A parametric family of quartic Thue equations

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Abstract

In this paper we prove that the Diophantine equation

$$x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4 = 1,$$

where $c \geq 3$ is an integer, has only the trivial solutions $(\pm 1, 0)$, $(0, \pm 1)$.

Using the method of Tzanakis, we show that solving this quartic Thue equation reduces to solving the system of Pellian equations

$$(2c+1)U^2 - 2cV^2 = 1, \quad (c-2)U^2 - cZ^2 = -2,$$

and we prove that all solutions of this system are given by $(U, V, Z) = (\pm 1, \pm 1, \pm 1)$.

1 Introduction

In 1909, Thue [23] proved that an equation $F(x, y) = m$, where $F \in \mathbf{Z}[X, Y]$ is a homogeneous irreducible polynomial of degree $n \geq 3$ and $m \neq 0$ a fixed integer, has only finitely many solutions. His proof was not effective. In 1968, Baker [2] gave an effective bound based on the theory of linear forms in logarithms of algebraic numbers. In recent years general powerful methods have been developed for the explicit solution of Thue equations (see [19, 26, 6]), following from Baker's work.

Thomas [22] was first who investigated a parametrized family of Thue equations. Since then, several families have been studied (see [12] for references). In particular, quartic families have been considered in [7, 12, 13, 15, 18, 20, 24, 27, 28].

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In this paper, we consider the equation

$$x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4 = 1, \tag{1}$$

and we prove that for $c \geq 3$ it has no solution except the trivial ones: $(\pm 1, 0)$, $(0, \pm 1)$.

We will apply the method of Tzanakis. In [25], Tzanakis considered the equations of the form $f(x, y) = m$, where f is a quartic form which corresponding quartic field $\mathbf{K}$ is Galois and non-cyclic. By [17], this condition on $\mathbf{K}$ is equivalent with $\mathbf{K}$ having three quadratic subfields, which happens exactly when the cubic resolvent of the quartic Thue equation has three distinct rational roots. Assuming that $\mathbf{K}$ is not totally complex, we conclude that $\mathbf{K}$ is totally real, in fact, it is compositum of two real quadratic fields and it contains exactly three quadratic subfields, all of which are real. Tzanakis showed that solving the equation $f(x, y) = m$, under above assumptions on $\mathbf{K}$, reduces to solving a system of Pellian equations.

We will show that solving (1) by the method of Tzanakis reduces to solving the system

$$\begin{aligned} (2c+1)U^2 - 2cV^2 &= 1\,, \\ (c-2)U^2 - cZ^2 &= -2\,. \end{aligned}$$

We will find a lower bound for solutions of this system using the "congruence method" introduced in [11] and used also in [9, 10]. The comparison of this lower bound with an upper bound obtained from a theorem of Bennett [5] on simultaneous approximations of algebraic numbers finishes the proof for $c \geq 179559$. For $c \leq 179558$ we use a theorem a Baker and Wüstholz [4] and a version of the reduction procedure due to Baker and Davenport [3].

There are three reasons why we have chosen the family (1). First of all, for all members of this family ($c \neq 0, 1, 2$) the corresponding quartic field satisfies the above conditions, so the method of Tzanakis can be applied. Furthermore, the system of Pellian equations obtained in this way is very suitable for the application of both "congruence method" and Bennett's theorem.

Our main result is the following theorem.

Theorem 1 *Let $c \geq 3$ be an integer. The only solutions to (1) are $(x, y) = (\pm 1, 0)$ and $(0, \pm 1)$.*

Let us note that the statement of Theorem 1 is trivially true for $c = 0$ and $c = 1$. On the other hand, for $c = 2$ we have

$$x^4 - 8x^3y + 14x^2y^2 + 8xy^3 + y^4 = (x^2 - 4xy - y^2)^2 = 1,$$

and therefore in this case our equation has infinitely many solutions given by $x = \frac{1}{2}F_{3n+3}$, $y = \frac{1}{2}F_{3n}$.

For $c = 4$ we have

$$x^4 - 16x^3y + 26x^2y^2 + 16xy^3 + y^4 = (x^2 - 8xy - y^2)^2 - (6xy)^2 = 1,$$

which clearly implies $xy = 0$. Therefore we may assume that $c \neq 4$.

2 The method of Tzanakis

In this section we will describe the method of Tzanakis for solving quartic Thue equations which corresponding quartic field $\mathbf{K}$ has the properties described in Section 1.

Consider the quartic Thue equation

$$f(x, y) = m \tag{2}$$

$$f(x, y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4 \in \mathbf{Z}[x, y], \ \ a_0 > 0.$$

We assign to this equation the cubic equation

$$4\rho^3 - g_2\rho - g_3 = 0 \tag{3}$$

with roots opposite to those of the cubic resolvent of the quartic equation $f(x, 1) = 0$. Here $g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 \in \frac{1}{12}\mathbf{Z}$,

$$g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \in \frac{1}{432}\mathbf{Z}\,.$$

By [25], the conditions on $\mathbf{K}$ from Section 1 are equivalent to the fact that the cubic equation (3) has three rational roots ρ_1, ρ_2, ρ_3 and

$$\frac{a_1^2}{a_0} - a_2 \geq \max\{\rho_1, \rho_2, \rho_3\}. \tag{4}$$

Let $H(x,y)$ and $G(x,y)$ be the quartic and sextic covariants of $f(x,y)$, respectively (see [16, Chapter 25]), i.e.

$$\begin{aligned} H(x,y) &= -\frac{1}{144}\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} \in \frac{1}{48}\mathbf{Z}[x,y]\,, \\ G(x,y) &= -\frac{1}{8}\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{vmatrix} \in \frac{1}{96}\mathbf{Z}[x,y]\,. \end{aligned}$$

Then $4H^3 - g_2Hf^2 - g_3f^3 = G^2$. If we put $H = \frac{1}{48}H_0$, $G = \frac{1}{96}G_0$, $\rho_i = \frac{1}{12}r_i$, $i = 1,2,3$, then $H_0, G_0 \in \mathbf{Z}[x,y]$, $r_i \in \mathbf{Z}$, $i = 1,2,3$, and

$$(H_0 - 4r_1f)(H_0 - 4r_2f)(H_0 - 4r_3f) = 3G_0^2.$$

There exist positive square-free integers k_1, k_2, k_3 and quadratic forms G_1, G_2, $G_3 \in \mathbf{Z}[x,y]$ such that

$$H_0 - 4r_if = k_iG_i^2, \quad i = 1,2,3$$

and $k_1k_2k_3(G_1G_2G_3)^2 = 3G_0^2$. If $(x,y) \in \mathbf{Z} \times \mathbf{Z}$ is a solution of (2), then

$$k_2G_2^2 - k_1G_1^2 = 4(r_1 - r_2)m, \tag{5}$$

$$k_3G_3^2 - k_1G_1^2 = 4(r_1 - r_3)m. \tag{6}$$

In this way, solving the Thue equation (2) reduces to solving the system of Pellian equations (5) and (6) with one common unknown.

3 The system of Pellian equations

Let us apply the method from Section 2 to the equation

$$f(x,y) = x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4 = 1\,.$$

We have

$$\begin{aligned} g_2 &= \frac{1}{3}(21c^2 + 6c + 4), \\ g_3 &= -\frac{1}{27}(81c^3 + 99c^2 - 18c - 8), \end{aligned}$$

$$\rho_1 = \frac{1}{2}c + \frac{2}{3}, \quad \rho_2 = c - \frac{1}{3}, \quad \rho_3 = -\frac{3}{2}c - \frac{1}{3}.$$

The condition (4) is clearly satisfied.

Furthermore, we obtain

$$\begin{aligned} H_0 - 4r_1 f &= 24(c-2)(2c+1)(x^2+y^2)^2, \\ H_0 - 4r_2 f &= 48c(c-2)(x^2+xy-y^2)^2, \\ H_0 - 4r_3 f &= 24c(2c+1)(-x^2+4xy+y^2)^2. \end{aligned}$$

Hence we may take $k_1 = 6(c-2)(2c+1)$, $k_2 = 3c(c-2)$, $k_3 = 6c(2c+1)$, $G_1 = 2(x^2+y^2)$, $G_2 = 4(x^2+xy-y^2)$, $G_3 = 2(-x^2+4xy+y^2)$. Inserting this in (5) and (6) we obtain

$$cG_2^2 - (4c+2)G_1^2 = -8, \tag{7}$$

$$cG_3^2 - (c-2)G_1^2 = 8. \tag{8}$$

Let

$$U = \frac{G_1}{2} = x^2+y^2, \quad V = \frac{G_2}{4} = x^2+xy-y^2, \quad Z = \frac{G_3}{2} = -x^2+4xy+y^2.$$

Then from (7) and (8) we obtain the system of Pellian equations

$$(2c+1)U^2 - 2cV^2 = 1, \tag{9}$$

$$(c-2)U^2 - cZ^2 = -2. \tag{10}$$

Lemma 1 *Let $k \geq 2$ be an integer. If x and y are positive integers satisfying the Pellian equation*

$$(k-1)y^2 - (k+1)x^2 = -2,$$

then there exist integer $m \geq 0$ such that $x = x_m$ and $y = y_m$, where the sequences (x_m) and (y_m) are given by

$$x_0 = 1, \quad x_1 = 2k-1, \quad x_{m+2} = 2kx_{m+1} - x_m, \quad m \geq 0;$$

$$y_0 = 1, \quad y_1 = 2k+1, \quad y_{m+2} = 2ky_{m+1} - y_m, \quad m \geq 0.$$

Proof. See [8, p. 312]. ∎

From Lemma 1 it follows immediately

Lemma 2 *Let (U,V,Z) be positive integer solution of the system of Pellian equations (9) and (10). Then there exist nonnegative integers m and n such that*

$$U = v_m = w_n,$$

where the sequences (v_m) and (w_n) are given by

$$v_0 = 1, \quad v_1 = 8c+1, \quad v_{m+2} = (8c+2)v_{m+1} - v_m, \quad m \geq 0 \tag{11}$$

and

$$w_0 = 1, \quad w_1 = 2c-1, \quad w_{n+2} = (2c-2)w_{n+1} - w_n, \quad n \geq 0. \tag{12}$$

Therefore, in order to prove Theorem 1, it suffices to show that $v_m = w_n$ implies $m = n = 0$.

Solving recurrences (11) and (12) we find

$$\begin{aligned} v_m &= \frac{1}{2\sqrt{4c+2}}\Big[(2\sqrt{c}+\sqrt{4c+2})\Big(4c+1+2\sqrt{2c(2c+1)}\Big)^m \\ &- (2\sqrt{c}-\sqrt{4c+2})\Big(4c+1-2\sqrt{2c(2c+1)}\Big)^m\Big], \end{aligned} \tag{13}$$

$$\begin{aligned} w_n &= \frac{1}{2\sqrt{c-2}}\Big[(\sqrt{c}+\sqrt{c-2})\Big(c-1+\sqrt{c(c-2)}\Big)^n \\ &- (\sqrt{c}-\sqrt{c-2})\Big(c-1-\sqrt{c(c-2)}\Big)^n\Big]. \end{aligned} \tag{14}$$

4 Congruence relations

Lemma 3 *Let the sequences (v_m) and (w_n) be defined by (11) and (12). Then for all $m, n \geq 0$ we have*

$$v_m \equiv 4m(m+1)c+1 \pmod{64c^2}, \tag{15}$$

$$w_n \equiv (-1)^{n-1}[n(n+1)c-1] \pmod{4c^2}. \tag{16}$$

Proof. Both relations are obviously true for $m, n \in \{0,1\}$.
Assume that (15) is valid for $m-2$ and $m-1$. Then

$$\begin{aligned} v_m &= (8c+2)v_{m-1} - v_{m-2} \\ &\equiv (8c+2)[4m(m-1)c+1] - [4(m-1)(m-2)c+1] \\ &\equiv c[8+8m(m-1)-4(m-1)(m-2)]+1 \\ &= 4m(m+1)c+1 \pmod{64c^2}. \end{aligned}$$

Assume that (16) is valid for $n-2$ and $n-1$. Then

$$\begin{aligned} w_n &= (2c-2)w_{n-1} - w_{n-2} \\ &\equiv (2c-2)(-1)^n[n(n-1)c-1] - (-1)^{n-1}[(n-1)(n-2)c-1] \\ &\equiv c(-1)^{n-1}[2+2n(n-1)-(n-1)(n-2)] + (-1)^n \\ &= (-1)^{n-1}[n(n+1)c-1] \pmod{4c^2}. \end{aligned}$$

■

Suppose that m and n are positive integers such that $v_m = w_n$. Then, of course, $v_m \equiv w_n \pmod{4c^2}$. By Lemma 3, we have $(-1)^n \equiv 1 \pmod{2c}$ and therefore n is even.

Assume that $n(n+1) < \frac{4}{5}c$. Since $m \leq n$ we also have $m(m+1) < \frac{4}{5}c$. Furthermore, Lemma 3 implies

$$4m(m+1)c + 1 \equiv 1 - n(n+1)c \pmod{4c^2}$$

and

$$2m(m+1) \equiv -\frac{n(n+1)}{2} \pmod{2c}. \tag{17}$$

Consider the positive integer

$$A = 2m(m+1) + \frac{n(n+1)}{2}.$$

We have $0 < A < 2c$ and, by (17), $A \equiv 0 \pmod{2c}$, a contradiction.

Hence $n(n+1) \geq \frac{4}{5}c$ and it implies $n > \sqrt{0.8c} - 0.5$. Therefore we proved

Proposition 1 *If $v_m = w_n$ and $m \neq 0$, then $n > \sqrt{0.8\,c} - 0.5$.*

5 An application of a theorem of Bennett

It is clear that the solutions of the system (9) and (10) induce good rational approximations to the numbers

$$\theta_1 = \sqrt{\frac{2c+1}{2c}} \quad \text{and} \quad \theta_2 = \sqrt{\frac{c-2}{c}}.$$

More precisely, we have

Lemma 4 *All positive integer solutions (U,V,Z) of the system of Pellian equations (9) and (10) satisfy*

$$\begin{aligned} |\theta_1 - \frac{V}{U}| &< \frac{1}{4c} \cdot U^{-2}, \\ |\theta_2 - \frac{Z}{U}| &< \frac{1}{\sqrt{c(c-2)}} \cdot U^{-2}. \end{aligned}$$

Proof. We have

$$\begin{aligned} |\theta_1 - \frac{V}{U}| &= \left|\sqrt{\frac{2c+1}{2c}} - \frac{V}{U}\right| = \left|\frac{2c+1}{2c} - \frac{V^2}{U^2}\right| \cdot \left|\sqrt{\frac{2c+1}{2c}} + \frac{V}{U}\right|^{-1} \\ &< \frac{1}{2cU^2} \cdot \frac{1}{2} = \frac{1}{4c}U^{-2} \end{aligned}$$

and

$$\begin{aligned} |\theta_2 - \frac{Z}{U}| &= \left|\sqrt{\frac{c-2}{c}} - \frac{Z}{U}\right| = \left|\frac{c-2}{c} - \frac{Z^2}{U^2}\right| \cdot \left|\sqrt{\frac{c-2}{c}} + \frac{Z}{U}\right|^{-1} \\ &< \frac{2}{cU^2} \cdot \frac{1}{2}\sqrt{\frac{c}{c-2}} = \frac{1}{\sqrt{c(c-2)}}U^{-2}. \end{aligned}$$

■

The numbers θ_1 and θ_2 are square roots of rationals which are very close to 1. For simultaneous Diophantine approximations to such kind of numbers there are very useful effective results of Masser and Rickert [14] and Bennett [5]. Let us mention that the first effective results on simultaneous approximation to fractional powers of rationals close to 1 were given by Baker in [1]. We will use the following theorem of Bennett [5, Theorem 3.2].

Theorem 2 *If a_i, p_i, q and N are integers for $0 \le i \le 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \le j \le 2$, q nonzero and $N > M^9$, where*

$$M = \max_{0 \le i \le 2} \{|a_i|\},$$

then we have

$$\max_{0 \le i \le 2} \left\{ \left| \sqrt{1 + \frac{a_i}{N}} - \frac{p_i}{q} \right| \right\} > (130N\gamma)^{-1} q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33N\gamma)}{\log\left(1.7N^2 \prod_{0 \le i < j \le 2}(a_i - a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} & \textit{if } a_2-a_1 \geq a_1-a_0, \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0} & \textit{if } a_2-a_1 < a_1-a_0. \end{cases}$$

We will apply Theorem 2 with $a_0 = -4$, $a_1 = 0$, $a_2 = 1$, $N = 2c$, $M = 4$, $q = U$, $p_0 = Z$, $p_1 = U$, $p_2 = V$. If $c \geq 131073$, then the condition $N > M^9$ is satisfied and we obtain

$$(130 \cdot 2c \cdot \frac{400}{9})^{-1} U^{-\lambda} < \frac{1}{\sqrt{c(c-2)}} U^{-2}. \tag{18}$$

If $c \geq 172550$ then $2 - \lambda > 0$ and (18) implies

$$\log U < \frac{9.355}{2-\lambda}. \tag{19}$$

Furthermore,

$$\frac{1}{2-\lambda} = \frac{1}{1 - \frac{\log(\frac{26400}{9}c)}{\log(0.017c^2)}} < \frac{\log(0.017c^2)}{\log(0.00000579c)}.$$

On the other hand, from (14) we find that

$$w_n > \left(c - 1 + \sqrt{c(c-2)}\right)^n > (2c-3)^n,$$

and Proposition 1 implies that if $(m,n) \neq (0,0)$, then

$$U > (2c-3)^{\sqrt{0.8c}-0.5}.$$

Therefore,

$$\log U > (\sqrt{0.8c} - 0.5) \log(2c-3). \tag{20}$$

Combining (19) and (20) we obtain

$$\sqrt{0.8c} - 0.5 < \frac{9.355 \log(0.017c^2)}{\log(2c-3) \log(0.00000579c)} \tag{21}$$

and (21) yields to a contradiction if $c \geq 179559$. Therefore we proved

Proposition 2 *If c is an integer such that $c \geq 179559$, then the only solution of the equation $v_m = w_n$ is $(m,n) = (0,0)$.*

6 The Baker-Davenport method

In this section we will apply so called Baker-Davenport reduction method in order to prove Theorem 1 for $3 \leq c \leq 179558$.

Lemma 5 *If $v_m = w_n$ and $m \neq 0$, then*

$$0 < n \log\left(c-1+\sqrt{c(c-2)}\right) - m \log\left(4c+1+2\sqrt{2c(2c+1)}\right)$$
$$+ \log \frac{\sqrt{4c+2}(\sqrt{c}+\sqrt{c-2})}{\sqrt{c-2}(2\sqrt{c}+\sqrt{4c+2})} < 0.627\left(4c+1+2\sqrt{2c(2c+1)}\right)^{-2m}.$$

Proof. Let define

$$\begin{aligned} P &= \frac{2\sqrt{c}+\sqrt{4c+2}}{\sqrt{4c+2}}\left(4c+1+2\sqrt{2c(2c+1)}\right)^m, \\ Q &= \frac{\sqrt{c}+\sqrt{c-2}}{\sqrt{c-2}}\left(c-1+\sqrt{c(c-2)}\right)^n. \end{aligned}$$

From (13) and (14) it follows that the relation $v_m = w_n$ implies

$$P + \frac{1}{2c+1}P^{-1} = Q - \frac{2}{c-2}Q^{-1}.$$

It is clear that $Q > P$. Furthermore,

$$\frac{Q-P}{Q} = \frac{1}{Q}\Big(\frac{1}{2c+1}P^{-1} + \frac{2}{c-2}Q^{-1}\Big) < P^{-2}\Big(\frac{1}{2c+1} + \frac{2}{c-2}\Big) \leq \frac{15}{7}P^{-2}.$$

Since $m, n \geq 1$, we have $P > 8c+1 \geq 25$ and $\frac{Q-P}{Q} < \frac{1}{291}$. Thus we may apply [21, Lemma B.2], and we obtain

$$\begin{aligned} 0 &< \log\frac{Q}{P} = -\log\left(1-\frac{Q-P}{Q}\right) < 1.002 \cdot \frac{15}{7}P^{-2} \\ &= 2.148 \cdot \frac{4c+2}{(2\sqrt{c}+\sqrt{4c+2})^2}\left(4c+1+2\sqrt{2c(2c+1)}\right)^{-2m} \\ &< \frac{4.296\,(2c+1)}{16c}\left(4c+1+2\sqrt{2c(2c+1)}\right)^{-2m} \\ &< 0.627\left(4c+1+2\sqrt{2c(2c+1)}\right)^{-2m}. \end{aligned}$$

■

Now we will apply the following famous theorem of Baker and Wüstholz [4]:

Theorem 3 *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational integer coefficients $b_1, \ldots, b_l$ we have*

$$\log \Lambda \geq -18(l+1)!\, l^{l+1} (32d)^{l+2} h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log B\,,$$

where $B = \max\{|b_1|, \ldots, |b_l|\}$, and where d is the degree of the number field generated by $\alpha_1, \ldots, \alpha_l$.

Here

$$h'(\alpha) = \frac{1}{d} \max\{h(\alpha), |\log\,\alpha|, 1\}\,,$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α.

We will apply Theorem 3 to the form from Lemma 5. We have $l = 3$, $d = 4$, $B = n$,

$$\alpha_1 = c - 1 + \sqrt{c(c-2)}, \qquad \alpha_2 = 4c + 1 + 2\sqrt{2c(2c+1)},$$

$$\alpha_3 = \frac{\sqrt{4c+2}(\sqrt{c} + \sqrt{c-2})}{\sqrt{c-2}(2\sqrt{c} + \sqrt{4c+2})}\,.$$

Under the assumption that $3 \leq c \leq 179558$ we find that

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log 2c, \qquad h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < 7.0889.$$

Furthermore, $\alpha_3 < 1.419$, and the conjugates of α_3 satisfy

$$|\alpha_3'| = \frac{\sqrt{4c+2}(\sqrt{c} - \sqrt{c-2})}{\sqrt{c-2}(2\sqrt{c} + \sqrt{4c+2})} < 1,$$

$$|\alpha_3''| = \frac{\sqrt{4c+2}(2\sqrt{c} + \sqrt{4c+2})}{\sqrt{c-2}(\sqrt{c} + \sqrt{c-2})} < 9.869,$$

$$|\alpha_3'''| = \frac{\sqrt{4c+2}(\sqrt{c} + \sqrt{c-2})(2\sqrt{c} + \sqrt{4c+2})}{2\sqrt{c-2}} < 1436471.1\,.$$

Therefore,

$$h'(\alpha_3) < \frac{1}{4}\log[(c-2)^2 \cdot 1.419 \cdot 9.869 \cdot 1436471.1] < 10.254\,.$$

Finally,

$$\log[0.627\Big(4c + 1 + 2\sqrt{2c(2c+1)}\Big)^{-2m}] < -2m\log(8c) < -2m\log(2c)\,.$$

Hence, Theorem 3 implies

$$2m\log(2c) < 3.822 \cdot 10^{15} \cdot \frac{1}{2}\log(2c) \cdot 7.0889 \cdot 10.254 \cdot \log n$$

and

$$\frac{m}{\log n} < 6.946 \cdot 10^{16}\,. \tag{22}$$

By Lemma 5, we have

$$\begin{aligned} n\log(c-1+\sqrt{c(c-2)}) &< m\log(4c+1+2\sqrt{2c(2c+1)}) + 0.000931 \\ &< m\log[(4c+1+2\sqrt{2c(2c+1)})\cdot 1.000932] \end{aligned}$$

and

$$\frac{n}{m} < 2.474\,. \tag{23}$$

Combining (22) and (23), we obtain

$$\frac{n}{\log n} < 1.719 \cdot 10^{17}\,,$$

which implies $n < 7.471 \cdot 10^{18}$.

We may reduce this large upper bound using a variant of the Baker-Davenport reduction procedure [3]. The following lemma is a slight modification of [11, Lemma 5 a)]:

Lemma 6 *Assume that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 10M$ and let $\varepsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.*

If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < n - m\kappa + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \le m \le M\,.$$

We apply Lemma 6 with

$$\kappa = \frac{\log\alpha_2}{\log\alpha_1}, \qquad \mu = \frac{\log\alpha_3}{\log\alpha_1}, \qquad A = \frac{0.627}{\log\alpha_1},$$

$$B = \left(4c + 1 + 2\sqrt{2c(2c+1)}\right)^2 \quad \text{and} \quad M = 7.471 \cdot 10^{18}.$$

If the first convergent such that $q > 10M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent.

We performed the reduction from Lemma 6 for $3 \leq c \leq 179558$, $c \neq 4$. The use of the second convergent was necessary in 6810 cases (3.79 %), the third convergent was used in 143 cases (0.08 %), the forth in 22 cases and the fifth in seven cases ($c = 21027$, 22393, 41842, 56576, 75541, 96007, 157920). In all cases we obtained $m \leq 7$. More precisely, we obtained $m \leq 7$ for $c = 3$; $m \leq 6$ for $c \geq 5$; $m \leq 5$ for $c \geq 6$; $m \leq 4$ for $c \geq 13$; $m \leq 3$ for $c \geq 67$; $m \leq 2$ for $c \geq 724$. According to Proposition 1, this finishes the proof for $c \geq 79$. It is trivial to check that for $c \leq 78$ there is no solution of the equation $v_m = w_n$ with $(m, n) \neq (0, 0)$ in the above ranges.

Therefore, we proved

Proposition 3 *If c is an integer such that $3 \leq c \leq 179558$, then the only solution of the equation $v_m = w_n$ is $(m, n) = (0, 0)$.*

Proof of Theorem 1. The statement follows directly from Propositions 2 and 3. ∎

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