On a problem of Diophantus for higher powers

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Abstract. Let $k \geq 3$ be an integer. We study the possible existence of finite sets of positive integers such that the product of any two of them increased by 1 is a k-th power.

1. Introduction

The Greek mathematician Diophantus observed that the rational numbers $\frac{1}{16}$, $\frac{33}{16}$, $\frac{17}{4}$ and $\frac{105}{16}$ have the following property: the product of any two of them increased by 1 is a square of a rational number. Later, Fermat found a set of four positive integers with the above property, namely the set $\{1, 3, 8, 120\}$. We call a Diophantine m-tuple any set of m positive integers $a_1, \ldots, a_m$ such that $a_i a_j + 1$ is a perfect square whenever $1 \leq i < j \leq m$. It was known already to Euler that there are infinitely many Diophantine quadruples (see for instance [5, pp. 513–520]). Among the broad literature on that topic, let us mention that Baker & Davenport [3] proved that $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple, a result improved by Dujella & Pethő [10], who showed that even $\{1, 3\}$ cannot be extended to a Diophantine quintuple. The first absolute upper bound for the size of Diophantine m-tuples was given by the second author in [7], where it was proved that Diophantine 9-tuples do not exist. Very recently, he was able to considerably improve upon his result, by showing [9] that there exist no Diophantine sextuple and only finitely many Diophantine quintuples. However, the question of the existence of a Diophantine quintuple remains a challenging open problem. We refer to [6] for further references on this topic.

In the present work, we are interested in an analogous problem, namely the existence of sets $\{a, b, c\}$ of positive integers such that the three numbers $ab+1$, $ac+1$ and $bc+1$ are perfect k-th powers, for an integer $k \geq 3$. Examples of such triples for k=3 and $k = 4$ are given, respectively, by $\{2, 171, 25326\}$ and $\{1352, 9539880, 9768370\}$. To our knowledge, no example of such triple is known for $k \geq 5$. In order to investigate this question, we study a slightly more general problem, recently considered by Gyarmati [12]. Let $N \geq 1$ and $k \geq 3$ be integers. Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\{1, \ldots, N\}$ such that $ab + 1$ is a perfect k-th power whenever $a \in \mathcal{A}$ and $b \in \mathcal{B}$. What can be said about the cardinalities of the sets $\mathcal{A}$ and $\mathcal{B}$? Let $|\mathcal{S}|$ denote the cardinality of a finite set $\mathcal{S}$. Using elementary arguments, Gyarmati [12] proved that $\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq 1 + (\log\log N)/\log(k-1)$. As a corollary of our main result, we show that, except for small values of k, we have the considerably better

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estimate $\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq 2$. We also provide an absolute (i.e. independent of N) upper bound for $\min\{|\mathcal{A}|, |\mathcal{B}|\}$ for the other values of k.

Our proofs rest on classical tools of Diophantine approximation, namely the theory of linear forms in logarithms and sharp irrationality measures for certain k-th roots of rational numbers.

2. Statement of the results

Theorem 1. *Let $k \geq 3$ and $0 < a < b < c < d$ be integers such that the four numbers*

$$ac + 1, \quad ad + 1, \quad bc + 1 \quad \text{and} \quad bd + 1$$

are perfect k-th powers. Then we have $k \leq 176$.

Remark : The proof of Theorem 1 rests on the theory of linear forms in two logarithms of algebraic numbers, and heavily depends on a refinement obtained by Shorey [17], who was first to notice that one gets the best possible estimates when the algebraic numbers involved are close to 1. Shorey's trick has numerous applications (see [19] for a survey), for instance to the exponential Diophantine equations $ax^n - by^n = c$, $\frac{x^n-1}{x-1} = y^q$ and $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$, considered, respectively, in [15], [13] and [4]. The numerical value we get in Theorem 1 is remarkably small. This is due to the use of the sharp estimate of Mignotte [16] (see Lemma 2 below), and to the fact that our problem allows us to take a very large ray ρ in the application of Lemma 2.

As an immediate corollary, we derive from Theorem 1 new results on the generalization of the problem of Diophantus mentioned in the Introduction.

Corollary 1. *For any integer $k \geq 177$, there exist no set of four positive integers such that the product of any two of them increased by 1 is a perfect k-th power.*

Corollary 2 below considerably improved Theorem 1 of Gyarmati [12] when the integer k is not too small.

Corollary 2. *Let $k \geq 177$ be an integer and $\mathcal{A}$ and $\mathcal{B}$ be sets of positive integers such that $ab + 1$ is a perfect k-th power for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then we have*

$$\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq 2.$$

Corollary 2 follows easily from Theorem 1. Indeed, if $a_1 < a_2 < a_3$ (*resp.* $b_1 < b_2 < b_3$) belong to $\mathcal{A}$ (*resp.* to $\mathcal{B}$), then we have either $a_1 < a_2 < b_2 < b_3$ or $b_1 < b_2 < a_2 < a_3$, and we may apply Theorem 1.

Theorem 2. *Let $4 \leq k \leq 176$ be an integer. Assume that the integers $0 < a < b < c_1 < \ldots < c_m$ are such that $ac_i + 1$ and $bc_i + 1$ are perfect k-th powers for any $1 \leq i \leq m$. Then there exists an effectively computable constant $C_1(k)$, depending only on k, such that $m \leq C_1(k)$. More precisely, we may take $C_1(4) = 3$ and $C_1(k) = 2$ for $k \geq 5$.*

Remark : The proof of Theorem 2 depends on a result of Evertse [11] on Thue equations $aX^n + bY^n = c$, whose proof uses hypergeometric methods. For $k \geq 6$, we could also derive Theorem 2 from Theorem 1 of Baker [1].

Unfortunately, the proof of Theorem 2 gives nothing for $k = 3$. In that case, we need a stronger assumption.

Theorem 3. *Assume that the integers* $0 < a < b < c < d_1 < \ldots < d_m$ *are such that* $ad_i + 1$, $bd_i + 1$ *and* $cd_i + 1$ *are perfect cubes for any* $1 \leq i \leq m$. *Then* $m \leq 6$.

New results on the problem considered by Gyarmati and on the generalization of the problem of Diophantus follow from Theorems 2 and 3.

Corollary 3. *Let* $3 \leq k \leq 176$ *be an integer and* $\mathcal{A}$ *and* $\mathcal{B}$ *be sets of positive integers such that* $ab + 1$ *is a perfect* k*-th power for any* $a \in \mathcal{A}$ *and* $b \in \mathcal{B}$. *Then there exists an effectively computable constant* $C_2(k)$, *depending only on* k, *such that*

$$\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq C_2(k).$$

More precisely, we may take $C_2(3) = 8$, $C_2(4) = 4$ *and* $C_2(k) = 3$ *for* $k \geq 5$.

The statement of Corollary 3 follows directly from Theorems 2 and 3, as Corollary 1 follows from Theorem 1.

Corollary 4. *Let* $k \geq 2$ *be an integer. Assume that the integers* $0 < a_1 < a_2 < \ldots < a_m$ *are such that* $a_i a_j + 1$ *are perfect* k*-th powers whenever* $1 \leq i < j \leq m$. *Then there exists an effectively computable constant* $C_3(k)$, *depending only on* k, *such that* $m \leq C_3(k)$. *More precisely, we may take* $C_3(2) = 5$, $C_3(3) = 7$, $C_3(4) = 5$, $C_3(k) = 4$ *for* $5 \leq k \leq 176$ *and* $C_3(k) = 3$ *for* $k \geq 177$.

The statement of Corollary 4 for $k \geq 4$ follows directly from Corollaries 2 and 3. The statement for $k = 2$ is just the main result from [9], while the statement for $k = 3$ will be proved in Section 4 using a special gap principle.

One can obtain weeker results than in Theorem 1 by using a result of Shorey & Nesterenko [20] on irrationality measures of k-th roots of certain rational numbers, derived from a theorem of Baker [2]. Already in a few papers (see for instance [18], [13], [4] and the survey [19]), the authors have successfully combined this method with the theory of linear forms in logarithms. Here, we are able to complement Theorem 2 in the range $11 \leq k \leq 176$.

Theorem 4. *Let* $11 \leq k \leq 176$. *Then there are only finitely many quadruples of integers* $0 < a < b < c < d$ *such that the four numbers*

$$ac + 1, \quad ad + 1, \quad bc + 1 \quad \text{and} \quad bd + 1$$

are perfect k*-th powers.*

Remark : Let us mention that for $k = 3$, 4 and 6, there are triples $a < b < c$ of positive integers such that $ac + 1$ and $bc + 1$ are perfect k-powers. E.g. for $k = 6$ the triple $(a, b, c) = (8, 45, 91)$ has the above property. Moreover, for $k = 3$ and $k = 4$ there exist infinite families of such triples.

For $k = 3$, let (x_n, y_n) denote the sequence of the positive integer solutions of Pell equation $x^2 - 7y^2 = 1$ and let $n \equiv 2 \mod 7$. Then we may take $a = (x_n + 5y_n - 3)/14$, $b = (5x_n + 7y_n - 3)/2$ and $c = ((5x_n + 7y_n)^2 + 3)/4$.

For $k = 4$, we may take $a = (F_n^2 - 1)/5$, $b = L_n^2 - 1$ and $c = L_n^2 + 1$, where $n \equiv 2$ or 8 mod 10, while F_n, L_n denote, respectively, n-th Fibonacci and Lucas number.

Remark : The methods used to prove Theorems 1 and 2 can also be applied to investigate similar questions, like the existence of quadruples of positive integers $0 < a < b < c < d$ such that the product of any two of them increased by N is a k-th power, where N is a fixed non-zero integer. For instance, we can explicitely compute an integer $k_0(N)$, depending only on N, such that such quadruples do not exist whenever $k > k_0(N)$. The case $k = 2$ has been studied by the second author [8].

3. Auxiliary lemmas

Lemma 1. *Let $k \geq 3$ be an integer. Let $a < b < c_1 < c_2$ be positive integers such that $ac_i + 1$ and $bc_i + 1$ are k-th powers for $i \in \{1, 2\}$. Then we have $bc_2 > k^k c_1^{k-1} a^{k-1}$ and $c_2 > k^k c_1^{k-2} a^{k-1}$. Further, if $a_1 < a_2 < \cdots < a_7$ are positive integers such that $a_i a_j + 1$ is a perfect cube for all $1 \leq i < j \leq 7$, then $a_7 > 3^{45} a_3^9 a_1^{22}$. Finally, if $a < b < c < d_1 < d_2$ are positive integers such that $ad_i + 1$, $bd_i + 1$ and $cd_i + 1$ are perfect cubes for $i \in \{1, 2\}$, then $d_2 > 27 d_1^{3-\sqrt{2}}$.*

Proof : The first statement follows from the proof of [12, Theorem 1] applied to the sets $\{a, b\}$ and $\{c_1, c_2\}$.

Assume now that $k = 3$. Then, by the same result of Gyarmati, we have $a_4 a_2 > 3^3 a_3^2 a_1^2$ and $a_5^2 a_3^2 > 3^6 a_4^4 a_2^4$. Multiplying these two inequalities we obtain

$$a_5^2 > 3^9 a_4^3 a_2^3 a_1^2 > 3^9 (3^3 a_3^2 a_1^2)^3 a_1^2 = 3^{18} a_3^6 a_1^8.$$

Therefore, we get

$$a_5 > 3^9 a_3^3 a_1^4.$$

Now we have

$$a_6 > 3^3 a_5^2 a_2^2 / a_3 > 3^{21} a_3^6 a_1^8 a_2^2 / a_3 > 3^{21} a_3^5 a_1^{10}$$

and

$$a_7 > 3^3 a_6^2 a_2^2 / a_3 > 3^{45} a_3^9 a_1^{22}.$$

For the last statement of the lemma, first note that the Gyarmati's gap principle gives

$$bd_2 > 27 a^2 d_1^2 \tag{1}$$

and

$$cd_2 > 27 b^2 d_1^2. \tag{2}$$

Set $\varphi = 1 + \sqrt{2}$. If $b < a^{\varphi}$ or $c > b^{\varphi}$, the result follows from (1), resp. (2). Otherwise, we have $c > b^{\varphi} > a^{\varphi^2}$, which, combined with (1), yields the result.

□

We need the following refinement, due to Mignotte [16], of a theorem of Laurent, Mignotte & Nesterenko [14] on linear forms in two logarithms. For any non-zero algebraic number α, we denote by $\mathrm{h}(\alpha)$ its logarithmic absolute height. For instance, for any non-zero rational number p/q, written under its irreducible form, we have $\mathrm{h}(p/q) = \log \max\{|p|, |q|\}$.

Lemma 2. *Consider the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 and b_2 are positive integers. Suppose that α_1 and α_2 are multiplicatively independent. Put

$$D = [\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}] \,/\, [\mathbf{R}(\alpha_1, \alpha_2) : \mathbf{R}].$$

Let a_1, a_2, h, k be real positive numbers, and ρ a real number > 1. Put $\lambda = \log \rho$, $\chi = h/\lambda$ and suppose that $\chi \geq \chi_0$ for some number $\chi_0 \geq 0$ and that

$$\begin{aligned} h &\geq D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log \lambda + f(\lceil K_0 \rceil)\right) + 0.023, \\ a_i &\geq \max\bigl\{1, \rho\,|\log \alpha_i| - \log |\alpha_i| + 2D\mathrm{h}(\alpha_i)\bigr\}, \quad (i = 1, 2), \\ a_1 a_2 &\geq \lambda^2 \end{aligned}$$

where

$$f(x) = \log \frac{\bigl(1 + \sqrt{x-1}\bigr)\sqrt{x}}{x-1} + \frac{\log x}{6x(x-1)} + \frac{3}{2} + \log \frac{3}{4} + \frac{\log \frac{x}{x-1}}{x-1},$$

and

$$K_0 = \frac{1}{\lambda}\left(\frac{\sqrt{2+2\chi_0}}{3} + \sqrt{\frac{2(1+\chi_0)}{9} + \frac{2\lambda}{3}\Bigl(\frac{1}{a_1} + \frac{1}{a_2}\Bigr) + \frac{4\lambda\sqrt{2+\chi_0}}{3\sqrt{a_1 a_2}}}\right)^2 a_1 a_2.$$

Put

$$v = 4\chi + 4 + 1/\chi \quad \text{and} \quad m = \max\bigl\{2^{5/2}(1+\chi)^{3/2}, (1+2\chi)^{5/2}/\chi\bigr\}.$$

Then we have the lower bound

$$\begin{aligned} \log |\Lambda| \geq &-\frac{1}{\lambda}\left(\frac{v}{6} + \frac{1}{2}\sqrt{\frac{v^2}{9} + \frac{4\lambda v}{3}\Bigl(\frac{1}{a_1} + \frac{1}{a_2}\Bigr) + \frac{8\lambda m}{3\sqrt{a_1 a_2}}}\right)^2 a_1 a_2 \\ &- \max\Bigl\{\lambda(1.5 + 2\chi) + \log\Bigl(\bigl((2+2\chi)^{3/2} + (2+2\chi)^2\sqrt{k^*}\bigr)A + (2+2\chi)\Bigr), D\log 2\Bigr\} \end{aligned}$$

where

$$A = \max\{a_1, a_2\} \quad \text{and} \quad k^* = \frac{1}{\lambda^2}\left(\frac{1+2\chi}{3\chi}\right)^2 + \frac{1}{\lambda}\left(\frac{2}{3\chi} + \frac{2}{3}\frac{(1+2\chi)^{1/2}}{\chi}\right).$$

Proof : This is Theorem 2 of [16]. □

The proof of Theorems 2 and 3 depends on the following result of Evertse [11].

Lemma 3. *If a, b and n are positive integers with $n \geq 3$ and c is a positive real number, then there is at most one positive integral solution (x, y) to the inequality*

$$|ax^n - by^n| \leq c$$

with

$$\max\{|ax^n|, |by^n|\} > \beta_n c^{\alpha_n},$$

where α_n and β_n are effectively computable positive constants satisfying

$$\alpha_3 = 9, \quad \alpha_n = \max\left\{\frac{3n-2}{2(n-3)}, \frac{2(n-1)}{n-2}\right\} \qquad \text{for } n \geq 4$$

and

$$\beta_3 = 1152.2, \quad \beta_4 = 98.53, \quad \beta_n < n^2 \qquad \text{for } n \geq 5.$$

Proof : This is Theorem 2.1 of [11]. □

The proof of Theorem 4 uses an irrationality measure [20] of certain algebraic numbers derived from a Theorem of Baker [1], using some improvements from [2].

Lemma 4. *Let A, B, K and n be positive integers such that $A > B, K < n, n \geq 3$ and $\omega = (B/A)^{1/n}$ is not a rational number. For $0 < \phi < 1$, put*

$$\delta = 1 + \frac{2-\phi}{K}, \quad s = \frac{\delta}{1-\phi}$$

and

$$u_1 = 40^{n(K+1)(s+1)/(Ks-1)}, \quad u_2^{-1} = K2^{K+s+1}40^{n(K+1)}.$$

Assume that

$$A(A-B)^{-\delta}u_1^{-1} > 1. \tag{3}$$

Then

$$\left|\omega - \frac{p}{q}\right| > \frac{u_2}{Aq^{K(s+1)}}$$

for all integers p and q with $q > 0$.

Proof : This is Lemma 1 of Shorey & Nesterenko [20]. We notice that this has been refined by Hirata-Kohno in [13] but the statement of [20] is sufficient for our purpose. □

4. Proofs

Proof of Theorem 1 :

Let $0 < a < b < c < d$ be integers such that there exist positive integers r, s, t, u and $k \geq 2$ with

$$ac + 1 = r^k, \;\; ad + 1 = s^k, \;\; bc + 1 = t^k \quad \text{and} \quad bd + 1 = u^k.$$

Our aim is to prove that k is bounded by an absolute constant. Hence, we may assume that $k \geq 160$ and that, since

$$c^2 > bc + 1 \geq 3^k, \tag{4}$$

we have

$$d > c > 3^{80}. \tag{5}$$

We also observe that, by Lemma 1, we have

$$\log d > (k-2) \log c. \tag{6}$$

We set

$$\alpha_1 = \frac{ur}{st}, \quad \alpha_2 = \frac{b}{a} \cdot \frac{ac+1}{bc+1}$$

and we consider the linear form in logarithms

$$\Lambda = |\log \alpha_2 - k \log \alpha_1| = \left| \log \left(\frac{b}{a} \cdot \frac{ac+1}{bc+1} \right) - k \log \left(\frac{ur}{st} \right) \right|.$$

Before applying Lemma 2 with $b_2 = 1$ and $b_1 = k$ in order to bound Λ, we need some estimates.

Firstly, we have

$$|\alpha_2 - 1| = \alpha_2 - 1 = \frac{b-a}{abc+a} < \frac{1}{c}. \tag{7}$$

Secondly, from (5) and the upper bound

$$\left| \left(\frac{b}{a} \cdot \frac{ac+1}{bc+1} \right) - \left(\frac{ur}{st} \right)^k \right| = \left(\frac{r}{t} \right)^k \frac{b-a}{a(ad+1)} \leq \frac{(b-a)(ac+1)}{a(ad+1)(bc+1)} \leq \frac{1}{ad},$$

we deduce that

$$\Lambda \leq \frac{2}{ad}. \tag{8}$$

Let now define the quantities a_1, a_2, h, k, ρ appearing in Lemma 2.

We set

$$\rho = c \quad (\text{thus} \quad \lambda = \log c),$$

and, by (5) and (7), we may take

$$a_1 = 3 + \frac{2(k+1)}{k(k-2)} \log d \quad \text{and} \quad a_2 = 3 + 6 \log c.$$

Indeed, we easily see that $k\mathrm{h}(\alpha_1) = \mathrm{h}((bd+1)(ac+1)) \leq \log(c^3 d)$, whence by (6) we get $k\mathrm{h}(\alpha_1) \leq (1 + 3/(k-2)) \log d$.

Further, we see that one can take $h = \lambda/2$, since $c \geq 3^{k/2}$ by (4). We should also check that α_1 and α_2 are multiplicatively independent. However, a look at the proof of Theorem 1.5 of [16] shows that this is not needed. Indeed, we apply it with the choice $L = 3$, hence it is sufficient to check that the three numbers 1, α_1 and α_2 are distinct, which is clearly the case.

It follows from our choice of h that $\chi_0 = 1/2$, whence $v = 8$ and $m = 8\sqrt{2}$. Using (5) and (6), we get the lower bound

$$\log \Lambda \geq -\frac{1}{\log c}\left(\frac{4}{3} + \frac{1}{2}\sqrt{\frac{64}{9} + \frac{64}{9} + \frac{32\sqrt{2}}{3\sqrt{3}}}\right)^2 a_1 a_2 - 2.5 \log c - \log(20.8 a_1).$$

Combined with (8), after a few calculations, we obtain

$$\log d \leq 167 \frac{k+1}{k(k-2)} \log d + 254.9 + 2.5 \log c + \log\big((\log d)/k\big). \tag{9}$$

Using (4), (5) and (6), we infer from (9) that

$$1 \leq 167 \frac{k+1}{k(k-2)} + \frac{254.9}{\log d} + \frac{2.5}{k-2} + \frac{1}{k}\left(\frac{k}{\log d}\right) \log\left(\frac{\log d}{k}\right). \tag{10}$$

Since we have assumed $k \geq 160$, it follows from (4), (5), (6) and (10) that the integer k satisfies

$$k \leq 176,$$

as claimed. □

Proof of Theorem 2 :

Let $k \geq 5$ and assume that $m \geq 3$. Let $ac_{m-1} + 1 = x^k$ and $bc_{m-1} + 1 = y^k$. Then

$$bx^k - ay^k = b - a$$

and Lemma 3 implies

$$abc_{m-1} < k^2 b^{3.25}.$$

Hence, $c_{m-1} < k^2 b^{2.25}$. On the other hand, Lemma 1 implies that

$$c_{m-1} \geq c_2 > k^k c_1^{k-2} > k^5 b^3,$$

a contradiction.

Let $k = 4$. Then, as above, we obtain $c_{m-1} < 99b^4$. By Lemma 1, we have $c_2 > 256b^2$ and $c_3 > 256c_2^2 > 256^3 b^4$. Therefore $m - 1 \leq 2$ and $m \leq 3$.

□

Proof of Theorem 3 :

Let $ad_{m-1}+1=x^3$ and $bd_{m-1}+1=y^3$. As in the proof of Theorem 2, an application of Lemma 3 gives $abd_{m-1} < 1153b^9$ and

$$d_{m-1} < 1153\, b^8.$$

On the other hand, successive applications of Lemma 1 give

$$\begin{aligned} d_2 &> 27d_1^{3-\sqrt{2}} > 27\, b^{3-\sqrt{2}}, \\ d_3 &> 27d_2^{3-\sqrt{2}} > 4930\, b^{2.51}, \\ d_4 &> 27d_3^2 b^{-1} > 6\cdot 10^8\, b^{4.02}, \\ d_5 &> 27d_4^2 b^{-1} > 9\cdot 10^{18}\, b^{7.04}, \\ d_6 &> 27d_5^2 b^{-1} > 2\cdot 10^{39}\, b^{13.08}. \end{aligned}$$

Therefore, $m-1 \le 5$ and $m \le 6$.

□

Proof of Corollary 4 :

It suffices to prove the corollary for $k=3$. Let $a_1 < a_2 < \cdots < a_8$ be positive integers such that the product of any two of them increased by 1 is a perfect cube. As in the proof of Theorem 3, Lemma 3 implies $a_7 < 1153a_2^8$. From Lemma 1, we have $a_7 > 3^{45}a_2^9$, a contradiction.

□

Proof of Theorem 4 :

Let $11 \le k \le 176$ be an integer. We denote by $\kappa_1(k), \dots, \kappa_6(k)$ effectively computable positive constants which depend only on k. Assume that the integers $0 < a < b < c < d$ are such that there exist integers r, s, t and u with

$$ac+1=r^k, \;\; ad+1=s^k, \;\; bc+1=t^k \quad \text{and} \quad bd+1=u^k.$$

We will apply Lemma 4 with $K=2$ to the algebraic number

$$\omega = \left(\frac{a(bc+1)}{b(ac+1)}\right)^{1/k},$$

i.e. with $A=b(ac+1)$ and $A-B=b-a$. Firstly, we observe that

$$\left|\omega - \frac{st}{ru}\right| \le \frac{3}{ad}. \tag{11}$$

Let $\phi < 1/4$ be a (very) small positive number, and, with the notation of Lemma 4, set $\delta = 2-\phi/2$ and $s=\delta/(1-\phi)$.

The assumption (3) in the statement of Lemma 4 is fulfilled if

$$b(ac+1) > 40^{3k(s+1)(2s-1)}\,(b-a)^{2-\phi/2},$$

thus, since $c > b$, it is fulfilled as soon as $c > \kappa_1(k)$. Under this assumption, we infer from Lemma 4 and (11) that

$$\frac{3}{ad} \geq \left|\omega - \frac{st}{ru}\right| > \frac{\kappa_2(k)}{bac(ur)^{6+7\phi}}. \tag{12}$$

Recalling that $ur = (ac+1)^{1/k}(bd+1)^{1/k}$, it follows from (12) that

$$ad < \kappa_3(k)abc(ac)^{(6+7\phi)/k}(bd)^{(6+7\phi)/k}. \tag{13}$$

By Lemma 1, we have

$$bd > k^k c^{k-1} a^{k-1}. \tag{14}$$

Using $b < c$ and combining (13) and (14), we get

$$d^{k-6-7\phi} < \kappa_4(k)(bc)^{k+6+7\phi}a^{6+7\phi} < \kappa_4(k)(ac)^{2k+12+14\phi} < \kappa_5(k)d^{(2k+12+14\phi)/(k-2)},$$

whence we deduce that $d < \kappa_6(k)$, since $k \geq 11$, as claimed. □

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