A proof of the Hoggatt-Bergum conjecture

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Abstract

It is proved that if k and d are positive integers such that the product of any two distinct elements of the set

$$\{F_{2k},\, F_{2k+2},\, F_{2k+4},\, d\}$$

increased by 1 is a perfect square, than d has to be $4F_{2k+1}F_{2k+2}F_{2k+3}$. This is a generalization of the theorem of Baker and Davenport for $k = 1$.

1 Introduction

Diophantus studied the following problem: Find four (positive rational) numbers such that the product of any two of them increased by 1 is a perfect square. He obtained the following solution: $\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}$ (see [5]). The first set of four positive integers with the above property was found by Fermat, and it was $\{1, 3, 8, 120\}$. Euler gave the solution $\{a, b, a+b+2r, 4r(r+a)(r+b)\}$, where $ab+1 = r^2$ (see [4]). For $a = F_{2k}$ and $b = F_{2k+2}$ we obtain the well-known generalization of the Fermat set in the term of Fibonacci numbers (see [9, Theorem 1.2]):

Theorem 1 *For $k \geq 1$, the four numbers F_{2k}, F_{2k+2}, F_{2k+4} and $d = 4F_{2k+1}F_{2k+2}F_{2k+3}$ have the property that the product of any two of them increased by 1 is a perfect square.*

In [9], Hoggatt and Bergum conjectured that the value d of Theorem 1 is unique. The conjecture for $k = 1$ was proved in 1969 by Baker and

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Davenport [2]. The same result was proved later by Kanagasabapathy and Ponnudurai [11], Sansone [14], Grinstead [8] and Kedlaya [12]. In the present paper we prove the Hoggatt-Bergum conjecture for all positive integers k.

Definition 1 *A set of positive integers* $\{a_1, a_2, \dots, a_m\}$ *is said to have the property of Diophantus if* $a_i a_j + 1$ *is a perfect square for all* $1 \leq i < j \leq m$. *Such a set is called a Diophantine* m*-tuple.*

The main result of the present paper is the following theorem.

Theorem 2 *Let* k *be a positive integer. If the set* $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$ *has the property of Diophantus, then* d *has to be* $4F_{2k+1}F_{2k+2}F_{2k+3}$.

Corollary 1 *If* $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$ *is a Diophantine quadruple, then* d *cannot be a Fibonacci number.*

Proof. Jones [10] proved that

$$F_{6k+5} < F_{2k+1}F_{2k+2}F_{2k+3} < F_{6k+6} \, ,$$

and the assertion follows directly from Theorem 2. ∎

The proof of Theorem 2 is devided into several parts. We may assume that $k \geq 2$ and we first prove the theorem for $k \geq 49$. We transform our problem in the problem of finding the intersection of two binary recurrence sequaences. Then we transform the exponential equation into inequality for linear forms in three logarithms of algebraic numbers. A comparison of the theorem of Baker and Wüstholz [3] with the lower bounds for the solution obtained from the congruence condition modulo $2F_{2k}F_{2k+2}$ finishes the proof for $k \leq 49$. We prove the statement for $2 \leq k \leq 48$ by a version of the reduction procedure due to Baker and Davenport [2].

2 Preliminaries

Let $k \geq 2$ be a positive integer. Let $a = F_{2k}$, $b = F_{2k+2}$, $c = F_{2k+4}$. Then $c = 3b - a$. Furthermore,

$$ab + 1 = (b-a)^2, \quad ac + 1 = b^2, \quad bc + 1 = (a+b)^2.$$

Assume that d is a positive integer such that $\{a, b, c, d\}$ has the property of Diophantus. It implies that there exist positive integers x, y and z such that the following holds:

$$ad+1=x^2, \quad bd+1=y^2, \quad (3b-a)d+1=z^2.$$

Eliminating d, we obtain the following system of Pellian equations:

$$ay^2 - bx^2 = a-b\,, \tag{1}$$

$$az^2 - (3b-a)x^2 = 2a-3b\,. \tag{2}$$

Since $a<b<4a$, [10, Theorem 8] implies that all solutions of (1) are given by $x=v_m$, $m\in\mathbf{Z}$, where two-sided recurrence sequence (v_m) is defined by

$$v_0=1, \quad ,v_1=b, \quad v_{m+2}=2(b-a)v_{m+1}-v_m, \quad m\in Z. \tag{3}$$

Since $b+\sqrt{ac}$ is a non-trivial unit of norm 1 in the number ring $\mathbf{Z}[\sqrt{ac}]$, the theory of Pellian equations guaranties that there is a finite set $\{z_0^{(i)}\sqrt{a}+x_0^{(i)}\sqrt{c},\ i=1,\dots,i_0\}$ of elements of $\mathbf{Z}[\sqrt{ac}]$ such that if (z,x) is any solution of (2) in integers then

$$z\sqrt{a}+x\sqrt{c}=(z_0^{(i)}\sqrt{a}+x_0^{(i)}\sqrt{c})(b+\sqrt{ac})^n$$

for some index i and integer $n\geq 0$. In this case, $z=w_n^{(i)}$, where the sequence $(w_n^{(i)})$ is defined by

$$w_0^{(i)}=x_0^{(i)}, \quad w_1^{(i)}=bx_0^{(i)}+az_0^{(i)}, \quad w_{n+2}^{(i)}=2bw_{n+1}^{(i)}-w_n^{(i)}. \tag{4}$$

We call the set $\{z_0^{(i)}\sqrt{a}+x_0^{(i)}\sqrt{c},\ i=1,\dots,i_0\}$ of solutions of (2) fundamental if we choose representatives so that the $|z_0^{(i)}|$ are minimal.

By [13, Theorem 108a] we have the following estimate for the fundamental solutions of (2):

$$0<x_0^{(i)}\leq\sqrt{\frac{a(3b-2a)}{2(b-1)}}=\sqrt{\frac{(b+1)(3b-2a)}{2(3b-a)}}<\sqrt{\frac{b+1}{2}}\leq\sqrt{\frac{3a}{2}}<a\,. \tag{5}$$

From (3) and (4) it follows easily by induction that

$$v_{2m}\equiv 1 \pmod{a}, \quad v_{2m+1}\equiv b \pmod{a},$$
$$w_{2n}^{(i)}\equiv x_0^{(i)} \pmod{a}, \quad w_{2n+1}^{(i)}\equiv bx_0^{(i)} \pmod{a}.$$

Hence, if the equation $v_m=w_n^{(i)}$ has a solution in integers m and n, then we must have $x_0^{(i)}\equiv 1 \pmod{a}$ or $x_0^{(i)}\equiv b \pmod{a}$ or $bx_0^{(i)}\equiv 1 \pmod{a}$

or $bx_0^{(i)} \equiv b \pmod{a}$. Now the estimate (5) implies that $x_0^{(i)} = 1$ or $x_0^{(i)} = b-2a$.

But if $k \geq 3$, then $a \geq 8$ and $\sqrt{\frac{3a}{2}} > b-2a > \frac{\sqrt{5}-1}{2}a$ leads to a contradiction. If $k=2$, than $a=3$, $b=8$, and $2 = b-2a \geq \sqrt{\frac{(b+1)(3b-2a)}{2(3b-a)}} = \frac{2\sqrt{3}}{\sqrt{7}}$ leads to a contradiction again. Therefore $x_0^{(i)} = 1$ and $z_0^{(i)} = \pm 1$.

We have thus proved the following lemma.

Lemma 1 *Let x, y, z be positive integer solutions of the system of Pellian equations (1) and (2). Then there exist integers m and n such that*

$$x = v_m = w_n \,,$$

where (v_m) is given by (3), and the two-sided sequence (w_n) is defined by

$$w_0 = 1, \quad w_1 = a+b, \quad w_{n+2} = 2bw_{n+1} - w_n, \quad n \in \mathbf{Z}\,.$$

3 Congruence condition modulo $2F_{2k}F_{2k+2}$

Lemma 2 *Let m and n be integers such that $v_m = w_n$. Then $m \equiv 0$ or $2 \pmod{2F_{2k+2}}$.*

Proof. Let us consider the sequences $(v_m \bmod 2ab)$ and $(w_n \bmod 2ab)$. It follows easily by induction that

$$v_{2m} \equiv 2mb^2 - (2m-1) \pmod{2ab}, \quad v_{2m+1} \equiv b + 2(m-1)a \pmod{2ab},$$

$$w_{4n} \equiv 1 \pmod{2ab}, \quad w_{4n+2} \equiv 2b^2 - 1 \pmod{2ab},$$

$$w_{4n+1} \equiv a+b \pmod{2ab}, \quad w_{4n+3} \equiv b-a \pmod{2ab}.$$

We need to consider four cases.

1) If $v_{2m} \equiv w_{2n}$, then we have two possibilities:

   a) $2mb^2 - (2m-1) \equiv 1 \pmod{2ab}$
   It implies $2ma(3b-a) \equiv 0 \pmod{2ab}$ and $2m \equiv 0 \pmod{2b}$.

   b) $2mb^2 - (2m-1) \equiv 2b^2 - 1 \pmod{2ab}$
   It implies $(2m-2)a(3b-a) \equiv 0 \pmod{2ab}$ and $2m \equiv 2 \pmod{2b}$.

2) If $v_{2m} = w_{2n+1}$, then $2mb^2 - (2m-1) \equiv b \pm a \pmod{2ab}$. It implies $b \equiv 1 \pmod{a}$. Thus $b-2a=1$, which is possible only if $a=1$, $b=3$, contradicting the assumption that $k \geq 2$.

3) If $v_{2m+1} = w_{2n}$, then $b + 2(m-1)a \equiv 1 \pmod{2ab}$. We have again $b \equiv 1 \pmod{a}$ which leads to a contradiction as in 2).

4) If $v_{2m+1} = w_{2n+1}$, then $b + 2(m-1)a \equiv b \pm a \pmod{2ab}$. It implies $2(m-1) \equiv \pm 1 \pmod{2b}$, a contradiction.

■

4 Linear forms in three logarithms

In order to apply Baker's method it is convenient to consider a two-sided sequence as two ordinary sequences. Therefore, insted of the sequences $(v_m,\, m \in \mathbf{Z})$ we will consider two sequences $(v_m,\, m \geq 0)$ and $(v_m,\, m \leq 0)$, and similary for the sequence $(w_n,\, n \in \mathbf{Z})$. Thus, we will actually consider four equations of the form $v_m = w_n$.

Lemma 3 *If $v_m = w_n$, and $m \neq 0$, then*

$$\begin{aligned} 0 &< m \log(b - a + \sqrt{ab}) - n \log(b + \sqrt{ac}) + \log \frac{\sqrt{c}(\pm\sqrt{a} + \sqrt{b})}{\sqrt{b}(\pm\sqrt{a} + \sqrt{c})} \\ &< 4(b - a + \sqrt{ab})^{-2m}. \end{aligned}$$

Proof. We have by Lemma 1 that

$$\begin{aligned} v_m &= \frac{1}{2\sqrt{b}}[(\pm\sqrt{a} + \sqrt{b})(b - a + \sqrt{ab})^m - (\pm\sqrt{a} - \sqrt{b})(b - a - \sqrt{ab})^m]\,, \\ w_n &= \frac{1}{2\sqrt{c}}[(\pm\sqrt{a} + \sqrt{c})(b + \sqrt{ac})^n - (\pm\sqrt{a} - \sqrt{c})(b - \sqrt{ac})^n]\,. \end{aligned}$$

If we put

$$P = \frac{\pm\sqrt{a} + \sqrt{b}}{\sqrt{b}}(b - a + \sqrt{ab})^m, \qquad Q = \frac{\pm\sqrt{a} + \sqrt{c}}{\sqrt{c}}(b + \sqrt{ac})^n,$$

then the relation $v_m = w_n$ implies

$$P + \frac{b-a}{b}P^{-1} = Q + \frac{c-a}{c}Q^{-1}. \tag{6}$$

It is clear that $P > 1$ and $Q > 1$, and from

$$P - Q > \frac{b-a}{b}(Q^{-1} - P^{-1}) = \frac{b-a}{a}(P - Q)P^{-1}Q^{-1}$$

it follows that $P > Q$. We may assume that $m \geq 2$. Thus, we have $P \geq \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}} \cdot 4ab > 2(3-\sqrt{5})ab$. On the other hand, $\frac{c-a}{a} \leq 7$ and we obtain that $P > \frac{c-a}{a}$. Relation (6) implies, $Q > P - \frac{c-a}{c}Q^{-1} > P - \frac{c-a}{c}$. Hence,

$$\begin{aligned} P - Q &= \frac{c-a}{c}Q^{-1} - \frac{b-a}{b}P^{-1} < \frac{c-a}{c}(P - \frac{c-a}{c})^{-1} - \frac{b-a}{b}P^{-1} \\ &< P^{-1} - \frac{b-a}{b}P^{-1} = \frac{a}{b}P^{-1}, \end{aligned}$$

and finally

$$\begin{aligned} 0 < \log \frac{P}{Q} &= -\log(1 - \frac{P-Q}{P}) \\ &< \frac{a}{b}P^{-2} + \frac{a^2}{b^2}P^{-4} < \frac{3a}{2b}P^{-2} = \frac{3a}{2(\sqrt{b}-\sqrt{a})^2}(b-a-\sqrt{ab})^{2m} \\ &< \frac{3(3+\sqrt{5})}{4}(b-a+\sqrt{ab})^{-2m} < 4(b-a+\sqrt{ab})^{-2m}. \end{aligned}$$

■

Now we apply the following theorem of Baker and Wüstholz [3]:

Theorem 3 *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational coefficients $b_1, \ldots, b_l$ we have*

$$\log \Lambda \geq -18(l+1)!\, l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B\,,$$

where $B = \max(|b_1|, \ldots, |b_l|)$, and where d is the degree of the number field generated by $\alpha_1, \ldots, \alpha_l$.

Here

$$h'(\alpha) = \frac{1}{d}\max\left(h(\alpha), |\log\,\alpha|, 1\right),$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α.

In the present situation we have $l = 3$, $d = 4$, $B = m$, and

$$\alpha_1 = b - a + \sqrt{ab}, \quad \alpha_2 = b + \sqrt{ac}, \quad \alpha_3 = \frac{\sqrt{c}(\pm\sqrt{a}+\sqrt{b})}{\sqrt{b}(\pm\sqrt{a}+\sqrt{c})},$$

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < 1.05\log a, \qquad h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < 1.27\log a,$$

$$h'(\alpha_3) = \frac{1}{4}\log[bc(c-a)(\sqrt{a}+\sqrt{b})^2] < 2.52\log a,$$

$$\log 4(b-a+\sqrt{ab})^{-2m} < \log a^{-2m} = -2m \log a.$$

Hence,

$$2m \log a < 3.822 \cdot 10^{15} \cdot 3.361 \cdot log^3 a \log m,$$

and

$$\frac{m}{\log m} < 6.423 \cdot 10^{15} \log^2 a. \tag{7}$$

Now we will apply Lemma 2. We want to prove that $v_0 = w_0 = 1$ and $v_2 = w_{-2} = 2b^2 - 2ab - 1$ are the only solutions of the equation $v_m = w_n$, $m, n \in \mathbf{Z}$. These solutions correspond to $d = 0$ and $d = 4F_{2k+1}F_{2k+2}F_{2k+3}$.

Let d be a positive integer such that the set $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$ has the property of Diophantus and assume that $d \neq 4F_{2k+1}F_{2k+2}F_{2k+3}$. Let $ad + 1 = x^2$. Then, by Lemma 1, $x = v_m = w_n$ and $m \neq 0, 2$. Lemma 2 implies that

$$|m| \geq 2b - 2 > 4a. \tag{8}$$

If we compare (7) and (8), we obtain

$$\frac{m}{\log^3 m} < 6.423 \cdot 10^{15},$$

which implies $m < 8 \cdot 10^{20}$, $a = F_{2k} < 2 \cdot 10^{20}$ and finally $k \leq 48$.

Thus we proved Theorem 2 for $k \geq 49$.

5 The Baker-Davenport reduction procedure

It remains to prove Theorem 2 for $2 \leq k \leq 48$ and we will do it using a version of the reduction procedure due to Baker and Davenport [2]. The followinf lemma is a slight modification of [7, Lemma 5] and its proof is completely analogous.

Lemma 4 *Suppose that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\varepsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.*

a) *If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < m\kappa - n + \mu < AB^{-m} \tag{9}$$

in integers m and n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leq m \leq M\,.$$

b) *Let $r = \lfloor \mu q + \frac{1}{2} \rfloor$. If $2p - 2q + r = 0$, then there is no solution of inequality (9) in integers m and n with*

$$\max\left(\frac{\log(3Aq)}{\log B}, 2\right) < m \leq M\,.$$

c) *If $p - q - r = 0$, then there is no solution of inequality (9) in integers m and n with*

$$\frac{\log(3Aq)}{\log B} \leq m \leq M\,.$$

We apply Lemma 4 with $\kappa = \frac{\log\alpha_1}{\log\alpha_2}$, $\mu = \frac{\log\alpha_3}{\log\alpha_2}$, $A = \frac{4}{\log\alpha_3}$, $B = (b - a + \sqrt{ab})^2$ and $M = 8 \cdot 10^{20}$. If the first convergent such that $q > 6M$ does not satisfy the conditions **a)**, **b)** or **c)** of Lemma 4, then we use the next convergent. We have to consider $4 \cdot 47 = 188$ cases, and the use of next convergent is necessary in 10 cases. In all cases the reduction gives new bound $m \leq M_0$, where $M_0 \leq 12$. The next step of the reduction (the applying of Lemma 4 with $M = M_0$) gives $m \leq 2$ in all cases, which completes the proof of Theorem 2.

6 Final remarks

The Hoggatt-Bergum conjecture is a special case of a more general conjecture. Namely, Arkin, Hoggatt and Strauss [1] proved that every Diophantine triple $\{a, b, c\}$ can be extended to the Diophantine quadruple $\{a, b, c, d\}$. More precisely, if

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2,$$

then we can take

$$d = a + b + c + 2abc \pm 2rst.$$

The conjecture is that, in the above notation, d has to be $a+b+c+2abc\pm 2rst$.

In [12], Kedlaya verified the conjecture for the triples $\{1, 3, 120\}$, $\{1, 8, 15\}$, $\{1, 8, 120\}$, $\{1, 15, 24\}$, $\{1, 24, 35\}$ and $\{2, 12, 24\}$. In [6], the conjecture was verified for all triples of the form $\{k, k + 2, 4k + 4\}$, and in [7] for all Diophantine triples of the form $\{1, 3, c\}$.

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