A bijective proof of Riordan's theorem on powers of Fibonacci numbers

Andrej Dujella
Department of Mathematics, University of Zagreb, Bijenička cesta 30
10000 Zagreb, CROATIA

Abstract

Let $F_k(x) = \sum_{n=0}^{\infty} F_n^k x^n$. Using the interpretation of Fibonacci numbers in the terms of Morse codes, we give a bijective proof of Riordan's formula

$$(1 - L_k x + (-1)^k x^2) F_k(x) = 1 + kx \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(-1)^j}{j} a_{kj} F_{k-2j}((-1)^j x),$$

where $L_k = F_k + F_{k-2}$, and a_{kj} is defined by means of

$$(1 - x - x^2)^{-j} = \sum_{k=2j}^{\infty} a_{kj} x^{k-2j}.$$

The Fibonacci numbers F_n may be defined by $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$. Their generating function is

$$F_1(x) = \sum_{n=0}^{\infty} F_n x^n = (1 - x - x^2)^{-1}.$$

More generally we may put

$$F_k(x) = \sum_{n=0}^{\infty} F_n^k x^n.$$

Riordan [2] has proved that $F_k(x)$ satisfies the following recurrence relation:

$$(1 - L_k x + (-1)^k x^2) F_k(x) = 1 + kx \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(-1)^j}{j} a_{kj} F_{k-2j}((-1)^j x), \tag{1}$$

$k \geq 1$, where $L_k = F_k + F_{k-2}$ are Lucas numbers, and a_{kj} is defined by means of

$$(1 - x - x^2)^{-j} = \sum_{k=2j}^{\infty} a_{kj} x^{k-2j}.$$

It is easy to verify that

$$a_{kj} = \frac{j}{k} \sum_{m=1}^{\lfloor k/2 \rfloor} b_{mk} \binom{m}{j},$$

where $b_{mk} = \frac{k}{m}\binom{k-m-1}{m-1}$. Therefore, relation (1) is an immediate consequence of the relation

$$F_n^k = L_k F_{n-1}^k - (-1)^k F_{n-2}^k + \delta_{n,0} + \sum_{m,j \geq 1} b_{mk} \binom{m}{j} (-1)^{jn} F_{n-1}^{k-2j}, \qquad (2)$$

where $\delta_{n,0} = 1$ if $n = 0$, and $\delta_{n,0} = 0$ if $n > 0$.

The purpose of the present paper is to give a bijective proof of relation (2).

Our starting point is the proof due to Werman and Zeilberger [3] of Cassini's identity:

$$F_n^2 - F_{n-1} F_{n+1} = (-1)^n.$$

Their proof is based on the fact that $F_n = |A(n)|$, where $A(n) = \{(a_1, \ldots, a_r) \ : \ r \geq 0,\, a_i = 1 \text{ or } 2,\, a_1 + \cdots a_r = n\}$. They defined the bijection $\pi_n : A(n) \times A(n) \setminus (e, e) \to A(n-1) \times A(n+1) \setminus (e, e)$, where $e = (2, \ldots, 2)$, as follows. Let $((a_1, \ldots, a_r), (b_1, \ldots, b_s)) \in A(n) \times A(n)$ and look for the first 1 in $a_1, b_1, a_2, b_2, \ldots$. If the first 1 is an a_k, then delete $a_k = 1$ from the first vector and insert it between b_{k-1} and b_k in the second vector. If the first 1 is a b_k, then exchange a_k and b_k.

An equivalent interpretation of Fibonacci numbers is that F_n is the number of Morse code sequences of length n, where each dot contributes 1 to the length and each dash contributes 2 (see [1, p. 302]). There is a similar interpretation of Lucas numbers. Namely, L_n is the number of cyclic Morse codes of length n, i. e. we allow a dash connecting the last and the first positions. Equivalently, $L_n = |B(n)|$, where $B(n) = \{(b_1, \ldots, b_r) \ : \ r \geq 0,\, b_i = 1 \text{ or } 2,\, b_1 + \cdots b_r = n \text{ or } n-2\}$.

Relation (2) suggest that the numbers F_n^k and $L_k F_{n-1}^k$ are close to each other (the same follows also from Binet's formula). Let

$$M_n^{(k)} = \underbrace{A(n) \times \cdots \times A(n)}_{k \text{ times}}, \quad N_n^{(k)} = \underbrace{A(n-1) \times \cdots \times A(n-1)}_{k \text{ times}} \times B(k).$$

Since $|M_n^{(k)}| = F_n^k$, $|N_n^{(k)}| = L_k F_{n-1}^k$, we will try to construct a function $m_n^k : N_n^{(k)} \to M_n^{(k)}$ which would be an "almost" bijection.

Denote by ρ_n the inverse of π_n. Let $(A_1, \ldots, A_k, B) \in N_n^{(k)}$, $A_i = (a_1^i, \ldots, a_{r_i}^i)$, $i = 1, \ldots, k$, $B = (b_1, \ldots, b_s)$.

Let $c_j = b_1 + \cdots + b_j + \varepsilon$, $j = 1, \ldots, s$, where $\varepsilon = 0$ if $b_1 + \cdots + b_s = k$, and $\varepsilon = 1$ if $b_1 + \cdots + b_s = k - 2$. Then we define the function m_n^k by

$$m_n^k(A_1, \ldots, A_k, B) = (D_1, \ldots, D_k),$$

such that if $b_j = 1$ then $D_{c_j} = (a_1^{c_j}, a_2^{c_j}, \ldots, a_{r_{c_j}}^{c_j}, 1)$, if $b_j = 2$ then $(D_{c_j - 1}, D_{c_j}) = \rho_n(A_{c_j - 1}, (a_1^{c_j}, \ldots, a_{r_{c_j}}^{c_j}, 2))$, and if $\varepsilon = 1$ then $(D_k, D_1) = \rho_n(A_k, (a_1^1, a_2^1, \ldots, a_{r_1}^1, 2))$.

Using the Morse code interpretation, we illustrate this function by an example:

$$\begin{array}{lcl cl}
- \; - \; \cdot & & - \; - \; \cdot & & - \; - \; \cdot \\
\cdot \; - \; \cdot & & \cdot \; - \; \cdot & & \cdot \; \cdot \; - \; \cdot \\
\cdot \; \cdot \; \cdot \; \cdot \; | & \mapsto & \cdot \; \cdot \; \cdot \; \cdot \; - & \mapsto & \cdot \; \cdot \; \cdot \; - \\
- \; \cdot \; \cdot & & - \; \cdot \; \cdot & & - \; - \; \cdot \\
- \; - \; | & & - \; - \; - & & - \; \cdot \; -
\end{array}$$

The function m_n^k is "almost" bijection, but in general it is not nor injection nor surjection, and moreover it is not defined on entire set $N_n^{(k)}$. Let us consider this three problem more precisely.

1) Domain of $\mathbf{m_n^k}$.

Let $\mathcal{D}(\rho_n)$ and $\mathcal{D}(m_n^k)$ denote the domains of the functions ρ_n and m_n^k.

If n is even, then $\mathcal{D}(\rho_n) = A(n-1) \times A(n+1)$ and consequently $\mathcal{D}(m_n^k) = N_n^{(k)}$.

If n is odd, then $\mathcal{D}(\rho_n) = A(n-1) \times A(n+1) \setminus \{(e, e)\}$. Thus m_n^k is not defined on configurations which contain a subconfiguration of the form

$$\begin{array}{ccc}
- \; - \; - & \ldots & - \; - \; - \\
- \; - \; - & & - \; - \; - \; |
\end{array}$$

Let B_{mk} denotes the number of cyclic Morse codes of the length k with m dashes. Then by the principle of inclusion-exclusion we obtain:

$$|N_n^{(k)} \setminus \mathcal{D}(m_n^k)| = \sum_{m \geq 1} b_{mk} (m \cdot F_{n-1}^{k-2} - \binom{m}{2} \cdot F_{n-1}^{k-4} + \binom{m}{4} \cdot F_{n-1}^{k-6} - \cdots).$$

Furthermore,

$$b_{mk} = \binom{k-m}{m} + \binom{k-1-m}{m-1} = \frac{k}{m}\binom{k-m-1}{m-1}.$$

2) Injection.

Let $x = (A_1, \ldots, A_k, B)$, $x' = (A'_1, \ldots, A'_k, B') \in N_n^{(k)}$, $x \neq x'$, and $m_n^k(x) = m_n^k(x') = (D_1, \ldots, D_k)$.

Assume that $B \neq B'$ and $B \neq (2, \ldots, 2)$. Then there exist indices i, j such that $b_i = 1$, $b'_j = 2$ and $c_i = c'_j - 1$. Now we have: $b_{i+1} = 2$, $c_{i+1} = 2$, $b_{i+2} = 2$, $c_{i+2} = 2$, ... (the addition in indices is modulo k) which leads to a contradiction (k would be simultaneously even and odd). Since ρ_n is an injection we conclude that if $x \neq x'$ and $m_n^k(x) = m_n^k(x')$, then $B = (2, \ldots, 2)$, $b_1 + \cdots + b_s = k$, $B' = (2, \ldots, 2)$, $b'_1 + \cdots + b'_{s'} = k - 2$, and $D_i = (d_1^i, \ldots, d_{t_{i-1}}^i, 2)$, $i = 1, \ldots, k$.

This, first of all, implies that k is even. Thus, if k is odd then m_n^k is an injection. Let k be even. Since ρ_n is an injection, we have $|(m_n^k)^{-1}(y)| \leq 2$, for all $y \in M_n^{(k)}$. Furthermore, $|\{y : |(m_n^k)^{-1}(y)| = 2\}| = F_{n-2}^k - Z_{nk}$, where

$$\begin{aligned} Z_{nk} = |\{y = (D_1, \ldots, D_k) \in M_n^{(k)} \, : \, D_i = (d_1^i, \ldots, d_{t_{i-1}}^i, 2), \\ i = 1, \ldots k, \, y \notin m_n^k(N_n^{(k)})\}|. \end{aligned} \tag{3}$$

3) Surjection.

First of all, observe that if k is odd then $y \notin m_n^k(N_n^{(k)})$ for all y of the form $(D_1, \ldots, D_k)$, $D_i = (d_1^i, \ldots, d_{t_{i-1}}^i, 2)$, $i = 1, \ldots, k$, and the number of such y's is F_{n-2}^k.

Besides of this, observe that $\rho_n(A(n-1) \times A(n+1)) = A(n) \times A(n) \setminus \{(e, e)\}$ if n is even. Thus, if n is even then we have some more elements in $M_n^{(k)}$ which are not in $m_n^k(N_n^{(k)})$. The number of such elements is

$$\sum_{m \geq 1} b_{mk}(m \cdot F_{n-1}^{k-2} + \binom{m}{2} \cdot F_{n-1}^{k-4} + \binom{m}{4} \cdot F_{n-1}^{k-6} + \cdots) - Z_{nk}, \tag{4}$$

where Z_{nk} is defined by (3). Indeed, given a cyclic Morse code of the length k with m dashes, choose j dashes and put Morse code consisting of $\frac{n}{2}$ dashes in corresponding $2j$ rows. In remaining $k-2j$ rows, we have F_{n-1}^{k-2j} possibilities for the first $n-1$ columns and the we apply the function m_n^{k-2j} to these $n-1$ columns and the remainder of the given cyclic Morse code in the last column. In this way, we obtain all elements with prescribed property, and the elements with dashes in the last column are counted twice. This proves formula (4).

From **1)**, **2)** and **3)** it follows that formula (2) is valid, and the proof is complete.

Acknowledgements. I would like to thank Professor D. Svrtan for drawing my attention to this topic.

References

[1] R.L. Graham, D.E. Knuth and O. Patashnik, Concrete Mathematics, 2nd ed. (Addison-Wesley, Reading, 1994).

[2] J. Riordan, Generating functions for power of Fibonacci numbers, Duke Math. J. 29 (1962), 5–12.

[3] M. Werman and D. Zeilberger, A bijective proof of Cassini's Fibonacci identity, Discrete Math. 58 (1986), 109.