AN INTERESTING PROPERTY OF A RECURRENCE RELATED TO THE FIBONACCI SEQUENCE

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1   Introduction

The sequence of Fibonacci numbers with even subscripts (F_{2n}) has one remarkable property. If we choose three successive elements of this sequence, than the product of any two of them increased by 1 is a perfect square. Indeed,

$$F_{2n} \cdot F_{2n+2} + 1 = F_{2n+1}^2, \quad F_{2n} \cdot F_{2n+4} + 1 = F_{2n+2}^2.$$

This property was studied and generalized by several authors (see references). Let us just mention that Hoggatt and Bergum [8] proved that the number $d = 4F_{2n+1}F_{2n+2}F_{2n+3}$ has the property that $F_{2n} \cdot d + 1$, $F_{2n+2} \cdot d + 1$ and $F_{2n+4} \cdot d + 1$ are perfect squares, and Dujella [7] proved that the positive integer d with the above property is unique.

The purpose of this paper is to characterize linear binary recursive sequences which possess the similar property as the above property of Fibonacci numbers.

We will consider binary recursive sequences of the form

$$G_{n+1} = AG_n - G_{n-1}, \tag{1}$$

where A, G_0, G_1 are integers. We call the sequence (G_n) *nondegenerated* if $|G_0| + |G_1| > 0$ and the quotient of the roots $\alpha, \beta \in \mathbf{C}$ of the characteristic equation of G_n,

$$x^2 - Ax + 1 = 0,$$

is not a root of unity. Let $D = A^2 - 4$, $C = G_1^2 - AG_0G_1 + G_0^2$. Then nondegeneracy implies $|A| \geq 3$ and $C \neq 0$. Solving the recurrence (1) we obtain

$$G_n = \frac{a\alpha^n - b\beta^n}{\alpha - \beta},$$

where $a = G_1 - G_0\beta$, $b = G_1 - G_0\alpha$.

Definition 1 *Let k be an integer. A sequence (G_n) is said to have the property $P(k)$ if $G_nG_{n+1} + k$ and $G_nG_{n+2} + k$ are perfect squares for all $n \geq 0$.*

With this notation, we may say that the sequence (F_{2n}) has the property $P(1)$.

Our main result is the following theorem.

Theorem 1 *Let (G_n) be a nondegenerated binary recursive sequence given by (1). If G_n has the property $P(k)$ for some $k \in \mathbf{Z}$, then $A = 3$ and $k = G_0^2 - 3G_0G_1 + G_1^2$.*

Remark 1 The sequences from Theorem 1 have the form

$$G_n = G_1F_{2n} - G_0F_{2n-2}$$

and for $G_0 = 0$ and $G_1 = 1$ we obtain exactly the sequence (F_{2n}). Note that the converse of Theorem 1 is also valid. This follows from the formula $G_nG_{n+2} + k = G_{n+1}^2$ proved below, and the general fact that if $ab + k = r^2$, then $a(a + b - 2r) + k = (a - r)^2$.

2 Proof of Theorem 1

Assume that k is an integer such that the sequence (G_n) has the property $P(k)$. It implies that $G_nG_{n+2} + k$ is a perfect square for all $n \geq 0$. On the other hand,

$$
\begin{aligned}
G_nG_{n+2} &= \frac{a^2\alpha^{2n+2} + b^2\beta^{2n+2} - ab(\alpha\beta)^n(\alpha^2 + \beta^2)}{(\alpha - \beta)^2} \\
&= \left(\frac{a\alpha^{n+1} - b\beta^{n+1}}{\alpha - \beta}\right)^2 - \frac{ab(\alpha\beta)^n(\alpha - \beta)^2}{(\alpha - \beta)^2} \\
&= G_{n+1}^2 - ab = G_{n+1}^2 - C.
\end{aligned}
$$

Hence, $G_{n+1}^2 - C + k$ is a perfect square for all $n \geq 0$. This implies that $k = C$.

Our problem is now reduced to finding sequences such that

$$G_n G_{n+1} + C$$

is a perfect square for all $n \geq 0$.

We have $G_{n+1}^2 - A G_n G_{n+1} + G_n^2 = C$ (see [9]). Denote

$$G_n G_{n+1} + C = G_n^2 - (A-1)G_n G_{n+1} + G_{n+1}^2$$

by H_n. It can be easily verified that the sequence (H_n) satisfies the recurrence relation

$$H_{n+1} = (A^2 - 2)H_n - H_{n-1} - C(A^2 - A - 4).$$

Finally, put

$$S_n = (A^2 - 4)H_n - C(A^2 - A - 4).$$

Then the sequence (S_n) satisfies the homogeneous recurrence relation

$$S_{n+1} = (A^2 - 2)S_n - S_{n-1}.$$

Denote the polynomial $(A^2 - 4)x^2 - C(A^2 - A - 4)$ by $R(x)$. Then our condition implies that for every $n \geq 0$ there exist $x \in \mathbf{Z}$ such that

$$S_n = R(x). \tag{2}$$

Therefore, the equation (2) has infinitely many solutions.

Let $D_1 = (A^2 - 2)^2 - 4 = A^2(A^2 - 4)$ and $C_1 = S_1^2 - S_0 S_2 = -(A^2 - 4)A^2 C^2$ be the discriminant and the characteristic of the sequence (S_n), respectively. Assume also that

$$S_n = \frac{a_1 \alpha^{2n} - b_1 \beta^{2n}}{\alpha^2 - \beta^2} \qquad \text{for some } a_1 \text{ and } b_1,$$

and put

$$T_n = a_1 \alpha^{2n} + b_1 \beta^{2n} \qquad \text{for all } n \geq 0.$$

Then, since

$$T_n^2 = D_1 S_n^2 + 4C_1 \qquad \text{for all } n \geq 0,$$

and since the equation $S_n = R(x)$ has infinitely many integer solutions (n, x), it follows that the equation

$$y^2 = D_1 R(x)^2 + 4C_1$$

has infinitely many integer solutions (x, y). By a well known theorem of Siegel [20], we get that the polynomial $F(X) = D_1 R(X)^2 + 4C_1$ has at most two simple roots. Since F is of degree 4, it follows that F must have a double root. Notice that $F'(X) = 2D_1 R(X)R'(X) = 4(A^2 - 4)D_1 R(X)X$. Certainly, F and R cannot have a common root because this would imply that $C_1 = 0$, which is impossible since (G_n) is nondegenerated. Hence, $F(0) = 0$, which is equivalent to

$$A^2(A^2 - 4)[C(A^2 - A - 4)]^2 - 4A^2(A^2 - 4)C^2 = 0. \qquad (3)$$

Formula (3) implies that

$$A^2 - A - 4 = \pm 2.$$

If $A^2 - A - 4 = 2$ then $A = 3$ or $A = -2$, and if $A^2 - A - 4 = -2$ then $A = 2$ or $A = -1$. Since we assumed that the sequence (G_n) is nondegenerated, i.e. $|A| \geq 3$, we conclude that $A = 3$. ∎

Remark 2 In degenerated cases with $A = 0, \pm 1, \pm 2$, the sequence (G_n) also may have the property $P(k)$ for some $k \in \mathbf{Z}$. For example, for $A = 2$ the sequence $G_n = a$ has the property $P(b^2 - a^2)$; for $A = 0$ the sequence $G_{2n} = 0$, $G_{4n+1} = 2ab$, $G_{4n+3} = -2ab$ has the property $P((a^2 + b^2)^2)$; for $A = -1$ the sequence $G_{3n} = a$, $G_{3n+1} = b$, $G_{3n+2} = -a - b$ has the property $P(a^2 + ab + b^2)$. Here a and b are arbitrary integers.

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