G_j are T_j-periodic, where T_1/T_2 is irrational.

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Example. Using suitably chosen generalized Cantor sets $C^{(m_1,a_1)}$ and $C^{(m_2,a_2)}$, it is possible to achieve that for their (disjoint) union A, T_1/T_2 is even transcendental. We use Gel'fond–Schneider's theorem from number theory, 1934.
We say that A is transcendentally 2-quasiperiodic set.

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It is possible to construct transcendentally n-quasiperiodic sets for any $n \geq 2$, and even for $n = \infty$. We use Baker's theorem from number theory.

Fractal zeta functions

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Fractal zeta functions

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