# LAPLACE TRANSFORMS AND EXPONENTIAL BEHAVIOR OF REPRESENTING MEASURES 

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#### Abstract

In this article behavior of measures on $[0, \infty)$ is studied by considering their Laplace transforms. We present a unified approach that covers many cases when Karamata's and de Haan's Tauberian theorems apply. If the Laplace transform can be extended to a complex half-plane containing the imaginary axis, we prove that the tail of the representing measure has exponential decay and establish the precise rate of the decay. We translate this result to the language of Bernstein functions and give two applications in the theory of non-local equations.


## 1. Introduction and main result

The Laplace transform represents a powerful integral transform in analysis. Besides transforming operations which appear naturally in analysis (e.g. convolution or differentiation) to simple algebraic operations, the real power of this integral transform can be seen through Tauberian theorems. Theorems belonging to this class determine asymptotic behavior of a function (or a measure) from asymptotic properties of its Laplace transform.

A classical example is Karamata's Tauberian theorem, which says that regular variation of the Laplace transform implies regular variation of the distribution function of the measure (cf. [Fel71, Section XIII.5] or [BGT87, Section 1.7]). In this article we formulate and prove a Tauberian type theorem that also treats measures with density that is not necessary of regular variation (e.g. densities with exponential or logarithmic behavior). In particular, we will see that exponential decay of density occurs when it is possible to extend its Laplace transform to a complex half-plane containing the imaginary axis.

Although the article is written analytically, our motivation comes from probability theory. When considering jump processes, it is often important to know the behavior of measures that govern jumps. Typical examples are rotationally invariant symmetric stable process and variance gamma process. The density of the jump measure of the former process does not have exponential decay, while the one of the latter process has. By using our approach one can obtain behavior of the jump measure of a large class of jump processes which includes previously mentioned examples.

[^0]Let us be more precise now. Recall that the Laplace transform of a measure $\nu$ on $[0, \infty)$ is a function $\mathcal{L} \nu:(0, \infty) \rightarrow[0, \infty]$ defined by

$$
(\mathcal{L} \nu)(\lambda):=\int_{[0, \infty)} e^{-\lambda t} \nu(d t) \text { for } \lambda>0 .
$$

If the integral converges for all $\lambda>0$, the Laplace transform belongs to the class of completely monotone functions, which is a class of functions defined by

$$
\begin{aligned}
\mathcal{C M}:=\{f:(0, \infty) \rightarrow \mathbb{R}: & f \text { is a } C^{\infty} \text { function and } \\
& \left.(-1)^{n} f^{(n)}(\lambda) \geqslant 0 \text { for all } n \in \mathbb{N} \cup\{0\}, \lambda>0\right\} .
\end{aligned}
$$

The converse is also true; for any $f \in \mathcal{C M}$ there exists a unique measure $\nu$ on $[0, \infty)$ so that $\mathcal{L} \nu=f$. These two statements are known as Bernstein's theorem (cf. [Fel71, Theorem XIII.4.1] or [SSV12, Theorem 1.4]). The measure $\nu$ will be called the representing measure of $f$.

Before stating our conditions and the main result, we introduce a notion of extension of a completely monotone function.

Let $f \in \mathcal{C M}$. Since all derivatives of $f$ are monotone, we can set $f^{(n)}(0+):=\lim _{\lambda \rightarrow 0+} f^{(n)}(\lambda) \in$ $[-\infty, \infty]$ for all $n \in \mathbb{N} \cup\{0\}$. Let

$$
\omega_{0}:=\omega_{0}^{f}:=\left\{\begin{array}{lc}
\inf \left\{\lambda \in \mathbb{R}: \sum_{n=0}^{\infty} \frac{f^{(n)}(0+)}{n!}(-\lambda)^{n} \text { converges }\right\} & \text { if }\left|f^{(n)}(0+)\right|<\infty \\
0 & \text { for all } n \in \mathbb{N} \cup\{0\} \\
& \text { otherwise }
\end{array}\right.
$$

where we have used the convention that the infimum of the empty set is infinite. Now we can define an extension $f_{e}:\left(\omega_{0}, \infty\right) \rightarrow(0, \infty)$ of the function $f$ by setting

$$
f_{e}(\lambda)= \begin{cases}\sum_{n=0}^{\infty} \frac{f^{(n)}(0+)}{n!}(-\lambda)^{n} & \omega_{0}<\lambda \leqslant 0  \tag{1.1}\\ f(\lambda) & \lambda>0\end{cases}
$$

Remark 1.1. It may happen that all right derivatives $f^{(n)}(0+)$ exist, but $\omega_{0}=0$. E.g. for $\alpha \in(0,1), \nu(d t):=e^{-t^{\alpha}} d t$ and $f(\lambda):=(\mathcal{L} \nu)(\lambda)=\int_{0}^{\infty} e^{-\lambda t-t^{\alpha}} d t$ it follows that $(-1)^{n} f^{(n)}(0+)=\int_{0}^{\infty} t^{n} e^{-t^{\alpha}} d t$ is finite for any $n \in \mathbb{N}$, but $\omega_{0}^{f}=0$, since

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(0+)}{n!}(-\lambda)^{n}=\int_{0}^{\infty} e^{-\lambda t-t^{\alpha}} d t=\infty \text { for all } \lambda<0
$$

Remark 1.2. In general theory of Laplace transform, $\omega_{0}$ is also known as the abscissa of convergence of the integral

$$
\begin{equation*}
\int_{[0, \infty)} e^{-z t} \nu(d t) \text { for } z \in \mathbb{C} . \tag{1.2}
\end{equation*}
$$

More precisely, the region of convergence of the integral (1.2) in $\mathbb{C}$ is the half-plane $M=$ $\left\{z \in \mathbb{C}: \Re z>\omega_{0}\right\}$ and the integral in (1.2) defines an analytic function on $M$ with
singularity at $z=\omega_{0}$ (cf. [Wid46, Corollary II. 1 and Theorems II.5a, II.5b]). This implies that the extension $f_{e}$ defined by (1.1) is the restriction of the analytic function

$$
F: M \rightarrow \mathbb{C}, \quad F(z):=\int_{[0, \infty)} e^{-z t} \nu(d t), z \in M
$$

to the real interval $\left(\omega_{0}, \infty\right)$. We show in Lemma 2.1 that $f_{e}(\lambda)=\int_{[0, \infty)} e^{-\lambda t} \nu(d t)$ for all $\lambda>\omega_{0}$ without using this general theory of Laplace transform.

Throughout the paper the following conditions concerning $f \in \mathcal{C} \mathcal{M}$ will be used:
(A-1) the representing measure $\nu$ of $f$ has a density with respect to the Lebesgue measure, i.e. there exists a function

$$
\nu:(0, \infty) \rightarrow(0, \infty) \text { such that } \nu(d t)=\nu(t) d t ;
$$

(A-2) $\omega_{0}>-\infty$ and $t \mapsto e^{-\omega_{0} t} t^{-1} \nu(t)$ is non-increasing;
(A-3) there exist constants $\theta>0,0 \leqslant \Lambda_{1}<\Lambda_{2} \leqslant \infty$ and $\gamma>0$ such that

$$
\frac{f_{e}^{\prime}\left(\lambda x+\omega_{0}\right)}{f_{e}^{\prime}\left(\lambda+\omega_{0}\right)} \leqslant \theta x^{-\gamma} \text { for all } x \geqslant 1 \text { and } \lambda \in\left(\Lambda_{1}, \Lambda_{2}\right) .
$$

Let us comment condition (A-3) (in Sections 3 and 4 we will see how to treat conditions (A-1) and (A-2)). A function $\psi:(0, \infty) \rightarrow(0, \infty)$ is said to vary regularly at infinity (at the origin) with index $\rho \in \mathbb{R}$ if

$$
\lim _{\substack{\lambda \rightarrow \infty \\(\lambda \rightarrow 0+}} \frac{\psi(\lambda x)}{\psi(\lambda)}=x^{\rho} \text { for all } x>0
$$

A function that varies regularly with index 0 is also said to vary slowly. By Potter's Theorem (cf. [BGT87, Theorem 1.5.6 (iii)]) it will follow that $f \in \mathcal{C} \mathcal{M}$ satisfies (A-3) if $f^{\prime}$ varies regularly with index $\rho<0$ at the origin (take $\Lambda_{1}=0$ and $\Lambda_{2}<\infty$ ) or at infinity (take $\Lambda_{1}>0$ and $\Lambda_{2}=\infty$ ).

Now we can state the main theorem.
Theorem 1.3. Let $f \in \mathcal{C M}$ and assume that it satisfies ( $A-1$ ).
(i) If (A-2) holds, then there is a constant $c_{1}>0$ such that

$$
\nu(t) \leqslant-c_{1} t^{-2} f_{e}^{\prime}\left(t^{-1}+\omega_{0}\right) e^{\omega_{0} t} \text { for all } t>0 .
$$

(ii) If (A-2) and ( $A-3$ ) hold, then there exist constants $c_{2}>0$ and $\delta \in(0,1)$ such that

$$
\nu(t) \geqslant-c_{2} t^{-2} f_{e}^{\prime}\left(t^{-1}+\omega_{0}\right) e^{\omega_{0} t} \quad \text { for all } t \in\left(\delta \Lambda_{2}^{-1}, \delta \Lambda_{1}^{-1}\right) .
$$

(iii) For any $\omega<\omega_{0}$

$$
\limsup _{t \rightarrow \infty} \frac{\nu(t)}{e^{\omega t}}=+\infty .
$$

Remark 1.4. (a) Theorem 1.3 (i)-(ii) says that exponential decay of the density of $\nu$ is connected to the fact that the Laplace transform of $\nu$ can be analytically extended to a half-plane in $\mathbb{C}$ that contains the imaginary axis $\{z \in \mathbb{C}: \Re z=0\}$.
(b) Theorem 1.3 (iii) implies that in case $\omega_{0}=0$ there is no exponential decay of $\nu$. Nevertheless, subexponential decay is possible, as Remark 1.1 shows.
(c) If $\omega_{0}=-\infty$, superexponential behavior can occur (e.g. $\nu(d t)=e^{-t^{2}} d t$ ).
(d) In case $\omega_{0}=0$, Theorem 1.3 represents a unified approach that covers many cases when classical Tauberian theorems apply (cf. Corollary 1.6 and the discussion following it).

In the following examples our Theorem 1.3 is useful, since no closed expression of the density of the representing measure is known.
Example 1.5. It can be checked that (A-1)-(A-3) hold in this example with $\Lambda_{1}=0$ and $\Lambda_{2}=\infty$ and thus Theorem 1.3 can be applied (cf. Remark 4.4).
(a) $f(\lambda)=\frac{1}{1+\lambda^{\alpha}} \quad(0<\alpha<1)$
$\nu(t)= \begin{cases}t^{-1+\alpha} & 0<t<1 \\ t^{-1-\alpha} & t \geqslant 1\end{cases}$
(b) $f(\lambda)=\frac{1}{1+\log (1+\lambda)}$
$\nu(t)= \begin{cases}\frac{t^{-1}}{\log \left(1+t^{-1}\right)^{2}} & 0<t<1 \\ e^{-\left(1-e^{-1}\right) t} & t \geqslant 1\end{cases}$
(c) $f(\lambda)=\frac{1}{1+\log \left(1+\lambda^{\alpha}\right)} \quad(0<\alpha<1)$
$\nu(t)= \begin{cases}\frac{t^{-1}}{\log \left(1+t^{-1}\right)^{2}} & 0<t<1 \\ t^{-1-\alpha} & t \geqslant 1\end{cases}$

As mentioned earlier, Theorem 1.3 covers some cases where classical Tauberian theorems also apply. Before discussing this, we record the following
Corollary 1.6. Let $f \in \mathcal{C} \mathcal{M}$ be such that $f^{\prime}$ varies regularly at infinity (at the origin) with index $-\rho-1$, where $0 \leqslant \rho \leqslant 1$.

If the representing measure of $f$ has a non-increasing density $\nu(t)$, then

$$
\nu(t)=-t^{-2} f^{\prime}\left(t^{-1}\right) \text { for all } t \in(0,1) \quad(\text { for all } t>1)
$$

Let $f \in \mathcal{C M}$ and assume that it satisfies the assumptions of Corollary 1.6. By [BGT87, Theorems 1.4.1 and 1.5.11 (and Proposition 1.5.9a if $\rho=0$ )],

$$
f(\lambda)= \begin{cases}\lambda^{-\rho} \ell(\lambda) & \rho>0 \\ \ell(\lambda) & \rho=0\end{cases}
$$

where $\ell:(0, \infty) \rightarrow(0, \infty)$ varies slowly at infinity (at the origin).
(a) For $0<\rho \leqslant 1$ we obtain the same behavior of the density as in Karamata's Tauberian theorem, since Corollary 1.6 implies

$$
\nu(t)=\rho t^{\rho-1} \ell\left(t^{-1}\right) \text { for all } t \in(0,1) \quad(\text { for all } t>1)
$$

(cf. [BGT87, Theorems 1.7.1 and 1.7.2]).
(b) Karamata's theory does not apply to the case $\rho=0$. Since $f^{\prime}$ varies regularly with index -1 , by the uniform convergence theorem for regularly varying functions (cf. [BGT87, Theorem 1.5.2]) we get

$$
\lim _{\substack{\lambda \rightarrow 0+\\(\lambda \rightarrow \infty)}} \frac{f\left(\frac{1}{\lambda x}\right)-f\left(\frac{1}{\lambda}\right)}{-\frac{1}{\lambda} f^{\prime}\left(\frac{1}{\lambda}\right)}=-\lim _{\substack{\lambda \rightarrow 0+\\(\lambda \rightarrow \infty)}} \int_{1}^{\frac{1}{x}} \frac{f^{\prime}\left(\frac{t}{\lambda}\right)}{f^{\prime}\left(\frac{1}{\lambda}\right)} d t=-\int_{1}^{\frac{1}{x}} \frac{d t}{t}=\log x
$$

for any $x>0$. Now de Haan's Tauberian theorem (cf. [BGT87, Theorems 3.6.8 and 3.9.1]) can be applied to obtain

$$
\lim _{\substack{t \rightarrow 0+\\(t \rightarrow \infty)}} \frac{\nu(t)}{-t^{-2} f^{\prime}\left(t^{-1}\right)}=1
$$

Note that these estimates are the same (up to constants) as in Corollary 1.6.
The following result can be considered as a converse to Theorem 1.3.
Theorem 1.7. Let $f \in \mathcal{C M}$ and assume that it satisfies ( $A-1$ ).
(i) If there exist $c>0, \alpha \in(0,1]$ and $\beta>0$ such that

$$
\nu(t) \leqslant c e^{-\beta t^{\alpha}} \quad \text { for all } t \geqslant 1,
$$

then $f^{(n)}(0+)$ is finite for any $n \in \mathbb{N}$. Moreover, in case $\alpha=1$, it follows that $\omega_{0} \leqslant-\beta$ and

$$
f_{e}(\lambda)= \begin{cases}\sum_{k=0}^{\infty} \frac{f^{(k)}(0+)}{k!}(-\lambda)^{k} & \lambda \in(-\beta, 0] \\ f(\lambda) & \lambda>0\end{cases}
$$

defines a $C^{\infty}$-extension of $f$ to $(-\beta, \infty)$.
(ii) If there exists $m \in \mathbb{N}$ such that

$$
\liminf _{t \rightarrow \infty} t^{m} \nu(t)>0 .
$$

then $f^{(n)}(0+)$ is infinite for any $n \geqslant m-1$.
Note that in case of subexponential behavior in Theorem 1.7, i.e. $\alpha \in(0,1)$, we can have $\omega_{0}^{f}=0$ (cf. Remark 1.1). If the density of the representing measure has a polynomial decay, then it cannot happen that all right derivatives $f^{(n)}(0+)$ are finite, as the following example shows.
Example 1.8. Let $f(\lambda)=\frac{\lambda}{1+\lambda}-2 \lambda \log \left(1+\lambda^{-1}\right)$. In this case it follows that (cf. [SSV12, p. 312, No. 46] and Section 4):

$$
\nu(t)=\frac{2-e^{-t}\left(t^{2}+2 t+2\right)}{t^{2}} \sim 2 t^{-2} \text { as } t \rightarrow \infty .
$$

A simple computation shows that $f(0+)=0$ and $f^{\prime}(0+)=-\infty$.
Theorem 1.3 has several applications. As a first application, we show that our result can be translated directly to the setting of Bernstein functions.

Another application is analysis of decay of solutions of some integro-differential equations in $\mathbb{R}^{d}$. To be more precise, let $f$ be a continuous function with a compact support. For a local equation

$$
-\Delta u+u=f \text { in } \mathbb{R}^{d}
$$

it is known that the solution has exponential decay (cf. [Eva98, pp. 187-188]).
Considering non-local equations, different behavior of solutions appears. For example, the solution to

$$
\begin{equation*}
(-\Delta)^{\alpha / 2} u+u=f \quad \text { in } \mathbb{R}^{d} \quad(0<\alpha<2) \tag{1.3}
\end{equation*}
$$

will not have exponential decay; we will see that $|u(x)| \leqslant c|x|^{-d-\alpha}$ for all $x \in \mathbb{R}^{d}$ and in the case of a non-negative $f$ it will follow that

$$
\limsup _{|x| \rightarrow \infty} \frac{u(x)}{e^{-\sigma|x|}}=\infty \text { for every } \sigma>0
$$

(cf. Example 5.4). On the other hand, the solution to

$$
\log (1-\Delta) u+u=f \text { in } \mathbb{R}^{d}
$$

will have exponential decay (cf. Example 5.5).
The last application of our results is the representation of the operators of the form $-\phi(-\Delta)$ as integral operators. Our approach covers many examples; among others, fractional Laplacian and relativistic Schrödinger operator. Essentially, the idea is to use subordination (of semigroups) to estimate the kernel of these integral operators. Although we use this idea in the setting of the Brownian semigroup, one can treat even more general semigroups and obtain relations between the domains of the infinitesimal generators of the semigroup and subordinate semigroup (cf. [Phi52, Hir72, Sch96]).

New examples in applications will be the ones with kernels that have exponential decay. In the representation of the operator $-\log (1-\Delta)$ precise asymptotic analysis of decay was first obtained in [ŠSV06] by using the probabilistic counterpart of subordination. This was a motivating example for investigation of a broader class of non-local operators having similar properties. In order to treat more examples, methods for obtaining behavior of the kernel of such non-local operators around the origin are also generalized in this paper (cf. [SV09, KSV12, KM12]) .

The paper is organized as follows. The main results are proved in Section 2. Sufficient conditions that guarantee existence of monotone density of the representing measure are main topic of Section 3. The main result of this section (cf. Proposition 3.1) may be also of independent interest. Another important class of functions, namely the class of Bernstein functions, is treated within this framework in Section 4. Applications to nonlocal potential equations are given in Section 5. In Section 6 we obtain asymptotical behavior of kernels of a class of non-local operators.
Notation. The Fourier transform of $f \in L^{1}\left(\mathbb{R}^{d}\right)$ is defined by

$$
\widehat{f}(\xi):=\int_{\mathbb{R}^{d}} e^{i \xi \cdot x} f(x) d x, \xi \in \mathbb{R}^{d}
$$

We use the same notation for the extension of the Fourier transform from $L^{1}\left(\mathbb{R}^{d}\right) \cap L^{2}\left(\mathbb{R}^{d}\right)$ to a unitary operator on $L^{2}\left(\mathbb{R}^{d}\right)$ (cf. [Fol99, Theorem 8.29]). The inverse Fourier transform is denoted by ${ }^{2}$.

We write $f(x)=g(x)$ for $x \in I$ if $\frac{f(x)}{g(x)}$ stays between two positive constants for every $x \in I$.

## 2. Completely monotone functions

The main goal of this section is to prove Theorem 1.3. We start with an auxiliary lemma that helps us to avoid general theory of Laplace transform (cf. Remark 1.2).

Lemma 2.1. Let $f \in \mathcal{C M}$ with the representing measure $\nu$ and let $f_{e}$ be its extension defined by (1.1). Then
(i) $f_{e} \in C^{\infty}\left(\omega_{0}, \infty\right)$
and for all $m \in \mathbb{N} \cup\{0\}$ and $\lambda>\omega_{0}$
(ii) $(-1)^{m} f_{e}^{(m)}(\lambda) \geqslant 0$
(iii) $f_{e}^{(m)}(\lambda)=(-1)^{m} \int_{[0, \infty)} t^{m} e^{-\lambda t} \nu(d t)$.

Proof. If $\omega_{0}=0$ there is nothing to prove, so we can assume that $\omega_{0}<0$.
(i) To check smoothness it is enough to check it at $\lambda=0$; this holds, since $f_{e}^{(m)}(0-)=$ $f^{(m)}(0+)$ for all $m \in \mathbb{N} \cup\{0\}$.
(ii), (iii) Let $m \in \mathbb{N} \cup\{0\}, n \in \mathbb{N}$ and $\lambda \in\left(-\omega_{0}, 0\right]$. Since $\omega_{0}<0$, it follows that $(-1)^{k} f^{(k)}(0+)=\int_{[0, \infty)} t^{k} \nu(d t)$ for every $k \in \mathbb{N} \cup\{0\}$ and so

$$
\begin{aligned}
(-1)^{m} f_{e}^{(m)}(\lambda) & =\sum_{k=0}^{\infty} \frac{(-1)^{m+k} f^{(m+k)}(0+)}{k!}(-\lambda)^{k} \\
& =\sum_{k=0}^{\infty} \int_{[0, \infty)} \frac{t^{m+k}(-\lambda)^{k}}{k!} \nu(d t)=\int_{[0, \infty)} t^{m} e^{-\lambda t} \nu(d t) \geqslant 0,
\end{aligned}
$$

where in the last equality we have used Beppo-Levi theorem.
Remark 2.2. Note that Lemma 2.1 (iii) shows, in particular, that

$$
f_{e}(\lambda)=\int_{[0, \infty)} e^{-\lambda t} \nu(d t) \text { for all } \lambda>\omega_{0}
$$

Proof of Theorem 1.3. (i) Let $\lambda>0$. Then

$$
\begin{align*}
-f_{e}^{\prime}\left(\lambda+\omega_{0}\right) & \geqslant \int_{0}^{\lambda^{-1}} e^{-\omega_{0} t} e^{-\lambda t} t \nu(t) d t & \quad \text { by Lemma } 2.1 \text { (iii) with } m=1] \\
& \geqslant \lambda \nu\left(\lambda^{-1}\right) e^{-\omega_{0} \lambda^{-1}} \int_{0}^{\lambda^{-1}} e^{-\lambda t} t^{2} d t &  \tag{A-2}\\
& \geqslant(3 e)^{-1} e^{-\omega_{0} t} \lambda^{-2} \nu\left(\lambda^{-1}\right) . & \quad[\text { by (A-2)] }
\end{align*}
$$

This gives the upper bound; take $t>0$ and set $\lambda=t^{-1}$ to deduce from the previous display that

$$
\begin{equation*}
\nu(t) \leqslant-3 e t^{-2} e^{\omega_{0} t} f_{e}^{\prime}\left(t^{-1}+\omega_{0}\right) \tag{2.1}
\end{equation*}
$$

(ii) Let $\delta \in(0,1)$ and $\lambda \in\left(\Lambda_{1}, \Lambda_{2}\right)$. Then

$$
\begin{array}{rlr}
\int_{\delta \lambda^{-1}}^{\infty} & e^{-\omega_{0} t} e^{-\lambda t} t \nu(t) d t= & \\
& =-f_{e}^{\prime}\left(\lambda+\omega_{0}\right)-\int_{0}^{\delta \lambda^{-1}} e^{-\omega_{0} t} e^{-\lambda t} t \nu(t) d t & \text { [by Lemma 2.1 (iii) with } m=1] \\
& \geqslant-f_{e}^{\prime}\left(\lambda+\omega_{0}\right)-3 e \int_{0}^{\delta \lambda^{-1}} e^{-\lambda t} t^{-1}\left(-f_{e}^{\prime}\left(t^{-1}+\omega_{0}\right)\right) d t & \\
& \geqslant-f_{e}^{\prime}\left(\lambda+\omega_{0}\right)-3 e \theta\left(-f_{e}^{\prime}\left(\lambda+\omega_{0}\right)\right) \lambda^{\gamma} \int_{0}^{\delta \lambda^{-1}} e^{-\lambda t} t^{\gamma-1} d t &  \tag{A-3}\\
& \geqslant-f_{e}^{\prime}\left(\lambda+\omega_{0}\right)-3 e \theta \gamma^{-1} \delta^{\gamma}\left(-f_{e}^{\prime}\left(\lambda+\omega_{0}\right)\right) . & {[\text { by (2.1)] (A-3)] }}
\end{array}
$$

Choosing $\delta \in(0,1)$ small enough so that $1-3 e \theta \gamma^{-1} \delta^{\gamma} \geqslant \frac{1}{2}$ one obtains

$$
\begin{align*}
\frac{1}{2}\left(-f_{e}^{\prime}\left(\lambda+\omega_{0}\right)\right) & \leqslant \int_{\delta \lambda^{-1}}^{\infty} e^{-\omega_{0} t} e^{-\lambda t} t \nu(t) d t \\
& \leqslant\left(\delta \lambda^{-1}\right)^{-1} \nu\left(\delta \lambda^{-1}\right) e^{-\omega_{0} \lambda^{-1}} \int_{\delta \lambda^{-1}}^{\infty} e^{-\lambda t} t^{2} d t  \tag{A-2}\\
& \leqslant 5 \delta^{-3}\left(\delta \lambda^{-1}\right)^{2} \nu\left(\delta \lambda^{-1}\right) e^{-\omega_{0} \delta \lambda^{-1}} .
\end{align*}
$$

Let $t \in\left(\delta \Lambda_{2}^{-1}, \delta \Lambda_{1}^{-1}\right)$. Then $\lambda=\delta t^{-1} \in\left(\Lambda_{1}, \Lambda_{2}\right)$ and thus the last display implies

$$
\begin{aligned}
\nu(t) & \geqslant \frac{\delta^{3}}{10} t^{-2}\left(-f_{e}^{\prime}\left(\delta t^{-1}+\omega_{0}\right)\right) e^{\omega_{0} t} \\
& \geqslant \frac{\delta^{\gamma+3}}{10 \theta} t^{-2}\left(-f_{e}^{\prime}\left(t^{-1}+\omega_{0}\right)\right) e^{\omega_{0} t} \quad[\mathrm{by}(\mathrm{~A}-3)]
\end{aligned}
$$

(iii) Assume that the claim is not true. Then there exist $\omega<\omega_{0}, c_{1}>0$ and $t_{0}>0$ so that

$$
\nu(t) \leqslant c_{1} e^{\omega t} \text { for all } t>t_{0} .
$$

By the dominated convergence theorem one concludes that $f^{(k)}(0+)$ is finite and $f^{(k)}(0+)=$ $\int_{0}^{\infty} t^{k} \nu(t) d t$ for all $k \in \mathbb{N} \cup\{0\}$. Using Beppo-Levi theorem it follows that

$$
\sum_{k=0}^{\infty} \frac{f^{(k)}(0+)}{k!}(-\lambda)^{k}=\int_{0}^{\infty} e^{-\lambda t} \nu(t) d t
$$

$$
\leqslant \nu\left(\left(0, t_{0}\right]\right)+c_{1} \int_{t_{0}}^{\infty} e^{\omega t} e^{-\lambda t} d t<\infty \text { for all } \lambda>\omega
$$

This is a contradiction with the definition of $\omega_{0}$.

Proof of Corollary 1.6. In this case (A-1) and (A-2) hold, since $\omega_{0}=0$. If $f^{\prime}$ varies regularly at infinity with index $-\rho-1$, Potter's Theorem (cf. [BGT87, Theorem 1.5.6 (iii)]) implies condition (A-3) with any $\gamma \in(0,1), \Lambda_{1}=1$ and $\Lambda_{2}=\infty$. Now we can apply Theorem 1.3. In case of regular variation at the origin the result can be obtained similarly (with $\Lambda_{1}=0$ and $\Lambda_{2}=1$ ).

Now the converse result will be proved.
Proof of Theorem 1.7. (i) By the assumption and dominated convergence theorem, $f^{(k)}(0+)$ is finite for any $k \in \mathbb{N}$ and

$$
(-1)^{k} f^{(k)}(0+)=\int_{0}^{\infty} t^{k} \nu(t) d t
$$

If $\alpha=1$, it follows that

$$
\sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} f^{(k)}(0+)=\int_{0}^{\infty} e^{-\lambda t} \nu(t) d t
$$

is finite for any $\lambda \in(-\beta, 0]$ and thus $\omega_{0} \leqslant-\beta$. Lemma 2.1 (i) implies that the extension $f_{e}$ is a $C^{\infty}$-function.
(ii) Let $n \geqslant m-1$. By the assumption there exist $t_{0} \geqslant 1$ and $c_{1}>0$ such that $\nu(t) \geqslant c_{1} t^{m}$ for all $t \geqslant t_{0}$. Then by Fatou's lemma

$$
(-1)^{n} f^{(n)}(0+) \geqslant \int_{0}^{t_{1}} \liminf _{\lambda \rightarrow 0+} e^{-\lambda t} t^{n} \nu(t) d t \geqslant c_{1} \int_{t_{0}}^{t_{1}} \frac{d t}{t} \rightarrow \infty \quad \text { as } \quad t_{1} \rightarrow \infty
$$

## 3. Existence of monotone densities

The purpose of this section is to investigate whether a measure on $[0, \infty)$ has a nonincreasing density by just considering its Laplace transform. This will be useful in checking condition (A-2).

Similar results already exist in literature. For example, in [Fel71, Corollary to Theorem 2 in XIII.4] sufficient and necessary conditions for existence of bounded densities are given.

On the other hand, in many cases of interest densities are unbounded. Here we give sufficient conditions for existence of non-increasing densities and discuss some particular cases when (A-2) holds.

Proposition 3.1. Let $f \in \mathcal{C M}$ satisfying (A-1) and assume that, for some $\lambda_{0}>0$, the following holds:
(i) $\frac{(-\lambda)^{n} f^{(n)}(\lambda)}{n!} \geqslant \frac{(-\lambda)^{n+1} f^{(n+1)}(\lambda)}{(n+1)!}$ for all $n \in \mathbb{N} \cup\{0\}$ and $\lambda>\lambda_{0}$;
(ii) $\lim _{\lambda \rightarrow \infty} f(\lambda)=0$.

Then the density of the representing measure of $f$ is a non-increasing function.
Proof. Let $\nu$ be the representing measure of $f$. The proof relies on the inversion formula for the Laplace transform (cf. [Fel71, Theorem 2 in XIII.2] or [SSV12, equation (1.3)]):

$$
\begin{equation*}
\nu((0, x])=\lim _{\lambda \rightarrow \infty} \sum_{n \leqslant \lambda x} \frac{(-\lambda)^{n} f^{(n)}(\lambda)}{n!} \text { for } x>0 \tag{3.1}
\end{equation*}
$$

Here we have used that every $x>0$ is a point of continuity of $\nu$, i.e. $\mu(\{x\})=0$, since $\nu$ has a density.

Let $F:(0, \infty) \rightarrow \mathbb{R}$ be defined by $F(x):=\nu((0, x])$ for $x>0$. The idea is to prove that $F$ is concave. Assume, for the moment, that this is done.

Since $F$ is non-decreasing, it is almost everywhere differentiable and its derivative $F^{\prime}$ satisfies

$$
F(x)=\int_{0}^{x} F^{\prime}(t) d t \text { for } x>0
$$

(cf. [Fol99, Theorem 3.23, Corollary 3.33]). On the other hand, concavity of $F$ implies that

$$
\mu(t):=\lim _{h \rightarrow 0+} \frac{F(t+h)-F(t)}{h}, t>0
$$

is a well defined non-increasing function. Since $F^{\prime}(t)=\mu(t)$ almost everywhere, it follows that $\mu$ is the density of the representing measure:

$$
\nu((0, x])=F(x)=\int_{0}^{x} \mu(t) d t
$$

It is left to prove that $F$ is concave. Take $x, y>0$ such that $x<y$. By the inversion formula (3.1),

$$
\begin{equation*}
\frac{F(x)+F(y)}{2}=\lim _{\lambda \rightarrow+\infty}\left[\sum_{n \leqslant \lambda \frac{x+y}{2}} \frac{(-\lambda)^{n} f^{(n)}(\lambda)}{n!}+I(x, y, \lambda)\right] \tag{3.2}
\end{equation*}
$$

with

$$
I(x, y, \lambda)=\frac{1}{2} \sum_{\lambda \frac{x+y}{2}<n \leqslant \lambda y} \frac{(-\lambda)^{n} f^{(n)}(\lambda)}{n!}-\frac{1}{2} \sum_{\lambda x<n \leqslant \lambda \frac{x+y}{2}} \frac{(-\lambda)^{n} f^{(n)}(\lambda)}{n!} .
$$

The number of summands in the sums in $I(x, y, \lambda)$ differs by at most one. This and condition (i) give

$$
I(x, y, \lambda) \leqslant \frac{(-\lambda)^{n_{+}} f^{(n+)}(\lambda)}{2 n_{+}!} \text {or } I(x, y, \lambda) \leqslant-\frac{(-\lambda)^{n_{-}} f^{(n-)}(\lambda)}{2 n_{-}!}
$$

with $n_{+}:=\lfloor\lambda y\rfloor$ and $n_{-}:=\left\lfloor\lambda \frac{x+y}{2}\right\rfloor$. In both cases, (i) implies $I(x, y, \lambda) \leqslant \frac{f(\lambda)}{2}$, which together with (ii) and (3.2) yields mid-point concavity:

$$
\frac{F(x)+F(y)}{2} \leqslant \lim _{\lambda \rightarrow \infty}\left[\sum_{n \leqslant \lambda \frac{x+y}{2}} \frac{(-\lambda)^{n} f^{(n)}(\lambda)}{n!}+\frac{1}{2} f(\lambda)\right]=F\left(\frac{x+y}{2}\right) .
$$

Therefore

$$
\begin{equation*}
F((1-\vartheta) x+\vartheta y) \geqslant(1-\vartheta) F(x)+\vartheta F(y) \tag{3.3}
\end{equation*}
$$

for all dyadic rational numbers $\vartheta \in D:=\left\{\frac{j}{2^{m}}: m \in \mathbb{N}, j \in\left\{0,1, \ldots, 2^{m}\right\}\right\}$ and $x, y>0$.
Let $\vartheta \in[0,1]$ and $x, y>0, x<y$. Choose a non-increasing sequence $\left(\vartheta_{n}\right)_{n}$ in $D$ so that $\lim _{n} \vartheta_{n}=\vartheta$. Then (3.3) and right-continuity of $F$ yield concavity:

$$
\begin{aligned}
F((1-\vartheta) x+\vartheta y) & =F(x+\vartheta(y-x))=\lim _{n} F\left(x+\vartheta_{n}(y-x)\right) \\
& \geqslant \lim _{n}\left(\left(1-\vartheta_{n}\right) F(x)+\vartheta_{n} F(y)\right)=(1-\vartheta) F(x)+\vartheta F(y) .
\end{aligned}
$$

Remark 3.2. (a) Condition (i) implies that $(-\lambda)^{n} f^{(n)}(\lambda) \geqslant \lambda^{-1}$ for all $\lambda>\lambda_{0}$ and $n \in \mathbb{N} \cup\{0\}$. In particular, $f(\lambda)=e^{-\lambda}=\int_{[0, \infty)} e^{-\lambda t} \delta_{\{1\}}(d t)$ does not satisfy this condition. (b) Condition (i) is equivalent to

$$
1+\frac{\lambda}{n+1} \frac{f^{(n+1)}(\lambda)}{f^{(n)}(\lambda)} \geqslant 0 .
$$

Noting that

$$
(n+1) f^{(n)}(\lambda)+\lambda f^{(n+1)}(\lambda)=(\lambda f(\lambda))^{(n+1)},
$$

it follows that $(\lambda f(\lambda))^{(n+1)}$ must have the same sign as $f^{(n)}(\lambda)$. Thus (i) is equivalent to

$$
(-1)^{n}(\lambda f(\lambda))^{(n+1)} \geqslant 0 \text { for all } n \in \mathbb{N} \cup\{0\} \text { and } \lambda>\lambda_{0} .
$$

In particular, (i) holds if $\lambda f(\lambda)$ is a Bernstein function (cf. Section 4 for a definition).
Corollary 3.3. Let $f \in \mathcal{C} \mathcal{M}$. Define a function $g:(0, \infty) \rightarrow \mathbb{R}$ by

$$
g(\lambda):=f_{e}\left(\lambda+\omega_{0}\right), \lambda>0 .
$$

Assume that $g$ satisfies the assumptions of Proposition 3.1. Then the representing measure of $f$ has a density $\nu(t)$ such that

$$
t \mapsto e^{-\omega_{0} t} \nu(t) \quad \text { is non-increasing. }
$$

Proof. It follows from the assumption and Proposition 3.1 that the representing measure of $g$ has a non-increasing density. Then by Lemma 2.1 (iii) we obtain

$$
\int_{0}^{\infty} e^{-\lambda t} \eta(t) d t=g(\lambda)=f_{e}\left(\lambda+\omega_{0}\right)=\int_{[0, \infty)} e^{-\lambda t} e^{-\omega_{0} t} \nu(d t) \text { for all } \lambda>0 .
$$

The uniqueness of Laplace transform (cf. [SSV12, Proposition 1.2]) implies that $e^{-\omega_{0} t} \nu(d t)=$ $\eta(t) d t$, which implies that the representing measure of $f$ has density $\nu(t)$ and

$$
t \mapsto e^{-\omega_{0} t} \nu(t)=\eta(t) \text { is non-increasing. }
$$

## 4. Bernstein functions

In some situations it is more convenient to state Theorem 1.3 for Bernstein functions, that is, the following class of functions

$$
\begin{aligned}
\mathcal{B} \mathcal{F}:=\{\phi:(0, \infty) \rightarrow[0, \infty): & \phi \text { is a } C^{\infty} \text { function and } \\
& \left.(-1)^{n-1} \phi^{(n)}(\lambda) \geqslant 0 \text { for all } n \in \mathbb{N}, \lambda>0\right\} .
\end{aligned}
$$

Every $\phi \in \mathcal{B F}$ can be uniquely represented by a triple ( $a, b, \mu$ ), where $a, b \geqslant 0$ and $\mu$ is a measure on $(0, \infty)$ satisfying $\int_{(0, \infty)}(1 \wedge t) \mu(d t)<\infty$, in the following way (cf. [SSV12, Theorem 3.2]):

$$
\begin{equation*}
\phi(\lambda)=a+b \lambda+\int_{(0, \infty)}\left(1-e^{-\lambda t}\right) \mu(d t) \text { for } \lambda>0 . \tag{4.1}
\end{equation*}
$$

The measure $\mu$ will be called the Lévy measure of $\phi$. It follows from the definition that $\phi \in \mathcal{B F}$ implies $\phi^{\prime} \in \mathcal{C} \mathcal{M}$. Taking derivative in (4.1) one gets

$$
\begin{equation*}
\phi^{\prime}(\lambda)=b+\int_{(0, \infty)} e^{-\lambda t} t \mu(d t) \text { for } \lambda>0, \tag{4.2}
\end{equation*}
$$

which shows that the representing measure of $\phi^{\prime}$ is given by $b \delta_{\{0\}}+t \mu(d t)$.
Let $\phi \in \mathcal{B F}$. Since derivatives of $\phi$ are monotone, we can define an extension $\phi_{e}$ similarly as in (1.1). Then $\phi_{e} \in C^{\infty}(-\beta, \infty)$ with $\beta:=-\omega_{0}^{\phi^{\prime}}$. Using similar ideas as in the proof of Lemma 2.1 it can be proved that $f(\lambda):=\phi_{e}\left(\lambda+\omega_{0}\right)$ is an extended Bernstein function in the sense of [SSV12, Remark 5.9], that is, $f:(0, \infty) \rightarrow \mathbb{R}$ is a $C^{\infty}$ function satisfying $(-1)^{n-1} f^{(n)}(\lambda) \geqslant 0$ for all $n \in \mathbb{N}$ and $\lambda>0$.

These observations yield the following corollary to Theorem 1.3.
Corollary 4.1. Let $\phi \in \mathcal{B F}$ and set $\beta:=-\omega_{0}^{\phi^{\prime}}$. Assume that its Lévy measure has a density $\mu(t)$ such that $t \mapsto e^{\beta t} \mu(t)$ is a non-increasing function. Let $\phi_{e}$ be an extension of $\phi$ defined by (1.1).
(i) There is a constant $c_{1}>0$ such that

$$
\mu(t) \leqslant-c_{1} t^{-3} \phi_{e}^{\prime \prime}\left(t^{-1}-\beta\right) e^{-\beta t} \text { for all } t>0 .
$$

(ii) If there exist constants $\theta>0,0 \leqslant \Lambda_{1}<\Lambda_{2} \leqslant \infty$ and $\gamma>0$ such that

$$
\frac{\phi_{e}^{\prime \prime}(\lambda x-\beta)}{\phi_{e}^{\prime \prime}(\lambda-\beta)} \leqslant \theta x^{-\gamma} \text { for all } x \geqslant 1 \text { and } \lambda \in\left(\Lambda_{1}, \Lambda_{2}\right)
$$

then there exist constants $c_{2}>0$ and $\delta \in(0,1)$ such that

$$
\mu(t) \geqslant-c_{2} t^{-3} \phi_{e}^{\prime \prime}\left(t^{-1}-\beta\right) e^{-\beta t} \text { for all } t \in\left(\delta \Lambda_{2}^{-1}, \delta \Lambda_{1}^{-1}\right) .
$$

In order to use this result, the class of complete Bernstein functions will be useful:

$$
\begin{gathered}
\mathcal{C B F}:=\{\phi \in \mathcal{B F}: \text { the Lévy measure } \mu \text { in the representation (4.1) } \\
\text { has a completely monotone density }\} .
\end{gathered}
$$

The class $\mathcal{C B F}$ is closed under composition, pointwise convergence of functions and the following property holds:

$$
\begin{equation*}
\phi \in \mathcal{C B F}, \phi \not \equiv 0 \text { if and only if } \phi^{\star}(\lambda):=\frac{\lambda}{\phi(\lambda)} \text { is in } \mathcal{C B F} \tag{4.3}
\end{equation*}
$$

(cf. [SSV12, Proposition 7.1, Corollary 7.6]).
Before giving some examples, we will prove a lemma that can be useful in checking the assumptions of Corollary 4.1.

Lemma 4.2. Let $\phi \in \mathcal{B F}$ be such that $a=b=0$ in the representation (4.1) and set $\beta:=-\omega_{0}^{\phi^{\prime}} \geqslant 0$. If $h_{a}(\lambda):=\phi_{e}(\lambda+a)-\phi_{e}(a)$ is in $\mathcal{C B F}$ for any $a>-\beta$ then the Lévy measure of $\phi$ has a density $\mu(t)$ such that

$$
\begin{equation*}
t \mapsto e^{\beta t} \mu(t) \text { is in } \mathcal{C M} . \tag{4.4}
\end{equation*}
$$

Proof. Denote the density of the Lévy measure of $h_{a}$ by $\rho_{a} \in \mathcal{C} \mathcal{M}$. By the uniqueness of the Laplace transform (cf. [SSV12, Proposition 1.2]) and

$$
\int_{0}^{\infty} e^{-\lambda t} t \rho_{a}(t) d t=h_{a}^{\prime}(\lambda)=\phi_{e}^{\prime}(\lambda+a)=\int_{0}^{\infty} e^{-\lambda t} e^{-a t} t \mu(d t) \text { for } t>0,
$$

it follows that $e^{-a t} \mu(d t)=\rho_{a}(t) d t$. Therefore, $\mu$ has a density $\mu(t)$ and $e^{-a t} \mu(t)=\rho_{a}(t)$ is in $\mathcal{C M}$. Since $\mathcal{C M}$ is closed under pontwise convergence of functions,

$$
e^{\beta t} \mu(t)=\lim _{a \rightarrow(-\beta)+} \rho_{a}(t) \text { is in } \mathcal{C M}
$$

Example 4.3. Let us consider some functions belonging to the class $\mathcal{C B F}$. In parts (b) and (c) closed expressions of Lévy densities are not known.
(a) $\phi(\lambda)=\log (1+a \lambda)$ is in $\mathcal{C B F}$ for any $a>0$, since

$$
\log (1+a \lambda)=\int_{0}^{\infty}\left(1-e^{-\lambda t}\right) \frac{e^{-a^{-1} t}}{t} d t
$$

(b) $\phi(\lambda)=\log (1+\log (1+\lambda))$ is in $\mathcal{C B F}$, since this class is closed under the operation of composition of functions.
In this case $\beta=1-e^{-1}$. Using the part (a) it follows that

$$
\phi_{e}(\lambda+a)-\phi_{e}(a)=\log \left(1+\frac{\log \left(1+\frac{\lambda}{1+a}\right)}{1+\log (1+a)}\right) \text { is in } \mathcal{C B F}
$$

for all $a>-\beta$ and so we can apply Lemma 4.2. Furthermore,

$$
-\phi_{e}^{\prime \prime}(\lambda-\beta)=\frac{1+[\log (1+e \lambda)]^{-1}}{\left(e^{-1}+\lambda\right)^{2} \log (1+e \lambda)}= \begin{cases}\lambda^{-2} & 0<\lambda \leqslant 1 \\ \lambda^{-2}[\log (1+\lambda)]^{-1} & \lambda>1\end{cases}
$$

Thus Corollary 4.1 can be applied (with $\Lambda_{1}=0, \Lambda_{2}=\infty$ and any $\gamma \in(0,2)$ ) to obtain

$$
\mu(t)= \begin{cases}\frac{1}{t \log \frac{1}{t}} & 0<t<1 \\ \frac{e^{-\left(1-e^{-1}\right) t}}{t} & t \geqslant 1 .\end{cases}
$$

(c) Example (b) can be generalized in the folowing way: define

$$
\phi_{1}(\lambda):=\log (1+\lambda) \text { and } \phi_{n+1}:=\phi_{n} \circ \phi_{1} \text { for } n \in \mathbb{N} .
$$

Using the approach from (b) it follows that the Lévy density $\mu_{n}(t)$ of $\phi_{n}$ satisfies

$$
\mu_{n}(t)= \begin{cases}\frac{1}{t} \prod_{i=1}^{n-1} \frac{1}{\phi_{k}\left(t^{-1}\right)} & 0<t<1 \\ \frac{e^{\phi}{ }^{-1}(-1) t}{t} & t \geqslant 1\end{cases}
$$

Note that $\lim _{n} \phi_{n}^{-1}(-1)=0$.
Remark 4.4. Now we can explain why the functions in Example 1.5 belong to the class $\mathcal{C M}$. Since $\lambda^{\alpha}, \log (1+\lambda)$ are in $\mathcal{C B F}$ and $\frac{1}{1+\lambda}$ is in $\mathcal{C} \mathcal{M}$, by using the fact that $\mathcal{C M} \circ \mathcal{B F} \subset$ $\mathcal{C M}$ (cf. [SSV12, Theorem 3.7]) it follows that $f \in \mathcal{C} \mathcal{M}$ in all examples. Noting that $\frac{\lambda}{1+\phi(\lambda)}$ is in $\mathcal{C B F}$, it follows that for some $a, b \geqslant 0$ and $\eta \in \mathcal{C M}$

$$
\frac{1}{1+\phi(\lambda)}=\frac{a}{\lambda}+b+\int_{0}^{\infty} \frac{1-e^{-\lambda t}}{\lambda} \eta(t) d t=\frac{a}{\lambda}+b+\int_{0}^{\infty} e^{-\lambda t} \nu(t, \infty) d t
$$

where $\nu$ is the measure with density $\eta$. Since $f(0)=1$, we conclude that $a=0$. Also, from $\lim _{\lambda \rightarrow \infty} f(\lambda)=0$ we get $b=0$. Thus, (A-1) holds for $f$.

To check (A-2), it is enough to note that $\lambda f_{e}(\lambda+a)$ is in $\mathcal{C B F}$ for every $a>\omega_{0}$. Since $\mathcal{C B F}$ is closed under pointwise convergence of functions, $\lambda f_{e}\left(\lambda+\omega_{0}\right)$ is in $\mathcal{C B F}$. Now we can use Remark 3.2 (b) and Corollary 3.3.

## 5. Applications to non-LOCAL EqUATIONS

As a first application of the main result we consider decay of solutions of the equation

$$
\begin{equation*}
\phi(-\Delta) u+u=f \text { in } \mathbb{R}^{d} \tag{5.1}
\end{equation*}
$$

where $\Delta$ is the Laplacian in $\mathbb{R}^{d}$ and $\phi \in \mathcal{B F}$.
The operator $\phi(-\Delta)$ should be understood in terms of the Fourier transform (as a pseudo-differential operator):

$$
\begin{equation*}
\phi \widehat{(-\Delta)} u(\xi)=\phi\left(|\xi|^{2}\right) \widehat{u}(\xi), \xi \in \mathbb{R}^{d} \tag{5.2}
\end{equation*}
$$

for $u \in D(\phi(-\Delta)):=\left\{u \in L^{2}\left(\mathbb{R}^{d}\right): \phi\left(|\xi|^{2}\right) \widehat{u}(\xi)\right.$ is in $\left.L^{2}\left(\mathbb{R}^{d}\right)\right\}$. Similarly as in Remark 4.4 one concludes that $\Phi(\lambda):=\frac{1}{1+\phi(\lambda)}$ is in $\mathcal{C} \mathcal{M}$. Let $\nu$ be its representing measure.

Assume in the rest of this section that $a=b=0$ in the representation (4.1) of $\phi$. Then $\nu(\{0\})=\lim _{\lambda \rightarrow \infty} \Phi(\lambda)=0$.

The fundamental solution of the equation (5.1) is a function $K: \mathbb{R}^{d} \backslash\{0\} \rightarrow \mathbb{R}$ defined by

$$
K(x)=\int_{(0, \infty)} p(t, x) \nu(d t), x \in \mathbb{R}^{d} \backslash\{0\}
$$

where $p:(0, \infty) \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ is the Gauss-Weierstrass kernel given by

$$
p(t, x):=(4 \pi t)^{-d / 2} e^{-\frac{|x|^{2}}{4 t}}
$$

Remark 5.1. (a) For every $t>0$

$$
\int_{\mathbb{R}^{d}} p(t, x) d x=1 \quad \text { and } \quad \widehat{p(t, \cdot)}(\xi)=e^{-t|\xi|^{2}}
$$

(b) The fundamental solution belongs to $L^{1}\left(\mathbb{R}^{d}\right)$, since $\int_{\mathbb{R}^{d}} K(x) d x=\nu(0, \infty)=\lim _{\lambda \rightarrow 0+} \Phi(\lambda)=$

1. Therefore, $\widehat{K}(\xi)=\int_{(0, \infty)} e^{-t|\xi|^{2}} \nu(d t)=\Phi(\xi)$.

If $f \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{2}\left(\mathbb{R}^{d}\right)$, then

$$
\begin{equation*}
u(x):=(K \star f)(x):=\int_{\mathbb{R}^{d}} K(x-y) f(y) d y \text { for } x \in \mathbb{R}^{d} \tag{5.3}
\end{equation*}
$$

defines the unique solution of (5.1). Indeed, by Remark 5.1 (b) and (5.2),

$$
[\phi(-\Delta) u] \hat{(\xi)}+\widehat{u}(\xi)=\left(\phi\left(|\xi|^{2}\right)+1\right) \hat{K}(\xi) \widehat{f}(\xi)=\widehat{f}(\xi)
$$

The main result of this section gives some estimates of the fundamental solution.
Proposition 5.2. Let $\phi \in \mathcal{B F}$ such that $a=b=0$ in its representation (4.1). Assume that $\Phi(\lambda):=\frac{1}{1+\phi(\lambda)}$ satisfies $(A-1)$ and (A-2) and let $\kappa:=\sqrt{-\omega_{0}^{\Phi}}$.
(i) If $\kappa>0$, then there is a constant $c_{1}>0$ such that

$$
K(x) \leqslant c_{1}|x|^{-\frac{d+3}{2}} \Phi_{e}^{\prime}\left(2 \kappa|x|^{-1}-\kappa^{2}\right) e^{-\kappa|x|} \quad \text { for all }|x| \geqslant 1
$$

(ii) In the case $\kappa=0$, there exists a constant $c_{2}>0$ such that

$$
K(x) \leqslant c_{2} r^{-d-2} \Phi^{\prime}\left(r^{-2}\right) \text { for all } x \in \mathbb{R}^{d}, x \backslash\{0\} .
$$

Moreover, for any $\sigma>0$

$$
\limsup _{|x| \rightarrow+\infty} \frac{K(x)}{e^{-\sigma|x|}}=+\infty .
$$

Remark 5.3. If $\lambda \mapsto \phi_{e}\left(\lambda-\kappa^{2}\right)$ is in $\mathcal{C B F}$, then so is $\lambda \Phi_{e}\left(\lambda-\kappa^{2}\right)=\frac{\lambda}{1+\phi_{e}\left(\lambda-\kappa^{2}\right)}$ by (4.3). Remark 3.2 (b) implies that the assumptions (A-1) and (A-2) hold.

Proof of Proposition 5.2. (i) Applying Proposition 1.3 (i) to $\Phi$ it follows that the density of the representing measure satisfies

$$
\nu(t) \leqslant-c_{1} t^{-2} \Phi_{e}^{\prime}\left(t^{-1}-\kappa^{2}\right) e^{-\kappa^{2} t} \text { for } t>0 .
$$

We aim to apply Lemma A. 1 with $f(t):=-t^{-2} \Phi_{e}^{\prime}\left(t^{-1}-\kappa^{2}\right), a:=\frac{d}{2}, b:=\kappa$ and $c:=2$. By Lemma 2.1 (ii) it follows that $-\Phi_{e}^{\prime}$ is non-increasing, which implies that $t \mapsto t^{2} f(t)$ is non-decreasing.

It is left to check that $f$ is non-increasing. Here we use some ideas from the proof of [SSV12, Theorem 11.3]. Assumptions (A-1) and (A-2) imply that $t \mapsto e^{\kappa^{2}} \nu(t)$ is a non-increasing function. Let $q:=\lim _{t \rightarrow \infty} e^{\kappa^{2} t} \nu(t)$ and let $\mu$ be a measure on $(0, \infty)$ defined by

$$
\mu(t, \infty):=e^{\kappa^{2} t} \nu(t)-q, t>0 .
$$

By Lemma 2.1 (iii),

$$
\begin{aligned}
\Phi_{e}\left(\lambda-\kappa^{2}\right) & =\int_{0}^{\infty} e^{-\lambda t} e^{\kappa^{2} t} \nu(t) d t \\
& =\frac{q}{\lambda}+\int_{0}^{\infty} e^{-\lambda t} \mu(t, \infty) d t=\frac{q}{\lambda}+\int_{(0, \infty)} \frac{1-e^{-\lambda t}}{\lambda} \mu(d t)
\end{aligned}
$$

and so

$$
-\lambda^{2} \Phi_{e}^{\prime}\left(\lambda-\kappa^{2}\right)=q+\int_{(0, \infty)}\left(1-(1+\lambda t) e^{-\lambda t}\right) \mu(d t)
$$

Since $\lambda \mapsto 1-(1+\lambda t) e^{-\lambda t}$ is non-decreasing for every $t>0$, it follows that $f$ is nonincreasing.
(ii) By Theorem 1.3 (i),

$$
\nu(t) \leqslant-c_{1} t^{-2} \Phi^{\prime}\left(t^{-1}\right)=c_{1} \frac{t^{-2} \phi^{\prime}\left(t^{-1}\right)}{\left(1+\phi\left(t^{-1}\right)\right)^{2}} \text { for } t>0 .
$$

The idea is to apply Lemma A. 2 with $f(t):=\frac{t^{-2} \phi^{\prime}\left(t^{-1}\right)}{\left(1+\phi\left(t^{-1}\right)\right)^{2}}, a:=\frac{d}{2}$ and $c:=2$. The assumptions that $f$ is non-increasing and $t \mapsto t^{2} f(t)$ is non-decreasing can be checked
similarly as in part (i). If $d=1,2$, then for all $t>0$ and $x \geqslant 1$,

$$
\frac{f(t x)}{f(t)}=x^{-2} \frac{\phi^{\prime}\left(t^{-1} x\right)}{\phi^{\prime}\left(t^{-1}\right)}\left(\frac{1+\phi\left(t^{-1}\right)}{1+\phi\left(t^{-1} x\right)}\right)^{2} \leqslant x^{-2}
$$

since $\phi$ is non-decreasing and $\phi^{\prime}$ is non-increasing. Therefore, $\gamma^{\prime}=2>1-\frac{d}{2}$ and we can apply the lemma with $R=0$.

Let $\sigma>0$. Since $\nu(t)$ is non-increasing,

$$
\frac{K(x)}{e^{-\sigma|x|}} \geqslant e^{\sigma|x|} \nu\left(2 \sigma^{-1}|x|\right) \int_{\sigma^{-1}|x|}^{2 \sigma^{-1}|x|} t^{-d / 2} e^{-\frac{|x|^{2}}{4 t}} d t \geqslant c_{3}|x|^{-\frac{d}{2}+1} \nu\left(2 \sigma^{-1}|x|\right) e^{\frac{\sigma|x|}{2}}
$$

Now we can use Theorem 1.3 (iii) to conclude that $\lim \sup e^{\sigma|x|} K(x)=\infty$.

$$
|x| \rightarrow \infty
$$

Proposition 5.2 can be used to investigate decay of the solution to (5.1). Two examples of non-local equations with different behavior of solutions will be given.

Example 5.4. Let $0<\alpha<2$ and consider

$$
(-\Delta)^{\alpha / 2} u+u=f
$$

with $f \in C_{c}\left(\mathbb{R}^{d}\right)$.
Here $\Phi(\lambda)=\frac{1}{1+\lambda^{\alpha / 2}}$ and $\kappa=0$. Proposition 5.2 (ii) implies that $K(x) \leqslant|x|^{-d-\alpha}$ for $x \in \mathbb{R}^{d} \backslash\{0\}$ and thus, by (5.3),

$$
|u(x)| \leqslant c|x|^{-d-\alpha} \quad \text { for } \quad x \in \mathbb{R}^{d}
$$

Moreover, if $f$ is non-negative, then for any $\sigma>0$

$$
\limsup _{|x| \rightarrow \infty} \frac{u(x)}{e^{-\sigma|x|}}=\infty
$$

in other words, the solution does not have exponential decay.
Example 5.5. For the equation

$$
\log (1-\Delta) u+u=f \text { in } \mathbb{R}^{d}
$$

with $f \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ we have

$$
\Phi_{e}(\lambda)=\frac{1}{1+\log (1+\lambda)} \text { for } \lambda>\omega_{0}^{\Phi}=e^{-1}-1
$$

Proposition 5.2 (i) implies that the solution has an exponential decay; for every $a<$ $\sqrt{1-e^{-1}}$ there exists a constant $c=c(a)>0$ so that

$$
K(x) \leqslant c e^{-a|x|} \text { for }|x| \geqslant 1
$$

This implies

$$
|u(x)| \leqslant c e^{-a|x|} \text { for } x \in \mathbb{R}^{d}
$$

## 6. Kernels and non-LOCAL operators

A second application of the main result is a representation of the operator $-\phi(-\Delta)$ on $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. Since we have assumed $a=b=0$ in the representation $(4.1),-\phi(-\Delta)$ will be a non-local operator (cf. (6.2)).

Define an operator $A$ on $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ by

$$
A u(x):=\lim _{\varepsilon \rightarrow 0+} \int_{|y|>\varepsilon}(u(x+y)-u(x)) J(y) d y
$$

with $J: \mathbb{R}^{d} \backslash\{0\} \rightarrow(0, \infty)$ defined by

$$
\begin{equation*}
J(y):=\int_{0}^{\infty} p(t, y) \mu(d t) \tag{6.1}
\end{equation*}
$$

where $\mu$ is the Lévy measure of $\phi$ and $p$ is the Gauss-Weierstrass kernel. Using symmetry of the kernel $J$, operator $A$ can be rewritten as

$$
A u(x)=\int_{\mathbb{R}^{d} \backslash\{0\}}\left(u(x+y)-u(x)-\nabla u(x) \cdot y 1_{\{|y|<1\}}\right) J(y) d y .
$$

By Fubini theorem, symmetry and Remark 5.1 (a),

$$
\begin{aligned}
\widehat{A u}(\xi) & =\widehat{u}(\xi) \int_{(0, \infty)} \int_{\mathbb{R}^{d} \backslash\{0\}}\left(e^{-i y \cdot \xi}-1-\xi \cdot y 1_{\{|y|<1\}}\right) p(t, y) d y \mu(d t) \\
& \left.=\widehat{u}(\xi) \int_{(0, \infty)} \widehat{(p(t, \cdot)}(-\xi)-1\right) \mu(d t)=\widehat{u}(\xi) \int_{(0, \infty)}\left(e^{-t|\xi|^{2}}-1\right) \mu(d t) \\
& =-\widehat{u}(\xi) \phi\left(|\xi|^{2}\right)
\end{aligned}
$$

In the last equality we have used representation (4.1) of $\phi$ together with the assumption $a=b=0$. Therefore, for any $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$

$$
\begin{equation*}
-\phi(-\Delta) u(x)=\lim _{\varepsilon \rightarrow 0+} \int_{|y|>\varepsilon}(u(x+y)-u(x)) J(y) d y \tag{6.2}
\end{equation*}
$$

Using (6.1) one can analyze behavior of the kernel $J$ of the operator $-\phi(-\Delta)$.
Proposition 6.1. Let $\phi \in \mathcal{B F}$ and let $\beta:=-\omega_{0}^{\phi^{\prime}}$. Assume that all assumptions in Corollary 4.1 hold with $\Lambda_{1}=0$ and $\Lambda_{2}=\infty$. Furthermore, assume that

$$
\begin{equation*}
\lambda \mapsto-\lambda \phi_{e}^{\prime \prime}(\lambda-\beta) \text { is non-increasing } \tag{6.3}
\end{equation*}
$$

and, in the case $d \leqslant 2$ and $\beta=0$, there are constants $\theta^{\prime}>0$ and $\gamma^{\prime}<2+\frac{d}{2}$ so that

$$
\frac{\phi^{\prime \prime}(\lambda x)}{\phi^{\prime \prime}(\lambda)} \geqslant \theta^{\prime} x^{-\gamma^{\prime}} \text { for all } \lambda>0 \text { and } x \geqslant 1 .
$$

Then the following holds:
(i) $J(y)=-|y|^{-d-4} \phi^{\prime \prime}\left(|y|^{-2}\right)$ for all $0<|y| \leqslant 1$;
(ii) if $\beta>0$, then

$$
J(y)=-|y|^{-\frac{d}{2}-\frac{5}{2}} \phi_{e}^{\prime \prime}\left(|y|^{-1}-\beta\right) e^{-\sqrt{\beta}|y|} \quad \text { for all } \quad|y| \geqslant 1
$$

(iii) if $\beta=0$, then

$$
J(y)=-|y|^{-d-4} \phi^{\prime \prime}\left(|y|^{-2}\right) \text { for all }|y| \geqslant 1
$$

Proof. By Corollary 4.1 it follows that the Lévy density of $\mu$ satisfies

$$
\mu(t)=f(t) e^{-\beta t}, t>0
$$

with $f(t):=-t^{-3} \phi_{e}^{\prime \prime}\left(t^{-1}-\beta\right)$. We want to apply results from Appendix A with $a:=\frac{d}{2}$, $b:=\sqrt{\beta}$ and $c:=2$.

Similarly as in the proof of Proposition 5.2, from (A-1) and (A-2) we deduce that $t \mapsto e^{\beta t} \mu(t)$ is a non-increasing function and so there exists $q:=\lim _{t \rightarrow+\infty} e^{\beta t} \mu(t)$ and it is possible to define a measure $\nu$ on $(0, \infty)$ by $\nu(t, \infty):=e^{\beta t} \mu(t)-q$. Then Lemma 2.1 (iii) implies that, for any $\lambda>0$,

$$
\begin{aligned}
\phi_{e}^{\prime}(\lambda-\beta) & =q \int_{0}^{\infty} t e^{-\lambda t} d t+\int_{0}^{\infty} t e^{-\lambda t} \nu(t, \infty) d t \\
& =\frac{q}{\lambda^{2}}+\int_{(0, \infty)} \frac{1-e^{-\lambda t}(1+\lambda t)}{\lambda^{2}} \nu(d t) .
\end{aligned}
$$

Therefore,

$$
-\lambda^{3} \phi_{e}^{\prime \prime}(\lambda-\beta)=2 q+\int_{(0, \infty)}\left(2-\left((\lambda t)^{2}+2 \lambda t+2\right) e^{-\lambda t}\right) \nu(d t)
$$

Noting that $\lambda \mapsto 2-\left((\lambda t)^{2}+2 \lambda t+2\right) e^{-\lambda t}$ is non-decreasing for every $t>0$, it follows that $\lambda \mapsto-\lambda^{3} \phi_{e}^{\prime \prime}(\lambda-\beta)$ is also non-decreasing, and so $f$ is non-increasing. By the assumption (6.3) we see that $t \mapsto t^{2} f(t)$ is non-decreasing. Now Lemma A. 1 and Lemma A. 2 imply all claims of the proposition.
Example 6.2. Let $\phi(\lambda)=\lambda^{\alpha / 2} \log \left(1+\lambda^{1-\alpha / 2}\right)(0<\alpha<2)$. Then $\beta=0$ and $\phi \in \mathcal{C B F}$ (cf. [SSV12, Corollary 7.15 (iii)]). Since

$$
-\phi^{\prime \prime}(\lambda)= \begin{cases}\lambda^{-\alpha / 2} & 0<\lambda \leqslant 1 \\ \lambda^{-2+\alpha / 2} \log (1+\lambda) & \lambda>1\end{cases}
$$

from Proposition 6.1 we get

$$
J(y)= \begin{cases}|y|^{-d-\alpha} \log \left(1+|y|^{-2}\right) & 0<|y|<1 \\ |y|^{-d-4+\alpha} & |y| \geqslant 1\end{cases}
$$

Example 6.3. Let $\phi(\lambda)=\log (1+\log (1+\lambda))$. By Example 4.3 (b) and Proposition 6.1,

$$
J(y)= \begin{cases}|y|^{-d} \log \left(1+|y|^{-2}\right)^{-1} & 0<|y|<1 \\ |y|^{-\frac{d+1}{2}} e^{-\sqrt{1-e^{-1}}|y|} & |y| \geqslant 1\end{cases}
$$

Example 6.4 (Generalized relativistic Schrödinger operator). Let $\alpha \in(0,2)$ and $m>0$ and let $\phi(\lambda)=\left(\lambda+m^{2 / \alpha}\right)^{\alpha / 2}-m$. Here $\beta=m^{2 / \alpha}$ and both conditions in Lemma 4.2 hold. Therefore, one can use Proposition 6.1 to deduce

$$
J(y)= \begin{cases}|y|^{-d-\alpha} & 0<|y|<1 \\ |y|^{-\frac{d+\alpha+1}{2}} e^{-m^{1 / \alpha}|y|} & |y| \geqslant 1 .\end{cases}
$$

The relativistic Schrödinger operator corresponds to $\alpha=1$ and $d=3$.

## Appendix A. Two technical lemmas

Let $f:(0, \infty) \rightarrow(0, \infty)$ be a non-increasing function. Define $F:(0, \infty) \rightarrow(0, \infty)$ by

$$
F(r):=\int_{0}^{\infty} t^{-a} e^{-b^{2} t} e^{-\frac{r^{2}}{c^{2} t}} f(t) d t, r>0
$$

where $a, b \geqslant 0$ and $c>0$.
Lemma A.1. Assume that $b>0$ and that $t \mapsto t^{2} f(t)$ is a non-decreasing function. Then

$$
F(r)=r^{-a+\frac{1}{2}} f\left(\frac{r}{b c}\right) e^{-2 b c^{-1} r} \text { for all } r \geqslant 1 .
$$

Proof. First we note that

$$
F(r)=e^{-2 b c^{-1} r} \int_{0}^{\infty} t^{-a} e^{-\left(\frac{r}{c \sqrt{t}}-b \sqrt{t}\right)^{2}} f(t) d t
$$

Using change of variables $s=\frac{r}{c \sqrt{t}}-b \sqrt{t}$ we get $t=\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{2}$ and

$$
\begin{equation*}
F(r)=2 e^{-b c^{-1} r} \int_{-\infty}^{\infty}\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{-2 a+2} f\left(\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{2}\right) \frac{e^{-s^{2} d s}}{\sqrt{s^{2}+4 b c^{-1} r}} . \tag{A.1}
\end{equation*}
$$

Since

$$
\begin{equation*}
s \mapsto-s+\sqrt{s^{2}+4 b c^{-1} r} \text { is non-increasing } \tag{A.2}
\end{equation*}
$$

it follows that

$$
f\left(\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{2}\right) \leqslant f\left(\frac{r}{b c}\right) \text { for } s \leqslant 0
$$

and

$$
\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{4} f\left(\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{2}\right) \leqslant\left(\frac{r}{b c}\right)^{2} f\left(\frac{r}{b c}\right) \text { for } s \geqslant 0 .
$$

The last two observations and (A.1) yield

$$
\begin{aligned}
F(r) \leqslant & 2 e^{-2 b c^{-1} r} f\left(\frac{r}{b c}\right)\left[\int_{-\infty}^{0}\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{-2 a+2} \frac{e^{-s^{2} d s}}{\sqrt{s^{2}+4 b c^{-1} r}}\right. \\
& \left.+\left(\frac{r}{b c}\right)^{2} \int_{0}^{\infty}\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{-2 a-2} \frac{e^{-s^{2} d s}}{\sqrt{s^{2}+4 b c^{-1} r}}\right] .
\end{aligned}
$$

For the first integral we note that, by dominated convergence theorem,

$$
\begin{aligned}
\lim _{r \rightarrow+\infty} r^{a-\frac{1}{2}} & \int_{-\infty}^{0}\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{-2 a+2} \frac{e^{-s^{2} d s}}{\sqrt{s^{2}+4 b c^{-1} r}} \\
& =\lim _{r \rightarrow+\infty} \int_{-\infty}^{0}\left(\frac{-\frac{s}{\sqrt{r}}+\sqrt{\frac{s^{2}}{r}+4 b c^{-1}}}{2 b}\right)^{-2 a+2} \frac{e^{-s^{2} d s}}{\sqrt{\frac{s^{2}}{r}+4 b c^{-1}}}=\frac{b^{a-\frac{3}{2}} c^{a-\frac{1}{2}} \sqrt{\pi}}{4} .
\end{aligned}
$$

This together with a similar computation in the second integral yield a constant $c_{2}=$ $c_{2}(a, b, c)>0$ so that the following holds

$$
F(r) \leqslant c_{2} r^{-a+\frac{1}{2}} f\left(\frac{r}{b c}\right) e^{-2 b c^{-1} r} \text { for all } r \geqslant 1 .
$$

To obtain the lower bound, we use (A.1) and (A.2) to get

$$
\begin{aligned}
F(r) & \geqslant 2 e^{-2 b c^{-1} r} \int_{0}^{\infty}\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{-2 a+2} f\left(\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{2}\right) \frac{e^{-s^{2} d s}}{\sqrt{s^{2}+4 b c^{-1} r}} \\
& \geqslant 2 e^{-2 b c^{-1} r} \frac{f\left(\frac{r}{b c}\right)}{\sqrt{b c^{-1} r}} \int_{0}^{\infty}\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{-2 a+2} e^{-s^{2}} d s .
\end{aligned}
$$

Similarly as before, the dominated convergence theorem yields

$$
\lim _{r \rightarrow+\infty} r^{a-1} \int_{0}^{\infty}\left(\frac{-s+\sqrt{s^{2}+4 b c^{-1} r}}{2 b}\right)^{-2 a+2} e^{-s^{2}} d s=\frac{(b c)^{a-1} \sqrt{\pi}}{2} .
$$

and thus there is a constant $c_{1}=c_{1}(a, b, c)>0$ such that

$$
F(r) \geqslant c_{1} r^{-a+\frac{1}{2}} f\left(\frac{r}{b c}\right) e^{-2 b c^{-1} r} \text { for all } r \geqslant 1
$$

Lemma A.2. Assume that $b=0$ and that that $t \mapsto t^{2} f(t)$ is a non-decreasing function. If $a \leqslant 1$ we additionally assume that there exist constants $c^{\prime}>0, \gamma^{\prime}>1-a$ and $R \geqslant 0$ so that

$$
\begin{equation*}
\frac{f(t x)}{f(t)} \leqslant c^{\prime} x^{-\gamma^{\prime}} \text { for all } t>R \text { and } x \geqslant 1 . \tag{A.3}
\end{equation*}
$$

Then

$$
F(r)=r^{-2 a+2} f\left(r^{2}\right) \text { for all } r>R_{0},
$$

where $R_{0}:=\left\{\begin{array}{ll}\sqrt{R} & d=1,2 \\ 0 & d \geqslant 3\end{array}\right.$.
Proof. Since $t \mapsto t^{2} f(t)$ is non-decreasing,

$$
\begin{aligned}
\int_{0}^{r^{2}} t^{-a} e^{-\frac{r^{2}}{c^{2} t}} f(t) d t & \leqslant r^{4} f\left(r^{2}\right) \int_{0}^{r^{2}} t^{-a-2} e^{-\frac{r^{2}}{c^{2} t}} d t \\
& =c^{2 a+2} r^{-2 a+2} f\left(r^{2}\right) \int_{c^{-2}}^{\infty} s^{a} e^{-s} d s
\end{aligned}
$$

Since $f$ is decreasing, for $a>1$ we deduce $\int_{r^{2}}^{\infty} t^{-a} f(t) d t \leqslant f\left(r^{2}\right)(a-1)^{-1} r^{-2 a+2}$. In the case $a \leqslant 1$ we use (A.3) to get

$$
\begin{aligned}
\int_{r^{2}}^{\infty} t^{-a} f(t) d t & =f\left(r^{2}\right) \int_{r^{2}}^{\infty} t^{-a} \frac{f(t)}{f\left(r^{2}\right)} d t \\
& \leqslant c^{\prime} f\left(r^{2}\right) r^{2 \gamma^{\prime}} \int_{r^{2}}^{\infty} t^{-a-\gamma^{\prime}} d t \leqslant \frac{c^{\prime}}{a+1-\gamma^{\prime}} r^{-2 a+2} f\left(r^{2}\right),
\end{aligned}
$$

for any $r>\sqrt{R}$. This proves the upper bound.
Since $f$ is decreasing, the lower bound follows from

$$
\begin{aligned}
F(r) & \geqslant \int_{0}^{r^{2}} t^{-a} e^{-\frac{r^{2}}{c^{2}} t} f(t) d t \geqslant f\left(r^{2}\right) \int_{0}^{r^{2}} t^{-a} e^{-\frac{r^{2}}{t}} d t \\
& =c^{2 a-2} r^{-2 a+2} f\left(r^{2}\right) \int_{c^{-2}}^{\infty} s^{a-2} e^{-s} d s .
\end{aligned}
$$

## References

[BGT87] N. H. Bingham, C. M. Goldie, and J. L. Teugels, Regular variation, Cambridge University Press, Cambridge, 1987.
[Eva98] L. C. Evans, Partial differential equations, American Mathematical Society, Providence, Rhode Island, 1998.
[Fel71] W. Feller, An introduction to probability theory and its applications, John Wiley and Sons, New York, 1971.
[Fol99] G. B. Folland, Real analysis, John Wiley and Sons, New York, 1999.
[Hir72] F. Hirsch, Intégrales de résolvantes et calcul symbolique, Ann. Inst. Fourier(Grenoble) 22 (1972), 239-264.
[KM12] P. Kim and A. Mimica, Harnack inequalities for subordinate Brownian motions, Electron. J. Probab. 17 (2012), no. 37, 1-23.
[KSV12] P. Kim, R. Song, and Z. Vondraček, Potential theory for subordinate Brownian motions revisited, Stochastic Analysis and its applications to Mathematical Finance, Interdisciplinary Mathematical Sciences, vol. 13, World Scientific, 2012.
[Phi52] R. S. Phillips, On the generation of semigroups of linear operators, Pacific J. Math. 2 (1952), 343-369.
[Sch96] R. L. Schilling, On the domain of the generator of a subordinate semigroup, Potential Theory ICPT 94. Proceedings Intnl. Conf. Potential Theory, Kouty (CR) 1994 (J. Král et al., ed.), de Gruyter, 1996, pp. 449-462.
[ŠSV06] H. Šikić, R. Song, and Z. Vondraček, Potential theory of geometric stable processes, Prob. Theory Related Fields 135 (2006), 547-575.
[SSV12] R. L. Schilling, R. Song, and Z. Vondraček, Bernstein functions: theory and applications, Walter de Gruyter, Berlin, 2012.
[SV09] R. Song and Z. Vondraček, Potential theory of subordinate Brownian motion, In: Potential Analysis of Stable Processes and its Extensions, P. Graczyk, A. Stos, editors, Lecture Notes Math. 1980 (2009), 87-176.
[Wid46] D. V. Widder, Laplace transform, Princeton University Press, 1946.
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