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Fractal analysis of bifurcations of dynamical systems

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Theoretical and computational methods in dynamical systems and fractal geometry
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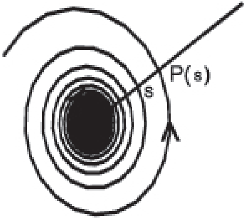
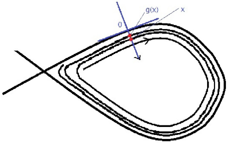
Motivation

- A natural idea is that "density" of orbit is related to quantity and quality of objects which could be produced by perturbation of the system.
- Classical fractal analysis associates box dimension and Minkowski content to measurable sets, which in some sense measures the "density" of a set.
- We study continuous systems by
  -spiral trajectories near focus, limit cycle and a polycycle
  -discrete system generated by Poincaré map
  -discrete system generated by unit-time map
- We also study oscillatory integrals and singularities of maps, by curve defined parametrically by the oscillatory integral

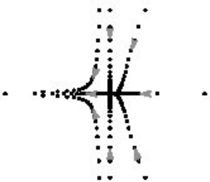
Examples of orbits



weak focus

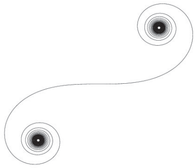


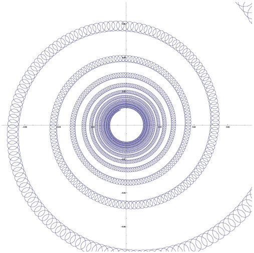
saddle-loop

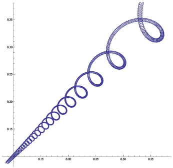


saddle-node

Examples of curves defined by oscillatory integrals $\int_{\mathbb{R}^n} e^{i\tau f(\mathbf{x})}\Phi(\mathbf{x})\, d\mathbf{x}$







clothoid, $f(x) = x^2$ **discontinuous amplitude,** $f(0) \neq 0$ **and** $f(0) = 0$

Definition of upper Minkowski content and upper box dimension

- *upper s-dimensional Minkowski content of the bounded set $A \in \mathbb{R}^N$*, $0 \leq s \leq N$: $\mathcal{M}^{*s}(A) = \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon(A)|}{\varepsilon^{N-s}}$

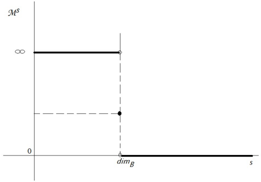


Figure: Minkowski content $\mathcal{M}^{*s}$ as function of $s \in [0, N]$

- *upper box dimension*: $\overline{\dim}_B A = \inf\{s \geq 0 \mid \mathcal{M}^{*s}(A) = 0\}$.

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- $\mathcal{M}^{*s}, \mathcal{M}_*^s \neq 0, \infty \Rightarrow |A_\varepsilon(A)| \simeq \varepsilon^{N-s}$, otherwise not comparable to any power of ε

Weak focus theorem,[Žubrinić, Ž, 2005]

- $$\left\{ \begin{array}{lcl} \dot{r} & = & r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}), \\ \dot{\varphi} & = & 1. \end{array} \right. \qquad (1)$$

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Theorem

(The case of weak focus)
Γ a part of a trajectory of (1) near the origin.
(a) $a_0 \neq 0$, then the spiral Γ is of exponential type, that is, comparable with $r = e^{a_0 \varphi}$, and hence $\dim_B \Gamma = 1$.
(b) k is fixed, $1 \leq k \leq l$, $a_l = 1$ and $a_0 = \cdots = a_{k-1} = 0$, $a_k \neq 0$. Then Γ is comparable with the spiral $r = \varphi^{-1/2k}$, and

$$d := \dim_B \Gamma = \frac{4k}{2k+1}.$$

Γ is Minkowski measurable with the value equal to explicit constant.

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- Minkowski content

$$\mathcal{M}^{1/2}(S^{g_1}(x_0)) > 0, \text{ but both } \mathcal{M}^{1/2}(S^{g_{2,3}}(x_0)) = 0$$

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- In nondiff. case find appropriate gauge functions (instead of powers) to compare $|A_\varepsilon|$ with $\to$ generalized Minkowski content (Lapidus)

The behaviour of the ε-neighbourhood of the orbit with respect to nondifferentiable generator

Theorem (Mardešić, Resman, Županović, 2012)

- *$f \in C^r(0,d)$, continuous on $[0,d)$, positive on $(0,d)$, $f(0) = f'(0) = 0$,*
- *f sublinear:*

$$m \leq x \cdot (\log f)'(x), \; x \in (0,d), \; m > 1.$$

Then

$$|A_\varepsilon(S^g(x_0))| \simeq f^{-1}(\varepsilon) \text{ as } \varepsilon \to 0.$$

$\star$ e.g. $f(x) = \frac{x}{-\log x}$ not sublinear, $\frac{|A_\varepsilon(S^g(x_0))|}{f^{-1}(\varepsilon)} \to \infty$ as $\varepsilon \to 0$.

Special case-differentiable generator

Corollary

f enough differentiable on $[0, d)$, *positive, strictly increasing on* $(0, d)$, $f(x) \simeq x^k$, $g = id - f$ *then* $|A_\varepsilon(S^g(x_0))| \simeq \varepsilon^{1/k}$ *and* $\dim_B(S^g(x_0)) = 1 - \frac{1}{k}$

Our admissible class of generating functions

- f with asymptotic development in Chebyshev scale at $x = 0$,

Definition (CHEBYSHEV SCALE; Mardešić: *Chebyshev systems and the versal unfolding of the cusp of order n*)

$\mathcal{I} = \{u_0, u_1, u_2, \ldots\}$, $u_i \in C[0, d) \cap C^r(0, d)$, $r \in \mathbb{N} \cup \{\infty\}$ *such that*

i) *Division/differentiation algorithm can be performed* r *times except possibly at* $x = 0$ ($\Rightarrow$ *extension by continuity to* 0):

$$\mathcal{I} = \{u_0, u_1, u_2, \ldots\} /{:}\, u_0 \Rightarrow D_0(\mathcal{I}) = \{\underbrace{1}_{D_0(u_0)}, \underbrace{\frac{u_1}{u_0}}_{D_0(u_1)}, \underbrace{\frac{u_2}{u_0}}_{D_0(u_2)}, \ldots\} \,/()'$$

$$\{D_0(u_1))', D_0(u_2)', \ldots\} /{:} D_0(u_1)' \Rightarrow D_1(\mathcal{I}) = \{\underbrace{1}_{D_1(u_1)}, \underbrace{\frac{(D_0(u_2))'}{(D_0(u_1))'}}_{D_1(u_2)}, \underbrace{\frac{(D_0(u_3))'}{(D_0(u_0))'}}_{D_1(u_3)}, \ldots\} /()'$$

$$\{D_1(u_2))', D_1(u_3)', \ldots\} /{:} D_1(u_2)' \Rightarrow D_2(\mathcal{I}) = \{\underbrace{1}_{D_2(u_2)}, \underbrace{\frac{(D_1(u_3))'}{(D_1(u_2))'}}_{D_2(u_3)}, \underbrace{\frac{(D_1(u_4))'}{(D_1(u_1))'}}_{D_2(u_4)}, \ldots\} /()'$$

ii) $D_i(u_{i+1})$ *strictly increasing*

iii) $D_j u_i(0) = 0$, $j < i$ *in the sense of limit*

$D_i(f) \ldots$ *$i-$th generalized derivative of f in the scale $\mathcal{I}$*

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Examples of Chebyshev scales

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- $\mathcal{I} = \{x^{\alpha_0}, x^{\alpha_1}, x^{\alpha_2}, \ldots\}$, $\alpha_i \in \mathbb{R}$, $0 < \alpha_0 < \alpha_1 < \alpha_2 < \ldots$

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- $\mathcal{I} = \{1, x(-\log x), x, x^2(-\log x), x^2, x^3(-\log x), x^3, \ldots\}$

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- $\mathcal{I} = \{1, x(-\log x), x, x^2(-\log x), x^2, x^3(-\log x), x^3, \dots\}$
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Generalized Minkowski content, critical Minkowski order (generalization of box dimension)

- $\mathcal{I} = \{u_0, u_1, \dots\}$ Chebyshev, $u_{i,\ i>0}$ positive, strictly increasing on $(0, d)$, f has development in $\mathcal{I}$

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Definition (generalized Minkowski content)

Upper generalized Minkowski content of $S^g(x_0)$ with respect to a Chebyshev scale $\{u_i,\ i = 1, 2, \ldots\}$

$$\mathcal{M}^*(S^g(x_0), u_i) = \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon(S^g(x_0))|}{u_i^{-1}(\varepsilon)}$$

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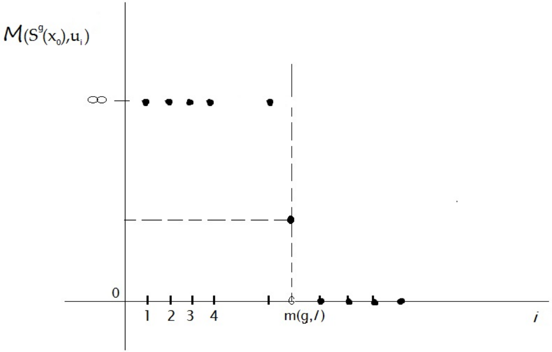

Figure: upper generalized Minkowski content as function of i.

Definition (critical Minkowski order)

$*$ *Upper critical order of g with respect to the scale $\mathcal{I}$:*

$$\overline{m}(g,\mathcal{I}) = \max\{i \geq 1 \mid \mathcal{M}^*(S^g(x_0), u_i) > 0\},$$

$*$ *(lower) critical order $\underline{m}(g,\mathcal{I})$, $m(g,\mathcal{I})$*

$*$ $m(g,\mathcal{I}) = i_0$ iff $|A_\varepsilon(S^g(x_0))| \simeq u_{i_0}^{-1}$

-g differentiable at zero:
development in $\mathcal{I} = \{1, x, x^2, \ldots\} \Rightarrow \dim_B(g) = 1 - \frac{1}{m(g,\mathcal{I})}$.

Multiplicity of fixed point zero of g-differentiable case and in a family

$g \in C^r[0, d)$, 0 fixed point; $f =$id$-g$

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- $\mu_0^{fix}(g) := \mu_0(f) = k$
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- (Mardešić: *Chebyshev systems and the versal unfolding of the cusp of order n*)

Connection multiplicity - critical Minkowski order

- (f_λ), asymptotic development in a family of T-scales $\mathcal{I}_\lambda$

Theorem (MRŽ, 2012)

$f = f_{\lambda_0}$ satisfies all assumptions of Theorem 2 and the upper power condition:

$$x \cdot (\log f(x))' \le M, \ \ x \in (0, d), \ \ \text{for some constant } M > 0.$$

Then the following is equivalent:

1. $D_i(f)(0) = 0$ *for* $i = 0, \ldots, k-1$ *and* $D_k(f)(0) > 0, \ \ k \ge 1$, $(f \simeq u_k, \ \ k \ge 1)$,
2. $|A_\varepsilon(S^g(x_0)| \simeq u_k^{-1}(\varepsilon)$,
3. $\mu_0^{fix}(g, (g_\lambda)) = k$,
4. $m(g, \mathcal{I}) = k$.

Deficiency of box dimension, nondifferentiable generators

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- $m(g_1, \mathcal{I}) = 3 > m(g_2, \mathcal{I}) = 2 > m(g_3, \mathcal{I}) = 1.$

Application of results

To find multiplicity of differentiable and nondifferentiable Poincaré maps around different limit periodic sets weak/strong focus, limit cycle, saddle loop, 2 saddle loop- example **stable homoclinic loop**

$$\begin{aligned} f_\lambda(x) &= \beta_0(\lambda) + \alpha_1(\lambda)[x\omega(x, \alpha_1(\lambda)) + g_1(x, \lambda)] + \\ &+ \beta_1(\lambda)x + \alpha_2(\lambda)[x^2\omega(x, \alpha_1(\lambda)) + g_2(x, \lambda)] + \beta_2(\lambda)x^2 + \ldots + \\ &+ \beta_n(\lambda)x + \alpha_n(\lambda)[x^n\omega(x, \alpha_1(\lambda)) + g_n(x, \lambda)] + \beta_n(\lambda)x^n + o(x^n), \\ &\omega(x, \alpha) = \begin{cases} \frac{x^{-\alpha}-1}{\alpha} & \text{if } \alpha \neq 0, \\ -\log x & \text{if } \alpha = 0, \end{cases} \qquad x \in (0, d), \end{aligned}$$

$g_i(x, \lambda)$ linear combination of monomials of the type $x^k\omega^l$ of strictly greater order than $x^i\omega$: $x^i\omega^j < x^k\omega^l$ if $(i < k)$ or $(i = k$ and $j > l)$.
$\star$ $\alpha_1(\lambda_0) = 0, \; \beta_0(\lambda_0) = 0.$
The corresponding family of Chebyshev scales:

$$\mathcal{I}_\lambda = \{1, x\omega(x, \alpha_1(\lambda)) + g_1(x, \lambda), x, x^2\omega(x, \alpha_1(\lambda)) + g_2(x, \lambda), x^2, \ldots\}.$$

Homoclinic loop

The development of f_{λ_0} around stable loop:

$$\begin{aligned} f_{\lambda_0}(x) &= \beta_1(\lambda_0)x + \alpha_2(\lambda_0)x^2\omega(x,0) + \alpha_3(\lambda_0)x^3\omega(x,0) + \ldots = \\ &= \beta_1(\lambda_0)x + \alpha_2(\lambda_0)x^2(-\log x) + \alpha_3(\lambda_0)x^3(-\log x) + \ldots (2) \end{aligned}$$

- If $f_{\lambda_0}(x) \simeq x^k$ as $x \to 0$, $k \geq 2$, then $m(g_{\lambda_0}, \mathcal{I}_{\lambda_0}) = 2k$.
- If $f_{\lambda_0} \simeq x^k(-\log x)$, $k \geq 2$, then $m(g_{\lambda_0}, \mathcal{I}_{\lambda_0}) = 2k - 1$.
- The cyclicity of the loop less than or equal to $2k$, $2k-1$; critical order recognizes cyclicity!
- $\dim_B(S^{g_{\lambda_0}}(x_0))$ in both cases $1 - 1/k$; box dimension does not recognize cyclicity!

Nilpotent singularities, [Horvat-Dmitrović, Ž, 2015]

system with nilpotent singularity at $(0,0)$

$$\begin{aligned} \dot{x} &= y + X(x,y), \\ \dot{y} &= Y(x,y), \qquad X, Y \in \mathcal{O}(|x,y|^2) \end{aligned} \tag{3}$$

- $y = f(x)$ is a solution of $y + X(x,y) = 0$ near $(0,0)$ and $f(0) = 0$,
- $y = f(x)$ ... characteristic curve
- $F(x) = Y(x, f(x)) = ax^m + \mathcal{O}(x^{m+1})$, $m \in \mathbb{N}$, $m \geq 2$, $a \neq 0$
- m-multiple nilpotent singularity
- $G(x) = (\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y})(x, f(x)) = bx^n + \mathcal{O}(x^{n+1})$, $n \in \mathbb{N}$, $n \geq 1, b \neq 0$

Nilpotent node

System with nilpotent node

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= ax^m + bx^n y \quad (4) \end{aligned}$$

Model system conditions

m odd, $a < 0$, n even and
$m > 2n + 1$ or $m = 2n + 1$ and
$b^2 + 4a(n + 1) \geq 0$

Model system (4) under above conditions has m-multiple nilpotent node at the origin. Parametric family of analytic systems with parameter δ is defined

$$\begin{aligned} \dot{x} &= y + X(x, y, \delta) \\ \dot{y} &= Y(x, y, \delta) \end{aligned} \qquad (5)$$

where $\delta = (\delta_1, \delta_2, \dots, \delta_l) \in D \subset \mathbb{R}^l$.

Characteristic unit-time map

Definition

Let $U : \mathbb{R}^2 \to \mathbb{R}^2$, $U(x, y) = (U_1(x, y), U_2(x, y))$ be an unit-time map of (4), and let $y = f(x)$ be a characteristic curve. Restriction of the map U_1 on the characteristic curve $U_1(x, f(x)) = C_h(x)$ is characteristic unit-time map. Box dimension of the orbit generated by the characteristic unit-time map near the origin is characteristic box dimension.

Proposition about characteristic box dimension

- Let a system (4) with nilpotent node at the origin satisfy the conditions for node. Let $U = (U_1, U_2)$ be a unit-time map of the system near the origin.
- Then the characteristic unit-time map C_h has a form $C_h(x) = x + \frac{a}{2}x^m + \mathcal{O}(x^{m+1})$ with the characteristic box dimension $\dim_{ch} U = 1 - \frac{1}{m}$
if and only if the origin is a m-multiple nilpotent node.

Remark, Liu, Li 2011, IJBC
We need l parameters to obtain l limit cycles from $2l+1$-multiple node.

Proposition about limit cycles

- Let a system (5) satisfy the conditions for node for $\delta = 0$, such that the characteristic box dimension is

$$\dim_{ch} U = 1 - \frac{1}{m}$$

  where $U = (U_1, U_2)$ is the unit-time map of the system near the origin.
- Then for $0 < |\delta_1| < |\delta_2| < \ldots |\delta_l| << 1$ system has at least $\lfloor \frac{m-1}{2} \rfloor$ limit cycles.

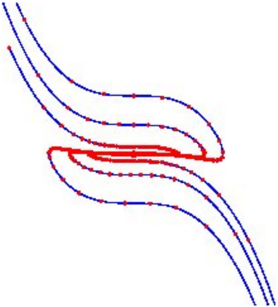
System with nilpotent node

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^5 - 4x^2 y. \qquad (6) \end{aligned}$$

Conditions

$m = 5,\ a < 0,\ n = 2,$
$m = 2n + 1,\ b^2 + 4a(n + 1) \geq 0$

- charact. curve $y = 0$,
- char. unit-time map $U_1(x, 0) = x - \frac{1}{2}x^5 + \mathcal{O}(x^6)$
- char. box dim. $\dim_{ch} U = 1 - \frac{1}{5} = \frac{4}{5}$
- separatrices $y = Cx^3 + \ldots$
- $\dim_B S_x = 1 - \frac{1}{3} = \frac{2}{3}$



Nilpotent focus

System with nilpotent focus at origin is of a form

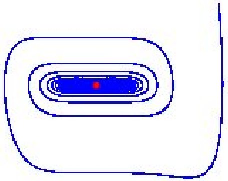
$$\begin{aligned} \dot{x} &= y + X(x, y) \\ \dot{y} &= Y(x, y) \end{aligned} \tag{7}$$

• Characteristic curve $y = f(x)$ is a solution of $y + X(x, y) = 0$ with $f(0) = 0$.

The system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^2 y - x^3 \end{aligned}$$

has a characteristic curve $y = 0$.



Family of nilpotent foci

Parametric family of analytic systems with parameter δ of a form

$$\begin{aligned} \dot{x} &= y + X(x,y,\delta) \\ \dot{y} &= Y(x,y,\delta), \end{aligned} \tag{8}$$

$\delta = (\delta_1, \delta_2, \ldots, \delta_l) \in D \subset \mathbb{R}^l$, where D is a simply connected domain, and $X, Y = \mathcal{O}(|x,y|^2)$.
The characteristic curve is $y = f(x,\delta)$,

$$\begin{aligned} F(x,\delta) &= \sum_{j\geq 2n-1} a_j(\delta)x^j, \quad n \geq 2, \quad a_{2n-1}(\delta) > 0, \\ G(x,\delta) &= \sum_{j\geq n-1} b_j(\delta)x^j, \quad b_{n-1}^2(\delta) - 4na_{2n-1}(\delta) < 0, \end{aligned}$$

We take the conditions regarding coefficients in the asymptotic expansion of the previous expressions, for all $\delta \in D$.

Cyclicity of nilpotent focus

Poincaré return map for (8) on the characteristic curve $y = f(x, \delta)$,

$$P(x, \delta) = x + \sum_{j \geq 1} v_j(\delta) x^j.$$

Theorem (Romanovski, Han (2012))

Let a family $\mathcal{X}_\delta$ defined by (8) has a nilpotent focus at the origin, let $P(x, \delta)$= Poincaré map on characteristic curve $y = f(x, \delta)$.
Let the family satisfy above conditions for all $\delta \in D$. Denote that $p_n = (1 + (-1)^n)/2$. If there is a integer $k \geq 1$ such that

$$\sum_{j=1}^{k+1} \left| v_{2j-1+p_n} \right| > 0, \;\; \forall \delta \in D$$

then there exists a neighbourhood U of the origin s.t. the family $\mathcal{X}_\delta$ has at most k limit cycles in U for all $\delta \in \bar{D} \subset D$, compact.

Box dimension and cyclicity of nilpotent focus

Theorem

Cyclicity of nilpotent focus and box dimension

- Let $\Gamma(\delta_0)$ be a spiral trajectory of (8) near the origin for some $\delta_0 \in D$.
- Let $\bar{P}(x, \delta_0)$ be the Poincaré map of (8) near focus on the characteristic curve $y = f(x, \delta_0)$.
- Let the sequence $S(x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = P(x_n, \delta_0)$ (stable focus) or $x_{n+1} = P^{-1}(x_n, \delta_0)$(unstable focus), $x_1 \in (0, r)$ has the box dimension $\dim_B S(x_1) = 1 - \frac{1}{2k+1}$ or $1 - \frac{1}{2k+2}$.
- Then for all $\delta \in D$ near δ_0 the system has at most k limit cycles in the neighborhood of the origin.

Singularities of differentiable maps [Rolin, Vlah, Ž, 2015]

- We analyze critical points ($\nabla f = 0$) of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$.
- Standard approach is analysis of asymptotic behavior of oscillatory integrals

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(x)} \Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty,$$

with respect to parameter $\tau \in \mathbb{R}$. Notice $I : \mathbb{R} \to \mathbb{C}$.
- We examine geometrical properties of curve in the complex plane generated by $I(\tau)$ for $\tau \geq \tau_0 > 0$, and also of graphs of real and imaginary parts of $I(\tau)$.

$$X(\tau) := \mathrm{Re}\, I(\tau) = \int_{\mathbb{R}^n} \cos(\tau f(\mathbf{x}))\Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty,$$

$$Y(\tau) := \mathrm{Im}\, I(\tau) = \int_{\mathbb{R}^n} \sin(\tau f(\mathbf{x}))\Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty.$$

Standard assumptions on functions f and Φ

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(\mathrm{x})} \Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty, \quad \tau \in \mathbb{R}.$$

- Function $\Phi : \mathbb{R}^n \to \mathbb{R}$
  - is called the *amplitude function*,
  - is of class C^∞,
  - is a function with compact support,
  - point $\mathbf{0}$ is inside the compact support of function Φ.
- Function $f : \mathbb{R}^n \to \mathbb{R}$
  - is called the *phase function*,
  - point $\mathbf{0} \in \mathbb{R}^n$ is the critical point of function f,
  - is a *real analytic* function in the neighborhood of its critical point $\mathbf{0}$,
  - point $\mathbf{0}$ is *the only* critical point of function f inside the compact support of function Φ.

Oscillatory and curve dimensions

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(\mathrm{x})}\Phi(\mathbf{x})\,d\mathbf{x}, \quad \tau \to \infty,$$

$$X(\tau) := \operatorname{Re} I(\tau) = \int_{\mathbb{R}^n} \cos(\tau f(\mathbf{x}))\Phi(\mathbf{x})\,d\mathbf{x}, \quad \tau \to \infty,$$

$$Y(\tau) := \operatorname{Im} I(\tau) = \int_{\mathbb{R}^n} \sin(\tau f(\mathbf{x}))\Phi(\mathbf{x})\,d\mathbf{x}, \quad \tau \to \infty.$$

Under the standard assumptions on functions f and Φ we determine:

- *Oscillatory dimensions* of functions $X(\tau)$ and $Y(\tau)$, which are defined as the box dimension of graphs of functions

$$x(t) := X(1/t), \quad t \to 0, \qquad y(t) := Y(1/t), \quad t \to 0,$$

  and associated Minkowski contents.
- *Curve dimension* of function $I(\tau)$, which is defined as the box dimension of the curve defined in the complex plane by $I(\tau)$, for $\tau \geq \tau_0 > 0$, and associated Minkowski contents.

Results: Phase function of a single variable

Theorem ($n = 1$)

Let the standard assumptions on f and Φ hold, and let $f(0) \neq 0$. Let $f'(0) = f''(0) = \cdots = f^{(p-1)}(0) = 0$ and $f^{(p)}(0) \neq 0$, $p \geq 2$ (p is the order of degeneracy). Using well known asymptotic $I(\tau) \sim C_1 \cdot e^{i\tau f(0)} \cdot \tau^{-1/p}$, as $\tau \to \infty$, it follows:

- *Oscillatory dimension of both $X(\tau)$ and $Y(\tau)$ is $d' = \frac{3p-1}{2p}$ and associated graphs are Minkowski nondegenerate. Explicit lower and upper bounds on d'-dimensional lower and upper Minkowski contents depend only on $f(0)$, p and C_1.*
- *Curve dimension of $I(\tau)$ is $d = \frac{2p}{p+1}$, associated curve Γ is Minkowski measurable, and d-dimensional Minkowski content of Γ is*

$$\mathcal{M}^d(\Gamma) = C_1^{\frac{2p}{p+1}} \cdot \pi \cdot \left(\frac{\pi}{p \cdot f(0)} \right)^{-\frac{2}{p+1}} \cdot \frac{p+1}{p-1}.$$

Newton diagram-$\mathbb{R}$-nondegeneracy

We consider the power series of the phase f

$$f(x) = \sum a_k x^k$$

with real coefficients, having monomials

$$x^k = x_1^{k_1} \dots x_n^{k_n}$$

with multi-index $k = (k_1, \dots k_n)$.
Polynomial f_Δ that equals to the sum of monomials belonging to the Newton diagram, is called the principal part of the series.
The principal part f_Δ of the power series f with real coefficients is $\mathbb{R}$-nondegenerate if for every compact face γ of the Newton polyhedron of the series the polynomials

$$\partial f_\gamma / \partial x_1, \dots, \partial f_\gamma / \partial x_n$$

do not have common zeroes in $(\mathbb{R} \setminus 0)^n$.

Newton diagram-remoteness and multiplicity

The bisector intersects the boundary of the Newton polyhedron in exactly one point $(c, \dots, c)$, which is called center of the Newton polyhedron. The remoteness of the Newton polyhedron is equal to $r = -1/c$. If $r > -1$ the Newton polyhedron is *remote*, which means that it does not contain the point $(1, \dots, 1)$.
The remoteness of the critical point of the phase is the upper bound of the remotenesses of the Newton polyhedra of the Taylor series of the phase in all systems of local analytic coordinates with origin at the critical point.
We consider the open face which contains the center of the boundary of Newton polyhedron. The codimension of this face, less one, is called the multiplicity of the remoteness. If the face is a vertex then multiplicity is $n - 1$, and if the face is an edge then multiplicity is $n - 2$.

Results: Phase function of two variables

Theorem ($n = 2$)

Let $n = 2$, the standard assumptions on f and ϕ hold, and let $f(0) \neq 0$. Let β be the remoteness of the critical point of the phase function f. Let Γ be the curve defined by $X(\tau)$ and $Y(\tau)$, near the origin, with $I(\tau) \sim e^{i\tau f(0)}\left(C_{\beta,0}\tau^{\beta} + C_{\beta,1}\tau^{\beta}\log\tau\right)$ as $\tau \to \infty$. Then:

- *If $C_{\beta,1} = 0$ then oscillatory dimension of both X and Y is equal to*
  $d' = (\beta + 3)/2$
  and Minkowski nondegenerate. Curve dimension of I is
  $d = 2/(1-\beta)$
  and associated Minkowski content is

$$\mathcal{M}^d(\Gamma) = \left[\frac{|C_{\beta,0}|}{f(0)^{\beta}}\right]^{\frac{2}{1-\beta}} \cdot [-\beta]^{\frac{2\beta}{1-\beta}} \cdot \pi^{\frac{1+\beta}{1-\beta}} \cdot \frac{1-\beta}{1+\beta} \qquad (9)$$

- *If $C_{\beta,1} \neq 0$ then oscillatory and curve dimensions are the same as in previous case but Minkowski degenerate.*

Phase function of more than two variables

Phase function of $n > 2$ variables

- Let $n > 2$ the standard assumptions on f and ϕ hold, and let $f(0) \neq 0$. Let Γ be the curve defined by $X(\tau)$ and $Y(\tau)$, near the origin.
- If phase function f has the non-degenerated critical point then oscillatory and curve dimensions are equal to 1.

Phase function of more than two variables

Theorem ($n > 2$)

- *Let phase function f has the degenerated critical point* $I(\tau) \sim e^{i\tau f(0)} \sum_{\alpha<\beta} \sum_{k=0}^{n-1} C_{\alpha,i}(\phi)\tau^{\alpha} (\log \tau)^{k}$, *as* $\tau \to \infty$.
- *Let Newton diagram of the phase f be $\mathbb{R}$-nondegenerate and remote with remoteness β of the critical point. Then:*
- *If $C_{\beta,0} \neq 0$ and $C_{\beta,i} = 0$, $i = 1, \dots, K$ where K is multiplicity of remoteness, then oscillatory dimension of $X(\tau)$ and $Y(\tau)$ is equal to*
  $d' = (\beta + 3)/2$
  and Minkowski nondegenerate. Curve dimension of $I(\tau)$ is
  $d = 2/(1 - \beta)$
  and Minkowski content is given by (9).
- *If there exists $C_{\beta,i} \neq 0$, $i = 1, \dots, K$ then oscillatory and curve dimensions are the same as for the case where $C_{\beta,1} = 0$, and Minkowski degenerate.*

Further research with oscillatory integrals

- Caustics consisting of degenerate singularities are also intersting objects where bifurcations appear.
- Our goal would be to go toward classification of critical points using curve and oscillatory dimension and Minkowski contents of associated oscillatory integrals.
- Case with amplitude which is not C^∞.

Main references

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