

A variant of Wiener’s attack on RSA

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Abstract

Wiener’s attack is a well-known polynomial-time attack on a RSA cryptosystem with small secret decryption exponent d , which works if $d < n^{0.25}$, where $n = pq$ is the modulus of the cryptosystem. Namely, in that case, d is the denominator of some convergent p_m/q_m of the continued fraction expansion of e/n , and therefore d can be computed efficiently from the public key (n, e) .

There are several extensions of Wiener’s attack that allow the RSA cryptosystem to be broken when d is a few bits longer than $n^{0.25}$. They all have the run-time complexity (at least) $O(D^2)$, where $d = Dn^{0.25}$. Here we propose a new variant of Wiener’s attack, which uses results on Diophantine approximations of the form $|\alpha - p/q| < c/q^2$, and “meet-in-the-middle” variant for testing the candidates (of the form $rq_{m+1} + sq_m$) for the secret exponent. This decreases the run-time complexity of the attack to $O(D \log D)$ (with the space complexity $O(D)$).

1 Introduction

The most popular public key cryptosystem in use today is the RSA cryptosystem, introduced by Rivest, Shamir, and Adleman [10]. Its security is based on the intractability of the integer factorization problem.

The modulus n of a RSA cryptosystem is the product of two large primes p and q . The public exponent e and the secret exponent d are related by

$$ed \equiv 1 \pmod{\varphi(n)}, \tag{1}$$

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where $\varphi(n) = (p - 1)(q - 1)$. In a typical RSA cryptosystem, p and q have approximately the same number of bits, while $e < n$. The encryption and decryption algorithms are given by $C = M^e \bmod n$, $M = C^d \bmod n$.

To speed up the RSA decryption one may try to use small secret decryption exponent d . The choice of a small d is especially interesting when there is a large difference in computing power between two communicating devices, e.g. in communication between a smart card and a larger computer. In this situation, it would be desirable that the larger computer has a small public exponent, while the smart card has a small secret exponent, to reduce the processing required in the smart card for encryption, resp. decryption.

In 1990, Wiener [15] described a polynomial time algorithm for breaking a typical (i.e. p and q are of the same size and $e < n$) RSA cryptosystem if the secret exponent d has at most one-quarter as many bits as the modulus n . From (1) it follows that there is an integer k such that $ed - k\varphi(n) = 1$. Since $\varphi(n) \approx n$, we have that $\frac{k}{d} \approx \frac{e}{n}$. Wiener's attack is usually described in the following form (see [3, 11]):

If $p < q < 2p$, $e < n$ and $d < \frac{1}{3}\sqrt[4]{n}$, then d is the denominator of some convergent of the continued fraction expansion of $\frac{e}{n}$.

Indeed, under these assumptions it is easy to show that

$$\left| \frac{e}{n} - \frac{k}{d} \right| < \frac{1}{2d^2}.$$

By the classical Legendre's theorem, $\frac{k}{d}$ is some convergent $\frac{p_m}{q_m}$ of the continued fraction expansion of $\frac{e}{n}$, and therefore d can be computed efficiently from the public key (n, e) . Namely, the total number of convergents is of order $O(\log n)$, and each convergent can be tested in polynomial time.

In 1997, Verheul and van Tilborg [14] proposed an extension of Wiener's attack that allows the RSA cryptosystem to be broken when d is a few bits longer than $n^{0.25}$. For $d > n^{0.25}$ their attack needs to do an exhaustive search for about $2t + 8$ bits (under reasonable assumptions on involved partial convergents), where $t = \log_2(d/n^{0.25})$.

In [5], we proposed a slight modification of the Verheul and van Tilborg attack, based on Worley's result on Diophantine approximations [16], which implies that all rationals $\frac{p}{q}$ satisfying the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{c}{q^2}, \tag{2}$$

for a positive real number c , have the form

$$\frac{p}{q} = \frac{rp_{m+1} \pm sp_m}{rq_{m+1} \pm sq_m} \quad (3)$$

for some $m \geq -1$ and nonnegative integers r and s such that $rs < 2c$. It has been shown recently in [6] that Worley's result is sharp, in the sense that the condition $rs < 2c$ cannot be replaced by $rs < (2 - \varepsilon)c$ for any ε .

In both mentioned extensions of Wiener's attack, the candidates for the secret exponent are of the form $d = rq_{m+1} + sq_m$. Then we test all possibilities for d . The number of possibilities is roughly the product of the number of possibilities for r and the number of possibilities for s , which is $O(D^2)$, where $d = Dn^{0.25}$. More precisely, the number of possible pairs (r, s) in the Verheul and van Tilborg attack is $O(D^2 A^2)$, where $A = \max\{a_i : i = m+1, m+2, m+3\}$ and the a_i 's are partial quotients in the continued fraction expansion of $\frac{e}{n}$, while in our variant the number of pairs is $O(D^2 \log A)$ (and also $O(D^2 \log D)$).

Another modification of the Verheul and van Tilborg attack has been recently proposed by Sun, Wu and Chen [13]. It requires (heuristically) an exhaustive search for about $2t - 10$ bits, so its complexity is also $O(D^2)$. We cannot expect drastic improvements here, since, by a result of Steinfeld, Contini, Wang and Pieprzyk [12], there does not exist an attack in this class with subexponential running time.

Boneh and Durfee [4] and Blömer and May [2] proposed attacks based on Coppersmith's lattice-based technique for finding small roots of modular polynomial equations using LLL-algorithm. The attacks work if $d < n^{0.292}$. The similar methods have been applied in [7, 1] to partial key exposure attack on RSA, i.e. in the situations where the secret key d is small and also a sufficient amount of d is given. The conjecture is that the right bound below which a typical version of RSA is insecure is $d < n^{0.5}$.

In the present paper, we propose a new variant of Wiener's attack. It also uses continued fractions and searches for candidates for the secret key in the form $d = rq_{m+1} + sq_m$. However, the searching phase of this variant is significantly faster. Its complexity is $O(D \log D)$, and it works efficiently for $d < 10^{30} n^{0.25}$. Although this bound is asymptotically weaker than the bounds in the above mentioned attacks based on the LLL-algorithm (note however that these bounds are not strictly proved since Coppersmith's theorem in the bivariate case is only a heuristic result - see also [8, 9]), for practical values of n (e.g. for 1024-bits) these bounds are of comparable size.

2 The Verheul and van Tilborg attack

In this section we briefly describe the Verheul and van Tilborg attack [14] and its modification from [5].

We assume that $p < q < 2p$ and $e < n$. Then it is easy to see that

$$\left| \frac{e}{n} - \frac{k}{d} \right| < \frac{2.122e}{n\sqrt{n}}. \quad (4)$$

Let m be the largest (odd) integer satisfying $\frac{p_m}{q_m} - \frac{e}{n} > \frac{2.122e}{n\sqrt{n}}$. Verheul and van Tilborg proposed to search for $\frac{k}{d}$ among the fractions of the form $\frac{rp_{m+1}+sp_m}{rq_{m+1}+sq_m}$. This leads to the system

$$\begin{aligned} rp_{m+1} + sp_m &= k, \\ rq_{m+1} + sq_m &= d. \end{aligned}$$

The determinant of the system satisfies $|p_{m+1}q_m - q_{m+1}p_m| = 1$, and therefore the system has (positive) integer solutions:

$$\begin{aligned} r &= dp_m - kq_m, \\ s &= kq_{m+1} - dp_{m+1}. \end{aligned}$$

If r and s are small, then they can be found by an exhaustive search. Let $[a_0; a_1, a_2, \dots]$ be the continued fraction expansion of e/n and $D = d/n^{0.25}$. In [5], the following upper bounds for r and s were derived:

$$\begin{aligned} r &< \max\{\sqrt{2.122(a_{m+3} + 2)}(a_{m+2} + 1)D, \sqrt{2.122(a_{m+2} + 2)}D\}, \\ s &< \max\{2\sqrt{2.122(a_{m+3} + 2)}D, \sqrt{2.122(a_{m+2} + 2)}(a_{m+1} + 1)D\}. \end{aligned}$$

The modified attack proposed in [5] searches for $\frac{k}{d}$ among the fractions of the forms $\frac{rp_{m+1}+sp_m}{rq_{m+1}+sq_m}$, $\frac{rp_{m+2}-sp_{m+1}}{rq_{m+2}-sq_{m+1}}$ and $\frac{rp_{m+3}+sp_{m+2}}{rq_{m+3}+sq_{m+2}}$. The analysis results in bounds for r and s which are (almost) independent on the partial quotients a_m 's. Hence, in both attacks bounds for r and s are of the form $O(D)$, but in the case of [5] the implied constants are much smaller (indeed, the table in Section 4 shows that with high probability we have $r < 4D$ and $s < 4D$).

3 Testing the candidates

There are two principal methods for testing candidates for the secret exponent d .

Method I ([15]): Compute p and q , assuming d is the correct guess, using the following formulas:

$$\begin{aligned}\varphi(n) &= (de - 1)/k, & p + q &= n + 1 - \varphi(n), \\ (q - p)^2 &= (p + q)^2 - 4n, \\ p &= \frac{p + q}{2} - \frac{q - p}{2}, & q &= \frac{p + q}{2} + \frac{q - p}{2}.\end{aligned}$$

Method II ([11, Chapter 17]): Test the congruence $(M^e)^d \equiv M \pmod{n}$, for some random value of M , or simply for $M = 2$.

Both methods are very efficient. But in the situation where we have to test huge amount of candidates for d of the form $rq_{m+1} + sq_m$, there is a significant difference between them. With the Method I it seems that we cannot avoid testing separately all possible pairs (r, s) . On the other hand, here we present a new idea, which is to apply “meet-in-the-middle” to the Method II.

We want to test whether

$$2^{e(rq_{m+1} + sq_m)} \equiv 2 \pmod{n}. \quad (5)$$

Note that m is (almost) fixed. Indeed, let m' be the largest odd integer such that

$$\frac{p_{m'}}{q_{m'}} > \frac{e}{n} + \frac{2.122e}{n\sqrt{n}}.$$

Then $m \in \{m', m' + 1, m' + 2\}$ (see [5] for details).

Let $2^{eq_{m+1}} \pmod{n} = a$, $(2^{eq_m})^{-1} \pmod{n} = b$. Then we test the congruence

$$a^r \equiv 2b^s \pmod{n}. \quad (6)$$

We can do it by computing $a^r \pmod{n}$ for all r , sorting the list of results, and then computing $2b^s \pmod{n}$ for each s one at a time, and checking if the result appears in the sorted list.

This decreases the time complexity of the testings phase to $O(D \log D)$ (with the space complexity $O(D)$).

4 Implementation issues and improvements

The theoretic base for the extension of Wiener’s attack is Worley’s theorem on Diophantine approximations of the form (2). We have already mentioned a result from [6] which shows that Worley’s result is in some sense the best possible. However, some improvements are possible if we consider unsymmetrical variants of Worley’s result (with different bounds on r and s). Roughly speaking, in solutions of (2) in form (3), if $r < s$ then we may take $rs < c$ instead of $rs < 2c$. Due to such unsymmetrical results, a space-time tradeoff might be possible. The following table shows the chance of success of our attack for various (symmetrical and unsymmetrical) bounds on r and s . We can see that the result obtained for a smaller bound on r and a larger bound on s is better than the result with reversed bounds on r and s . In the implementations, this fact can be used to decrease the memory requirements (up to factor 16).

bound for r	bound for s	chance of success
$4D$	$4D$	98%
$2D$	$2D$	89%
D	D	65%
D	$4D$	86%
$4D$	D	74%
$D/2$	$2D$	70%
$2D$	$D/2$	47%
$D/4$	$4D$	54%
$4D$	$D/4$	28%

In the implementation of the proposed attack, we can use hash functions instead of sorting. Furthermore, it is not necessary to store all bits of $a^r \bmod n$ in the hash table. Indeed, values of $a^r \bmod n$ are from the set $\{0, 1, \dots, n\}$, and the number of r ’s is typically much smaller than n . Therefore, around $2 \log_2 D$ stored bits will suffice in order to avoid too many accidental collisions. Note that a reasonable number of collisions is not big problem here, since each such collision can be efficiently tested by Method I. Hash tables can be used to take into account the condition $\gcd(r, s) = 1$. This condition was easy to use in brute-force testing of all possible pairs (r, s) , but the direct application of our “meet-in-the-middle” variant seemingly ignores

it. But if we create rows in the hash table according to divisibility properties of exponents r modulo small primes, we may take again an advantage of this condition and speed up the algorithm up to 39%.

We have implemented several variants of the proposed attack in PARI and C++, and they work efficiently for values of D up to 2^{30} , i.e. for $d < 2^{30}n^{0.25}$.

For larger values of D the memory requirements become too demanding for ordinary computers.

The following table compares this bound with the bound of d in the best known attacks on RSA with small secret exponent based on LLL-algorithm.

$\log_2 n$	$\log_2(2^{30}n^{0.25})$	$\log_2(n^{0.292})$
512	158	150
768	222	224
1024	286	299
2048	542	598

The attack can be also slightly improved by using better approximations to $\frac{k}{d}$, e.g. $\frac{e}{n+1-2\sqrt{n}}$ instead of $\frac{e}{n}$. Namely,

$$\left| \frac{e}{n+1-2\sqrt{n}} - \frac{k}{d} \right| < \frac{0.1221 e}{n\sqrt{n}}. \quad (7)$$

Comparing (7) with (4), we see that by replacing $\frac{e}{n}$ by $\frac{e}{n+1-2\sqrt{n}}$ we can gain the factor 4 in bounds for r and s , so decreasing both, time and memory requirements.

With these improvements, for 1024-bits RSA modulus n , the range in which our attack can be applied becomes comparable and competitive with best known attacks based on the LLL-algorithm.

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