Fractal analysis of oscillatory solutions of a class of ordinary differential equations including the Bessel equation

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Overview of the presentation - Main results

- We analyze fractal properties of oscillatory solutions of
$$t^2x''(t)+t(2-\mu)x'(t)+(t^2-\nu^2)x(t)=0, \tag{1}$$
having parameters $\mu\in(0,2)$ and $\nu\in\mathbf{R}$ (it is *Bessel equation* for $\mu=1$).
- We use the concept of *fractal dimension*, examining a *phase portrait* of solutions.
- We use Minkowski–Bouligand dimension, known as *box-counting dimension*.
- Phase portraits of solutions of (1) are *spirals* in the plane near the origin.
- We examine fractal properties of *chirp-like functions* related to solutions of (1),
$$x_1(t)=p(t)\sin t,\quad x_2(t)=p(t)\cos t,$$
where $p(t)$ is "similar" to $t^{-\alpha}$, $\alpha\in(0,1)$.
- We examine fractal properties of solutions of 3D systems of ODEs having 3D spiral trajectories related to chirp-like solutions.
- We examine fractal properties of generalized Euler spirals defined parametrically by
$$\Gamma_q\cdots\begin{cases} x(t) &= \int_0^t \cos(q(s))\,ds \\ y(t) &= \int_0^t \sin(q(s))\,ds, \end{cases}$$
where $t\geq 0,\ q(t)$ is "similar" to t^p.

Minkowski content

Definition (ε-neighbourhood)

Let $A \subset \mathbb{R}^n$, A is bounded. The **ε-neighbourhood** *of set A is*

$$A_\varepsilon := \{y \in \mathbb{R}^n : d(y,A) < \varepsilon\}.$$

Definition (Lower an upper s-dimensional Minkowski content)

Lower s-dimensional Minkowski content *of bounded set $A \subset \mathbb{R}^n$, $s \geq 0$ is*

$$\mathscr{M}_*^s(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{n-s}}.$$

Upper s-dimensional Minkowski content $\mathscr{M}^{*s}(A)$, *$s \geq 0$ is*

$$\mathscr{M}^{*s}(A) := \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{n-s}}.$$

If $\mathscr{M}^{*s}(A) = \mathscr{M}_*^s(A)$, the common value is called the **s-dimensional Minkowski content of** *A, and is denoted by $\mathscr{M}^s(A)$.*

Box dimension 1/2

Definition (Lower and upper box dimension)

Lower box dimension *of bounded set* $A \subset \mathbb{R}$ *is*

$$\underline{\dim}_B A := \inf\{s \geq 0 : \mathscr{M}_*^s(A) = 0\} = \sup\{s \geq 0 : \mathscr{M}_*^s(A) = \infty\}.$$

Upper box dimension *of* A *is*

$$\overline{\dim}_B A := \inf\{s \geq 0 : \mathscr{M}^{*s}(A) = 0\} = \sup\{s \geq 0 : \mathscr{M}^{*s}(A) = \infty\}.$$

Generally $\underline{\dim}_B A \leq \overline{\dim}_B A$.

Definition (Box dimension)

If $\underline{\dim}_B A = \overline{\dim}_B A$ *we define the* **box dimension** *of* A *to be*

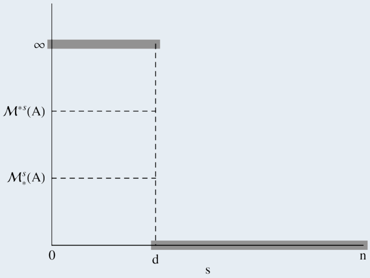
$$\dim_B A := \underline{\dim}_B A = \overline{\dim}_B A.$$

Box dimension 2/2

Definition (Minkowski nondegenerate or degenerate set)

If $0 < \mathcal{M}_*^d(A) \leq \mathcal{M}^{*d}(A) < \infty$ *for some* d*, then we say that* A *is* **Minkowski nondegenerate**. *In this case obviously* $d = \dim_B A$.

In the case when lower or upper d*-dimensional Minkowski contents of* A *are* 0 *or* ∞*, where* $d = \dim_B A$*, or* $\underline{\dim}_B A < \overline{\dim}_B A$*, we say that* A *is* **Minkowski degenerate**.



Definition (Minkowski measurable set)

If there exists $\mathcal{M}^d(A)$ *for some* d *and* $\mathcal{M}^d(A) \in (0, \infty)$*, we say that* A *is* **Minkowski measurable**.

Box dimension - examples

Examples of some sets and their box dimensions

Let $n = 2$, ambient space is $\mathbb{R}^2$.

- A is a single point, $\dim_B A = 0$
- A is a line segment, $\dim_B A = 1$
- A is a disk, $\dim_B A = 2$
- A is a smooth rectifiable curve, $\dim_B A = 1$
- A is a non-rectifiable power spiral, $0 < \alpha < 1$, given in polar coordinates by

$$r = \varphi^{-\alpha}, \quad \varphi \in [\varphi_0, \infty), \qquad \dim_B A = \frac{2}{1+\alpha} \in (1,2), \qquad \text{(C. Tricot, 1993)}$$

- A is a graph of a non-rectifiable (α, β)-chirp near the origin, $0 < \alpha < \beta$, given by

$$f(\tau) = \tau^{\alpha} \cos \tau^{-\beta} \quad \tau \in (0, \tau_0], \qquad \dim_B A = 2 - \frac{\alpha+1}{\beta+1} \in (1,2),$$

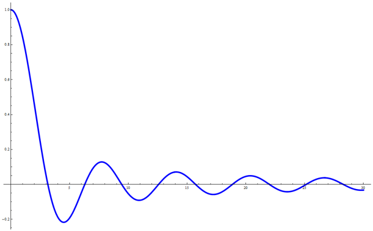
(C. Tricot, *Curves and Fractal Dimension*, 1993)

Oscillatory function near infinity

Definition (Oscillatory function near $t = \infty$)

Let $f : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, be a continuous function. $f(t)$ is an **oscillatory function** *near $t = \infty$ if there exists sequence $t_k \to \infty$ such that $f(t_k) = 0$, and restrictions $f|_{(t_k, t_{k+1})}$ intermittently change sign for $k \in \mathbb{N}$.*

Example: $f(x) = \frac{1}{x}\sin(x)$.
It is (α, β)-chirp near infinity, $f(t) = t^{-\alpha} \sin t^{\beta}$, $t \in [t_0, \infty)$, for $\alpha = \beta = 1$.

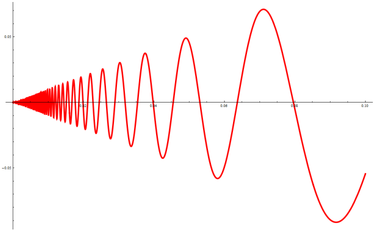


Oscillatory function near the origin

Definition (Oscillatory function near the origin)

Let $g : (0, \tau_0] \to \mathbb{R}$, $\tau_0 > 0$, *be a continuous function.* $g(t)$ *is an* **oscillatory function** *near the origin if there exists sequence* s_k *such that* $s_k \searrow 0$ *as* $k \to \infty$, $g(s_k) = 0$ *and restrictions* $g|_{(s_{k+1}, s_k)}$ *intermittently change sign for* $k \in \mathbb{N}$.

Example: $g(x) = x \sin(1/x)$.
Notice it is $(1,1)$-chirp near the origin.



Phase oscillatory function

Definition (Phase oscillatory function (Pašić, Žubrinić, Županović))

Let $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$ *and* $x \in C^1$. $x(t)$ *is a* **phase oscillatory** *function if set*

$$\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$$

in the plane is a spiral converging to the origin.

Definition (Spiral)

A **spiral** *is the graph of function* $r = f(\varphi)$, $\varphi \geq \varphi_1 > 0$, *in polar coordinates, where*

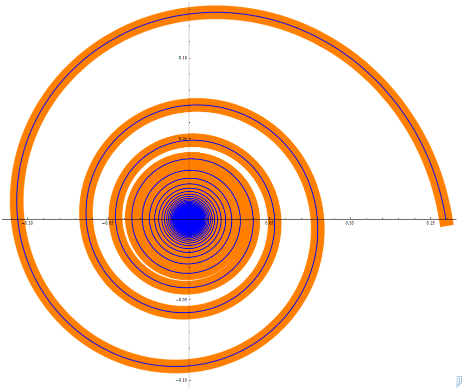
- $f : [\varphi_1, \infty) \to (0, \infty)$ *is such that* $f(\varphi) \to 0$ *as* $\varphi \to \infty$,
- f *is radially decreasing (ie, for any fixed* $\varphi \geq \varphi_1$ *the function* $\mathbb{N} \ni k \mapsto f(\varphi + 2k\pi)$ *is decreasing)* .

A mirror image of a spiral over the x-axes will be also called a spiral.

Phase dimension

Definition (Phase dimension (Pašić, Žubrinić, Županović))

The **phase dimension** $\dim_{ph}(x)$ *of phase oscillatory function* $x(t)$ *is the box dimension of corresponding spiral* $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$.

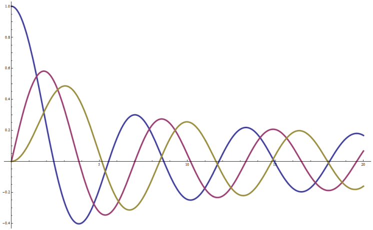


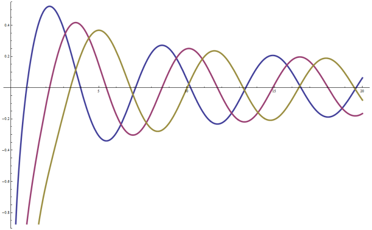
Bessel equation

Bessel equation of order ν

$$t^2x''(t) + tx'(t) + (t^2 - \nu^2)x(t) = 0, \quad \nu \in \mathbf{R}$$

- Linear second-order ordinary differential equation. Single parameter ν.
- Two linearly independent solutions are called Bessel functions $J_\nu(t)$ and $Y_\nu(t)$.
- The solutions are oscillatory functions near $t = \infty$.





Phase dimension of Bessel functions

Bessel system of order ν - substitution $y = \dot{x}$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (\frac{\nu^2}{t^2} - 1)x - \frac{1}{t}y, \quad \nu \in \mathbf{R} \end{aligned}$$

Theorem (Phase dimension of Bessel functions)

The phase dimension of $J_\nu(t)$ and $Y_\nu(t)$ is

$$\dim_{ph}(J_\nu) = \dim_{ph}(Y_\nu) = \frac{4}{3}, \quad \textit{for all } \nu \in \mathbf{R}$$

Remark

Problem: A spiral radius function is **non-monotone** - we get a **wavy spiral**.

Reflected Bessel equation

Substitution $t = \frac{1}{\tau}$ in Bessel equation:

Reflected Bessel equation

$$x''(\tau) + \frac{1}{\tau}x'(\tau) + \left(\frac{1}{\tau^4} - \frac{\nu^2}{\tau^2}\right)x(\tau) = 0, \quad \nu \in \mathbf{R}$$

- Solutions are oscillatory near the origin.

Generalized reflected Bessel equation

$$x''(\tau) + \frac{\mu}{\tau}x'(\tau) + \left(\frac{\lambda}{\tau^\sigma} - \frac{\nu^2}{\tau^2}\right)x(\tau) = 0, \quad \mu \in \mathbf{R},\ \lambda > 0,\ \sigma > 2,\ \nu \in \mathbf{R}$$

- Introduced by Pašić, Tanaka, 2011.
- Solutions are oscillatory near the origin.
- They determined box dimension of graphs of solutions - oscillatory dimension

Back to ∞

Substitution $\tau = \frac{1}{t}$ will get us back to ∞.

Generalized Bessel equation

$$t^2 x''(t) + t(2-\mu)x'(t) + (\lambda t^{\sigma-2} - \nu^2)x(t) = 0, \quad \mu \in \mathbf{R},\ \lambda > 0,\ \sigma > 2,\ \nu \in \mathbf{R}$$

- Two linearly independent solutions.
- Solutions are oscillatory near $t = \infty$.
- We would like to determine **phase dimension** of the solutions.
- $\mu = 1$, $\lambda = 1$, $\sigma = 4$, $\nu \in \mathbf{R}$ is the standard Bessel equation.
- We fix $\lambda = 1$, $\sigma = 4$.
- What is the phase dimension of the solutions depending on parameters μ and ν?

Phase dimension of solutions of generalized Bessel equation

Generalized Bessel equation for $\mu \in \mathbf{R}$, $\lambda = 1$, $\sigma = 4$ and $\nu \in \mathbf{R}$

$$t^2 x''(t) + t(2-\mu)x'(t) + (t^2 - \nu^2)x(t) = 0$$

Two linearly independent solutions we call **generalized Bessel functions**

$$\begin{aligned} \widetilde{J}_{\nu,\mu}(t) &= t^{\frac{\mu-1}{2}} J_{\widetilde{\nu}}(t), \\ \widetilde{Y}_{\nu,\mu}(t) &= t^{\frac{\mu-1}{2}} Y_{\widetilde{\nu}}(t), \quad \text{where } \widetilde{\nu} = \sqrt{\left(\frac{\mu-1}{2}\right)^2 + \nu^2}. \end{aligned}$$

Theorem (Phase dimension of generalized Bessel functions)

The phase dimension of $\widetilde{J}_{\nu,\mu}(t)$ and $\widetilde{Y}_{\nu,\mu}(t)$ is

$$\dim_{ph}(\widetilde{J}_{\nu,\mu}) = \dim_{ph}(\widetilde{Y}_{\nu,\mu}) = \frac{4}{4-\mu}, \quad \textit{for all } \mu \in (0,2),\ \nu \in \mathbf{R}.$$

The proof 1/4

Remarks

- If $\mu \geq 2$ then $\widetilde{J}_{\nu,\mu}(t)$ and $\widetilde{Y}_{\nu,\mu}(t)$ are not even phase oscillatory functions.
- If $\mu < 0$ then related spirals are rectifiable, phase dimension is equal to 1.

Sketch of the proof

1. Using asymptotic expansions of Bessel functions for large t, we carefully gather information on "shape" of related spiral $\Gamma_1 \ldots r = f(\varphi)$ in polar coordinates.
2. Γ_1 is **non-monotonically** converging to the origin - we call it **a wavy spiral**
3. We construct a new spiral Γ_2 that is "close" to spiral Γ_1 and prove the existence of bi-Lipschitz map F that maps Γ_1 to Γ_2.
4. We determine the box dimension of Γ_2 using a generalization of a result about the box dimension of spirals with a decreasing radius function, from Žubrinić, Županović, 2005.
5. Finally, we use a result about box dimension being invariant under bi-Lipschitz maps, from K. Falconer, *Fractal geometry*, 1990.

The proof 2/4 - Step 1

Asymptotic expansions of Bessel functions

Hankel's asymptotic expansions of Bessel functions $J_\nu(t)$ and $Y_\nu(t)$ for large t

$$
\begin{aligned}
J_\nu(t) &= \left(\frac{2}{\pi t}\right)^{\frac{1}{2}} \left[P_\nu(t)\cos\chi - Q_\nu(t)\sin\chi\right], \\
Y_\nu(t) &= \left(\frac{2}{\pi t}\right)^{\frac{1}{2}} \left[P_\nu(t)\sin\chi + Q_\nu(t)\cos\chi\right], \qquad \nu\in\mathbb{R}, \quad \chi = t - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi. \\
P_\nu(t) &= \sum_{k=0}^{N} (-1)^k \frac{(\nu,2k)}{(2t)^{2k}} + O\left(t^{-2N-2}\right) \quad \text{as } t\to\infty, \\
Q_\nu(t) &= \sum_{k=0}^{N} (-1)^k \frac{(\nu,2k+1)}{(2t)^{2k+1}} + O\left(t^{-2N-3}\right) \quad \text{as } t\to\infty, \quad \text{expansions to } N \text{ terms}, \\
(\nu,k) &= \frac{(-1)^k}{k!}\left(\frac{1}{2}-\nu\right)_k\left(\frac{1}{2}+\nu\right)_k = \frac{(4\nu^2-1)(4\nu^2-3^2)\cdots(4\nu^2-(2k-1)^2)}{2^{2k}k!}, \\
(\nu,0) &= 1.
\end{aligned}
$$

We must consider asymptotics of expressions like $r''(t) = \frac{d}{dt}\left(\frac{x(t)\dot{x}(t)+\dot{x}(t)\ddot{x}(t)}{\sqrt{x(t)^2+\dot{x}(t)^2}}\right)$, where $x(t) = \widetilde{J}_{\nu,\mu}(t) = t^{\frac{\mu-1}{2}} J_{\widetilde{\nu}}(t)$. For $N=2$ we easily get several hundred terms in fully expanded form!

The proof 3/4 - Step 2

Definition (Wavy function)

Let $r : [t_0, \infty) \to (0, \infty)$ be a C^1 function. Assume that $r'(t_0) \leq 0$. We say that $r = r(t)$ is a wavy function if the sequence (t_n) defined inductively by:

$$\begin{aligned} t_{2k+1} &:= \inf\{t : t > t_{2k}, r'(t) > 0\}, \quad k \in \mathbb{N}_0, \\ t_{2k+2} &:= \inf\{t : t > t_{2k+1}, r(t) = r(t_{2k+1})\}, \quad k \in \mathbb{N}_0, \end{aligned}$$

is well-defined, and satisfies the waviness condition:

$$\begin{cases} \text{(i)} & \text{The sequence } (t_n) \text{ is increasing and } t_n \to \infty \text{ as } n \to \infty. \\ \text{(ii)} & \text{There exists } \varepsilon > 0 \text{, such that for all } k \in \mathbb{N}_0 \text{ holds } t_{2k+1} - t_{2k} \geq \varepsilon. \\ \text{(iii)} & \text{For all } k \text{ sufficiently large it holds } \operatorname*{osc}_{t \in [t_{2k+1}, t_{2k+2}]} r(t) = o\left(t_{2k+1}^{-\alpha-1}\right), \ \alpha \in (0,1), \end{cases} \tag{2}$$

where $\operatorname*{osc}_{t \in I} r(t) = \max_{t \in I} r(t) - \min_{t \in I} r(t)$.

Definition (Wavy spiral)

Let a spiral Γ', given in polar coordinates by $r = f(\varphi)$, where f is a given function. If there exists increasing or decreasing function of class C^1, $\varphi = \varphi(t)$, such that $r(t) = f(\varphi(t))$ is a wavy function, then we say Γ' is a wavy spiral.

The proof 4/4 - Steps 3–5

Theorem (Box dimension of a wavy spiral)

Let $t_0 > 0$ *and assume* $r : [t_0, \infty) \to (0, \infty)$ *is a wavy function. Assume that* $\varphi : [t_0, \infty) \to [\varphi_0, \infty)$ *is an increasing function of class* C^1 *such that* $\varphi(t_0) = \varphi_0 > 0$ *and there exists* $\bar{\varphi}_0 \in \mathbb{R}$ *such that*

$$|(\varphi(t) - \bar{\varphi}_0) - (t - t_0)| \to 0 \;\; as \;\; t \to \infty. \tag{3}$$

Let $f : [\varphi_0, \infty) \to (0, \infty)$ *be defined by* $f(\varphi(t)) = r(t)$*. Assume that* Γ' *is a spiral defined in polar coordinates by* $r = f(\varphi)$*. Let* $\alpha \in (0, 1)$ *is the same value as in the definition of wavy function r, and assume* ε' *is such that* $0 < \varepsilon' < \varepsilon$*, where* ε *is determined by the definition of wavy function r. Assume that there exist positive constants* $\underline{m}$*,* $\overline{m}$*,* $\underline{a}'$ *and M such that for all* $\varphi \geq \varphi_0$*,*

$$\underline{m}\varphi^{-\alpha} \leq f(\varphi) \leq \overline{m}\varphi^{-\alpha}, \tag{4}$$

$$|f'(\varphi)| \leq M\varphi^{-\alpha-1}, \tag{5}$$

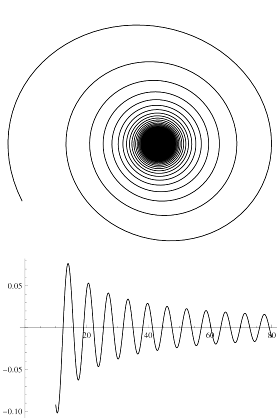
and for all $\triangle\varphi$*, such that* $\theta \leq \triangle\varphi \leq 2\pi + \theta$*, there holds*

$$\underline{a}'\varphi^{-\alpha-1} \leq f(\varphi) - f(\varphi + \triangle\varphi), \tag{6}$$

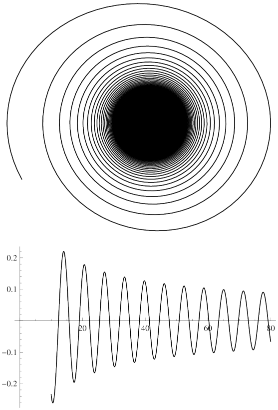
where $\theta := \min\{\varepsilon', \pi\}$*.*
Then Γ' *is a wavy spiral and*

$$\dim_B \Gamma' = \frac{2}{1+\alpha}.$$

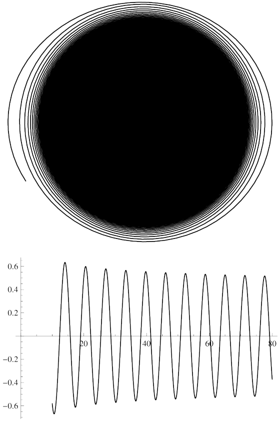
Generalized Bessel functions



$x_1(t) = \widetilde{J}_{5,0.2}(t)$,
$\dim_{ph}(x_1) = \frac{20}{19}$



$x_2(t) = \widetilde{J}_{5,1}(t)$,
$\dim_{ph}(x_2) = \frac{4}{3}$



$x_3(t) = \widetilde{J}_{5,1.8}(t)$,
$\dim_{ph}(x_3) = \frac{20}{11}$

$(\alpha, 1)$-chirp like equation

$(\alpha, 1)$-chirp equation

Generalized Bessel equation for $\mu \in (0, 2)$, $\lambda = 1$, $\sigma = 4$ and $\nu = \pm\frac{\sqrt{(2-\mu)\mu}}{2}$ becomes $(\alpha, 1)$-chirp like equation.

$$t^2 x''(t) + 2\alpha t x'(t) + (t^2 - \alpha(1-\alpha))x(t) = 0, \quad \alpha = \frac{2-\mu}{2} \in (0,1), \quad \mu \in (0,2)$$

- Two linearly independent solutions are $(\alpha, 1)$**-chirps**

$$x_1(t) = t^{-\alpha} \sin t, \qquad x_2(t) = t^{-\alpha} \cos t, \quad \alpha = \frac{2-\mu}{2} \in (0,1).$$

- Solutions are oscillatory near $t = \infty$.

Functions comparable of class k

Definition (Comparable of class k in the limit sense)

We write $f(t) \sim g(t)$ *as* $t \to \infty$ *if* $\frac{f(t)}{g(t)} \to 1$ *as* $t \to \infty$.
If $k \in \mathbb{N}$, *for* $f, g \in C^k$ *we write,*

$$f(t) \sim_k g(t) \text{ as } t \to \infty,$$

if $f^{(j)}(t) \sim g^{(j)}(t)$ *as* $t \to \infty$ *for all* $j = 0, 1, \ldots, k$.

For example, $\frac{(t-1)^{4-\alpha}}{t^4} \sim_3 t^{-\alpha}$ as $t \to \infty$, for $\alpha \in (0,1)$.

Definition (Comparable of class k)

We write $f(t) \simeq g(t)$ *as* $t \to \infty$ *if there exist* $C, D > 0$ *such that* $Cf(t) \le g(t) \le Df(t)$ *for all* t *sufficiently large. If* $k \in \mathbb{N}$, *for* $f, g \in C^k$ *we write*

$$f(t) \simeq_k g(t) \text{ as } t \to \infty,$$

if $f^{(j)}(t) \simeq g^{(j)}(t)$ *as* $t \to \infty$ *for all* $j = 0, 1, \ldots, k$.

$(\alpha, 1)$-chirp like functions

Definition ((α, β)-chirp-like function)

Functions of the form

$$x(t) = p(t)\sin(q(t)) \quad \text{or} \quad x(t) = p(t)\cos(q(t)),$$

where $p(t) \simeq t^{-\alpha}$, $q(t) \simeq_1 t^{\beta}$ *as* $t \to \infty$, *are called* (α, β)**-chirp-like functions** *near infinity.*

Theorem (Phase dimension of $(\alpha, 1)$-chirp like functions)

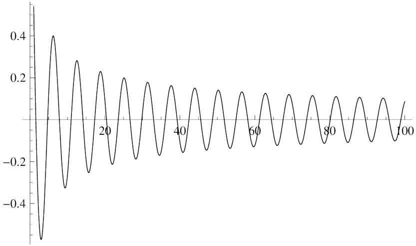
The phase dimension of $(\alpha, 1)$*-chirp-like functions*

$$x_1(t) = p(t)\sin t, \quad x_2(t) = p(t)\cos t,$$

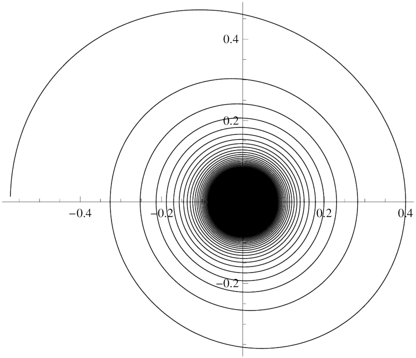
where $p(t) \sim_3 t^{-\alpha}$, $\alpha \in (0, 1)$ *is*

$$\dim_{ph}(x_1) = \dim_{ph}(x_2) = \frac{2}{1+\alpha}.$$

Graph of $x(t) = t^{-\frac{1}{2}} \cos t$ and plot in the phase plane



$x(t) = t^{-\frac{1}{2}} \cos t$



$\Gamma = \{(x(t), \dot{x}(t)) : t \geq t_0\}$
$\dim_{ph}(x) = \dim_B \Gamma = \frac{4}{3}$

(α, β)-chirp-like equation

$(\alpha, 1)$-chirp-like scalar equation

$$\ddot{x}(t) - \frac{2\,p'(t)}{p(t)}\dot{x}(t) + \left[1 + \frac{2p'^2(t)}{p^2(t)} - \frac{p''(t)}{p(t)}\right]x(t) = 0, \quad t \in [t_0, \infty),\ t_0 > 0,$$

$p : [t_0, \infty) \to \mathbb{R}$, of class C^2. The solution is $x(t) = C_1 p(t) \sin t + C_2 p(t) \cos t$.
For $p(t) \simeq t^{-\alpha}$, $x(t)$ is a linear combination of $(\alpha, 1)$-chirp-like functions near infinity.

$(\alpha, 1)$-chirp-like system

Substitutions $y = \dot{x}$ and $z = \frac{1}{t}$.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\left[1 + \frac{2p'^2(\frac{1}{z})}{p^2(\frac{1}{z})} - \frac{p''(\frac{1}{z})}{p(\frac{1}{z})}\right]x + \frac{2p'(\frac{1}{z})}{p(\frac{1}{z})}y \\ \dot{z} &= -z^2, \qquad z \in (0, \frac{1}{t_0}]. \end{aligned}$$

Box dimension of the 3D-system trajectories

Trajectory Γ of the solution of $(\alpha, 1)$-chirp-like system is, without loss of generality,

$$\begin{aligned} x(t) &= p(t)\sin t \\ y(t) &= p'(t)\sin t + p(t)\cos t \\ z(t) &= \frac{1}{t} \end{aligned}$$

Theorem (Trajectory in $\mathbb{R}^3$)

Let $p(t) \sim_3 t^{-\alpha}$ *as* $t \to \infty$, $\alpha > 0$.

(i) *Phase dimension of any solution of* $(\alpha, 1)$*-chirp-like scalar equation is equal to* $\dim_{ph}(x) = \frac{2}{1+\alpha}$ *for* $\alpha \in (0, 1)$.

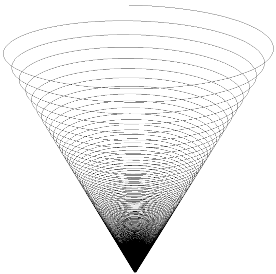
(ii) *Trajectory* Γ *of* $(\alpha, 1)$*-chirp-like system has box dimension* $\dim_B \Gamma = \frac{2}{1+\alpha}$ *for* $\alpha \in (0, 1)$.

(iii) *Trajectory* Γ *of* $(\alpha, 1)$*-chirp-like system for* $\alpha > 1$ *is rectifiable and* $\dim_B \Gamma = 1$.

3D-system trajectories 1/3

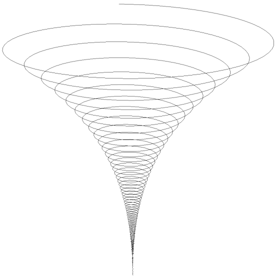
3D spiral trajectory of a solution of $(\alpha, 1)$-chirp-like system for $p(t) = t^{-\frac{1}{4}}$, Lipschitz case.

3D-system trajectories 2/3



3D spiral trajectory of a solution of $(\alpha, 1)$-chirp-like system for $p(t) = t^{-1}$, Lipschitz case.

3D-system trajectories 3/3



3D spiral trajectory of a solution of $(\alpha, 1)$-chirp-like system for $p(t) = t^{-3}$, Hölder case.

Poincaré map

Proposition (Poincaré map)

Assume Γ is the planar spiral that is the trajectory of any solution of $(\alpha, 1)$-chirp-like scalar equation near the origin. Let $P_\sigma : (0, \varepsilon_\sigma) \cap \Gamma \to (0, \varepsilon_\sigma) \cap \Gamma$ be the Poincaré map with respect to axis σ that passes through the origin.
Then map P_σ has the form $P_\sigma(r) = r + d_\sigma(r)$, where $-d_\sigma(r) \simeq r^{\frac{1}{\alpha}+1}$ as $r \to 0$.

Connection between the phase dimension and asymptotics of the Poincaré map of $(\alpha, 1)$-chirp-like scalar equation

- Phase dimension of any solution of $(\alpha, 1)$-chirp-like scalar equation is equal to $\dim_{ph}(x) = \frac{2}{1+\alpha}$ for $\alpha \in (0, 1)$.
- Map P_σ has the form $P_\sigma(r) = r + d_\sigma(r)$, where $-d_\sigma(r) \simeq r^{\frac{1}{\alpha}+1}$ as $r \to 0$.
- The connection is achieved indirectly through parameter α.

Box dimension of the clothoid

Definition

The **clothoid** or **Euler spiral** is a planar curve defined parametrically by

$$x(t) = \int_0^t \cos(s^2)\, ds,$$
$$y(t) = \int_0^t \sin(s^2)\, ds,$$

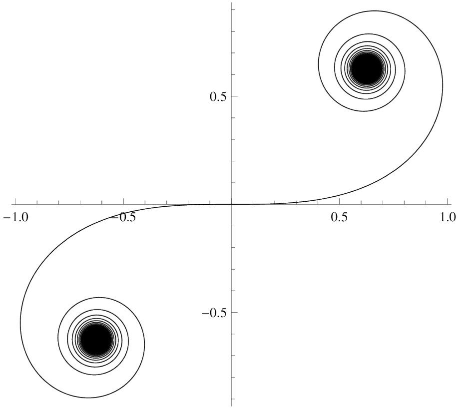
where $t \in \mathbb{R}$.

Theorem (Dimension of the clothoid (Korkut, Žubrinić and Županović (2009)))

Box dimension of the clothoid Γ is equal to $d = \frac{4}{3}$. Furthermore, Γ is Minkowski measurable and

$$\mathscr{M}^d(\Gamma) = 3 \cdot 2^{-\frac{2}{3}} \cdot \pi^{\frac{1}{3}}.$$

Clothoid or Euler spiral



The standard clothoid or Euler spiral Γ. Notice that $\dim_B \Gamma = \frac{4}{3}$.

Box dimension of the p-clothoid

Definition

By **p-clothoid**, $p > 1$, we mean a planar curve defined parametrically by

$$x(t) = \int_0^t \cos(s^p)\,ds, \quad y(t) = \int_0^t \sin(s^p)\,ds,$$

where $t \geq 0$.

Theorem (Dimension of p-clothoid (Korkut, Vlah, Žubrinić and Županović (2008)))

Let Γ_p be the p-clothoid, $p > 1$. Then $d = \dim_B \Gamma_p = \dfrac{2p}{2p-1}$. Furthermore, Γ_p is Minkowski measurable and

$$\mathscr{M}^d(\Gamma_p) = (2p-1)\left(p(p-1)^{p-1}\right)^{-2/(2p-1)} \pi^{1/(2p-1)}.$$

Box dimension of the q-clothoid

Definition (Clothoid generated by control function q - q-clothoid)

Let $q:(0,\infty)\to\mathbb{R}$ be a given function such that $q(t)\sim t^p$, $p>1$, when $t\to\infty$. By **the clothoid generated by control function q**, or **q-clothoid** Γ_q, we mean a planar curve defined parametrically by

$$\Gamma_q \cdots \begin{cases} x(t) &= \int_0^t \cos(q(s))\,ds \\ y(t) &= \int_0^t \sin(q(s))\,ds, \end{cases} \qquad \text{where } t\geq 0. \tag{7}$$

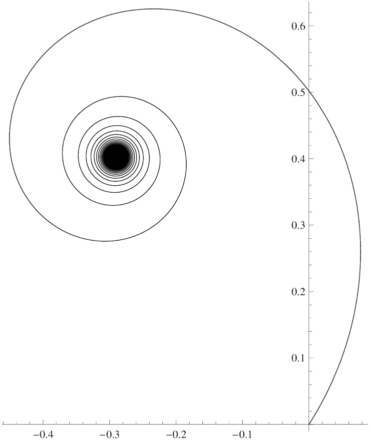
Theorem (Dimension of q-clothoid)

Assume that $q:(0,\infty)\to\mathbb{R}$ is increasing, convex, and of class C^5. Let

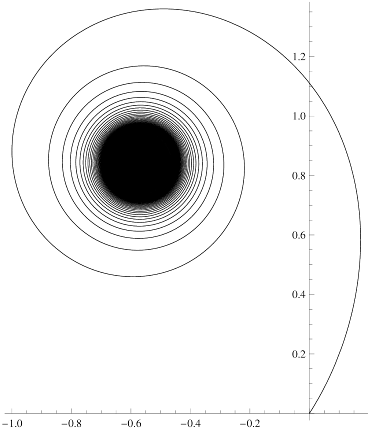
$$q(t)\sim_3 t^p,\quad q^{(4)}(t)=O(t^{p-4}),\quad q^{(5)}(t)=O(t^{p-5}),\quad \text{as } t\to\infty$$

be satisfied. Then $d=\dim_B\Gamma_q=\dfrac{2p}{2p-1}$. Furthermore, the spiral Γ_q is Minkowski measurable, and its d-dimensional Minkowski content is equal to the value $\mathcal{M}^d(\Gamma_p)$ from the previous theorem.

q-clothoids



q-clothoid for $q(s) = s^3 + 2s + 1$,
$\dim_B \Gamma_q = \frac{6}{5}$



q-clothoid for $q(s) = \sqrt{s^3 + 2s + 1}$,
$\dim_B \Gamma_q = \frac{3}{2}$

Main references

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