# NOTES ON FRAMES

Damir Bakić
University of Zagreb

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# Contents

# Introduction

Frames were introduced in 1952 by R.J. Duffin and A.C. Schaeffer in [64]. However, frames appeared implicitly in the literature even before. As an example here we mention only a paper by B. De Sz.Nagy from 1947; please see equation (5) on p 976 in [98]. It is generally acknowledged that frames became popular only 50 years after the work of Duffin and Schaeffer due mostly to the work of I. Daubechies, A. Grossmann, and I. Meyer ([59]). Today frames are unavoidable both in mathematics and engineering. The applications can be found in numerous areas such as operator theory, sampling theory, coding, signal reconstruction, denoising, robust transmission etc.

For an introduction to frame theory we refer the reader to [90] and [49]. For general theory of frames and, in particular, of finite frames we refer the reader to [51], resp [45]. Also the lists of references in [51] and [45] provide an excellent overview of relevant literature. A wealth of information can be found on the website http://www.framerc.org

This manuscript grew out of a set of notes prepared in 2016 for a one-semester graduate course on frames at the University of Zagreb.

In the first Chapter we present a review of basic background material concerning convergence of series, fundamental properties of bases in Banach spaces and Bessel sequences in Hilbert spaces. Chapter 2 is devoted to general theory of frames in abstract Hilbert spaces. Various aspects of frame theory are discussed; each particular subject in a separate section. In Chapter 3 we give a brief overview of finite frames. Only a few subjects (such as, for example, full spark frames) are discussed in some detail.

The last two chapters are devoted to special systems in $L^2(\mathbb{R})$, resp. $L^2(\mathbb{R}^N)$. In Chapter 4 we present some topics in wavelet theory. In particular, we describe translation invariant spaces with emphasis on the role played by frames. There is also an Appendix to Chapter 4 containing some technical results on integration and infinite sums. Finally, in Chapter 5 the fundamentals of Gabor systems are presented.

Each section ends with a set of exercises. Some of them are in fact (parts of) the results from the literature that are not included in the text, while the others serve as illustrations of the presented results. There is also a long list of references; however, it is by no means complete. Some historical remarks are given, but certainly a major revision and completion is needed in order to give proper credits to many authors which contributed to the theory with important results.

We should also mention that some of the important subjects from general theory of frames, such as the Feichtinger conjecture and the Paulsen problem are missing. These and some other topics will be discussed at seminar talks which are planned as a supplement of the course. Finally, it should be pointed out that some of the important chapters of frame theory, such as fusion frames and frames for Hilbert $C^*$-modules are not included. Hopefully, these chapters will find their place in some future expanded version of these notes.

We end this introduction with our notations and some notational conventions.

By $\mathbb{B}(X, Y)$ we denote the Banach space of all bounded operators of Banach spaces $X$ and $Y$. For $X = Y$ we write $\mathbb{B}(X)$. The range and the null-space of an operator $A$ will be denoted by $\mathrm{R}(A)$ and $\mathrm{N}(A)$, respectively.

Throughout these notes we work with real ($\mathbb{R}$) and complex ($\mathbb{C}$) spaces; we will denote the underlying field by $\mathbb{F}$ whenever our considerations apply to both real and complex case. The Lebesgue measure of a set $S$ will be denoted by $|S|$. By writing $S_1 \stackrel{0}{=} S_2$ we mean that the sets $S_1$ and $S_2$ differ by a set of measure 0.

The central object of our considerations are sequences of vectors in Hilbert spaces. We will write typically $(x_n)_n$ to denote a sequence of vectors (or scalars) assuming that the index set is the set $\mathbb{N}$ of all natural numbers or any of its subsets. So, writing $(x_n)_n$ we tacitly allow the possibility (depending on the context) that the sequence under consideration is finite. However, when summing up such sequences we will write $\sum_{n=1}^{\infty} x_n$ keeping in mind that the sum runs effectively over the index set under consideration.

Whenever the situation requires, we will use more precise notation, writing, for example, $(x_n)_{n=1}^{N}$, $(x_n)_{n=n_0}^{\infty}$, etc. and, correspondingly, $\sum_{n=1}^{N} x_n$, $\sum_{n=n_0}^{\infty} x_n$, etc.

Whenever the sequence under consideration is indexed by some set other than $\mathbb{N}$ or any of its subsets, we will use an appropriate precise notation such as, for example, $(\psi_{j,k})_{j,k\in Z}$ (which is the case that typically occurs in wavelet theory).

The same convention applies to the standard Hilbert space of square sumable sequences $\ell^2$. As a rule, $\ell^2$ denotes the space of sequences of scalars $\{(x_n)_n : \sum_{n=1}^{\infty} \|x_n\|^2 < \infty\}$, where the index set is $\mathbb{N}$ or some of its subsets. In particular, if the context allows, our sequences may be finite i.e. indexed, say, by $n \in \{1, 2, \dots, N\}$, $N \in \mathbb{N}$, and then we understand that $\ell^2 = \mathbb{F}^N$ equipped with the standard (Euclidean) inner product. In all other situations, when indexation is specific, naturally dictated by the context, we shall write $\ell^2(\mathbb{Z})$, $\ell^2(\mathbb{Z}\times\mathbb{Z})$, etc.

Throughout the text we will systematically use the abbreviation ONB for an orthonormal basis of a Hilbert space.

The rest of our notations is standard or will be explained at appropriate places in the text.

# 1 Unconditional convergence, Riesz bases, and Bessel sequences

## 1.1 Unconditional convergence of series in Banach spaces

**Definition 1.1.1.** *Let $(x_n)_n$ be a sequence in a normed space $X$. We say that the series $\sum_{n=1}^{\infty} x_n$ converges*

*(a) absolutely, if the series $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$,*

*(b) unconditionally, if the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges in $X$ for every permutation $\sigma$ of $\mathbb{N}$.*

We do not require that $\sum_{n=1}^{\infty} x_{\sigma(n)}$ must converge to the same value for every permutation $\sigma$; however, we will show that this is indeed the case.

We start with a well known result concerning absolutely convergent series in Banach spaces.

**Proposition 1.1.2.** *([10], Theorem 1.2.10.) Let $(x_n)_n$ be a sequence in a Banach space such that the series $\sum_{n=1}^{\infty} x_n$ converges absolutely. Then the series $\sum_{n=1}^{\infty} x_n$ converges in $X$ and*

$$\left\| \sum_{n=1}^{\infty} x_n \right\| \leq \sum_{n=1}^{\infty} \|x_n\|.$$

*If $X$ is a normed space in which every absolutely convergent series converges, then $X$ is complete, i.e. $X$ is a Banach space.*

**Theorem 1.1.3.** *Let $(x_n)_n$ be a sequence in a Banach space $X$. If $\sum_{n=1}^{\infty} x_n$ converges absolutely, then it converges unconditionally.*

**Proof.** Suppose that the series $\sum_{n=1}^{\infty} x_n$ converges absolutely and choose any permutation $\sigma$ of $\mathbb{N}$. Let $\epsilon > 0$. Since $\left(\sum_{n=1}^{N} \|x_n\|\right)_N$ is a Cauchy sequence, there exists $N_0$ such that

$$N > M \geq N_0 \Longrightarrow \sum_{n=M+1}^{N} \|x_n\| = \sum_{n=1}^{N} \|x_n\| - \sum_{n=1}^{M} \|x_n\| < \epsilon. \tag{1}$$

Let $N_1 = \max\{\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(N_0)\}$. That means that the set $\{1, 2, \ldots, N_1\}$ contains all $n$ with the property $1 \leq \sigma(n) \leq N_0$.

Let us now take $N > M \geq N_1$ and consider any index $n$ such that $M + 1 \leq n \leq N$. For such $n$ we have $n > N_1$, so by the preceding observation we have $\sigma(n) > N_0$. Hence, if we put $K = \min\{\sigma(M+1), \sigma(M+2), \ldots, \sigma(N)\}$ and $L = \max\{\sigma(M+1), \sigma(M+2), \ldots, \sigma(N)\}$, then $L > K > N_0$. From this we conclude

$$\begin{aligned}
\left\| \sum_{n=1}^{N} x_{\sigma(n)} - \sum_{n=1}^{M} x_{\sigma(n)} \right\| &= \left\| \sum_{n=M+1}^{N} x_{\sigma(n)} \right\| \\
&\leq \sum_{n=M+1}^{N} \|x_{\sigma(n)}\| \\
&\leq \sum_{n=K}^{L} \|x_n\| \overset{(1)}{<} \epsilon.
\end{aligned}$$

This shows that the sequence of partial sums $\left(\sum_{n=1}^{N} x_{\sigma(n)}\right)_N$ is a Cauchy sequence. $\square$

It is known that in finite-dimensional spaces unconditional convergence is equivalent to absolute convergence. This can be seen directly (see [10], Theorem 3.2.2), but we shall obtain this result as an easy consequence of a general theorem on unconditional convergence (see Theorem 1.1.12 and Corollary 1.1.15 below).

In general, the converse of the preceding theorem fails. We will show that absolute convergence is stronger than unconditional convergence in infinite-dimensional Hilbert spaces.

**Lemma 1.1.4.** *Let $(e_n)_n$ be an orthonormal (ON) sequence in a Hilbert space $H$, let $(c_n)_n$ be a sequence of scalars. Then the series $\sum_{n=1}^{\infty} c_n e_n$ converges if and only if $(c_n)_n \in \ell^2$. Moreover, the series $\sum_{n=1}^{\infty} c_n e_n$ converges if and only if it converges unconditionally.*

**Proof.** Denote by $f_N = \sum_{n=1}^{N} c_n e_n$ and $s_N = \sum_{n=1}^{N} |c_n|^2$, $N \in \mathbb{N}$, the relevant partial sums. Then we have for $N > M$

$$\|f_N - f_M\|^2 = \left\| \sum_{n=M+1}^{N} c_n e_n \right\|^2 = \sum_{n=M+1}^{N} |c_n|^2 = |s_N - s_M|.$$

This proves the first equivalence. Note that we do not need the completeness assumption in one direction.

Suppose now that the series $\sum_{n=1}^{\infty} c_n e_n$ converges in $H$. Then by the first part we have $(c_n)_n \in \ell^2$. Hence $\sum_{n=1}^{\infty} |c_n|^2 < \infty$. By Theorem 1.1.3 this series converges unconditionally; thus, $\sum_{n=1}^{\infty} |c_{\sigma(n)}|^2 < \infty$ for each permutation $\sigma$. This means that $(c_{\sigma(n)})_n \in \ell^2$. Clearly, the sequence $(e_{\sigma(n)})_n$ is ON. Hence, again by the first assertion of the lemma, the series $\sum_{n=1}^{\infty} c_{\sigma(n)} e_{\sigma(n)}$ converges. $\square$

**Remark 1.1.5.** We can now conclude that, in general, unconditional convergence does not imply absolute convergence (that is, the converse of Theorem 1.1.3 fails). To see this, let us take any sequence $(c_n)_n \in \ell^2 \setminus \ell^1$ - for example, $c_n = \frac{1}{n}$, $n \in \mathbb{N}$. By the preceding lemma, the series $\sum_{n=1}^{\infty} c_n e_n$ converges unconditionally but, clearly, it does not converge absolutely.

We now recall the most fundamental facts concerning orthonormal bases (ONB) for inner product spaces.

**Definition 1.1.6.** *An orthonormal sequence $(e_n)_n$ is an ONB for an inner product space $X$ if for each $x \in X$ there exists a sequence of scalars $(c_n)_n$ such that*

$$x = \sum_{n=1}^{\infty} c_n e_n. \tag{2}$$

**Remark 1.1.7.** (a) If a sequence $(e_n)_n$ is an ONB for an inner product space $X$, then $X$ is separable. (Clear.)

(b) If a sequence $(e_n)_n$ is an ONB for an inner product space $X$, then the series in (2) converges unconditionally (see Remark 1.1.5).

(c) The coefficients $c_n$ in (2) are of the form $c_n = \langle x, e_n \rangle$ for each $n$, and hence are uniquely determined by $x$ (this follows from the continuity of inner product in each argument).

(d) If $(e_n)_n$ is an ON sequence in an inner product space $X$ (not necessarily a basis), then every $x$ in $X$ satisfies the Bessel inequality: $\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2$.

(e) In each separable inner product space $X$ there exists an ON sequence $(e_n)_n$ with the property $\overline{\text{span}}\{e_n : n \in \mathbb{N}\} = X$ (we say that a sequence with this property is fundamental in $X$). In a separable inner product space each ON set is finite or countable ([10], Proposition 2.1.3).

**Theorem 1.1.8.** *([10], Theorem 2.1.7 and Theorem 2.1.13.) Let $(e_n)_n$ be an ON sequence in an inner product space $X$. Consider the following conditions:*

*(a) $(e_n)_n$ is an ONB for $X$.*

*(b) $(e_n)_n$ is fundamental in $X$.*

*(c) $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$ for every $x$ in $X$.*

*(d) $\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, e_n \rangle \langle e_n, y \rangle$ for all $x$ and $y$ in $X$.*

*(e) $(e_n)_n$ is maximal in $X$, i.e. if $x \in X$ is perpendicular to all $e_n$ then $x = 0$.*

*Then we have (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e). If $X$ is a Hilbert space, then condition (e) is equivalent to (a) - (d). In particular, each separable inner product space possesses an ONB.*

**Example 1.1.9.** It is known that the system $\left\{e_n = e^{2\pi i n t} : n \in \mathbb{Z}\right\}$ makes up an ONB for the Hilbert space $L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right)\right)$ (since it is an ON fundamental system). Hence each function $f$ from $L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right)\right)$ admits an expansion into its Fourier series as in (2). It is now important that the series (2) converges unconditionally. This enables us to organize the system $\left\{e^{2\pi i n t} : n \in \mathbb{Z}\right\}$ into a sequence by choosing any bijection $\sigma : \mathbb{N} \to \mathbb{Z}$. In this way we get

$$f = \sum_{n=1}^{\infty} \langle f, e_{\sigma(n)} \rangle e_{\sigma(n)} = \lim_{N \to \infty} \sum_{n=1}^{N} \langle f, e_{\sigma(n)} \rangle e_{\sigma(n)}.$$

One usually uses enumeration $e_0, e_{-1}, e_1, e_{-2}, e_2, \ldots$ and the corresponding sequence of partial sums. Since convergence is in this situation ensured, we are allowed to use any convenient subsequence of the sequence of partial sums. In particular, we can work with the subsequence of partial sums indexed by odd indices which gives us

$$f = \lim_{N \to \infty} \sum_{n=-N}^{N} \langle f, e_n \rangle e_n.$$

A similar maneuver is used whenever we work with an ONB indexed by some product set, e.g. $\mathbb{Z} \times \mathbb{Z}$ as it is the case with wavelets or Gabor systems.

We now need to recall the concept of sumability in normed spaces.

Let $(x_j)_J$ be a family of vectors in a normed space $X$. Consider the set $\mathcal{F}$ that consists of all finite subsets $F$ of $J$ directed by the relation $F_1 \le F_2 \Leftrightarrow F_1 \subseteq F_2$ and the net $(\sum_{j\in F} x_j)_{F\in\mathcal{F}}$. We say that the family $(x_j)_J$ is sumable if this net converges in $X$ and when this is the case, if we denote the limit by $x$, we write $\sum_{j\in J} x_j = x$.

In Banach spaces we have the following useful characterization of sumable families:

**Proposition 1.1.10.** *([10], Proposition 3.1.7.) Let $X$ be a Banach space. A family $(x_j)_{j\in J}$ in $X$ is sumable if and only if the following Cauchy condition is satisfied:*

$$\forall \epsilon > 0\ \exists G(\epsilon) \in \mathcal{F} \text{ such that } F \in \mathcal{F},\ F \subseteq J \setminus G(\epsilon) \Longrightarrow \left\| \sum_{j\in F} x_j \right\| < \epsilon. \tag{3}$$

In fact, it is easily seen that sumability implies (1.1.10) in all normed (not necessarily complete) spaces Given a sequence $(x_n)_n$ in $X$ we want to compare its sumability in the above sense with (various modes of) convergence of the corresponding series. First we need a lemma which provides us with an equivalent form of the Cauchy condition (3) in the case $J = \mathbb{N}$.

**Lemma 1.1.11.** *Let $(x_n)_n$ be a sequence in a normed space $X$. Then the Cauchy condition (3) is equivalent to*

$$\forall \epsilon > 0\ \exists N(\epsilon) \in \mathbb{N} \text{ such that } F \in \mathcal{F},\ \min F > N(\epsilon) \Longrightarrow \left\| \sum_{j\in F} x_j \right\| < \epsilon. \tag{4}$$

**Proof.** Assume (3) with $J = \mathbb{N}$. Choose any $\epsilon > 0$. Let $N(\epsilon) = \max G(\epsilon)$. Consider now any $F \in \mathcal{F}$ with the property $\min F > N(\epsilon)$. Then we also have $\min F > \max G(\epsilon)$ which implies $F \cap G(\epsilon) = \emptyset$. Applying (3), we conclude that $\left\| \sum_{j\in F} x_j \right\| < \epsilon$.

The converse is proved similarly. $\square$

Given a sequence $(x_n)_n$ in a normed space $X$, it is relatively easy to see that its sumability implies that the corresponding series is convergent. This means that sumability is a stronger condition than convergence of the corresponding series. A natural question then is: what about unconditional convergence? The theorem that follows provides us with several conditions that are equivalent to sumability.

**Theorem 1.1.12.** *Let $(x_n)_n$ be a sequence in a Banach space $X$. The following conditions are all equivalent:*

*(a)* $\{x_n : n \in \mathbb{N}\}$ *is sumable.*

*(b)* $\sum_{n=1}^{\infty} x_n$ *converges unconditionally.*

*(c)* $\sum_{n=1}^{\infty} x_{p(n)}$ *converges for every subsequence* $(x_{p(n)})_n$ *of* $(x_n)_n$.

*(d)* $\sum_{n=1}^{\infty} \varepsilon_n x_n$ *converges for every choice of signs* $\varepsilon_n = \pm 1$.

*(e)* $\sum_{n=1}^{\infty} \lambda_n x_n$ *converges for every bounded sequence of scalars* $(\lambda_n)_n$.

*(f)* $\sum_{n=1}^{\infty} |f(x_n)|$ *converges uniformly with respect to the closed unit ball in the dual space* $X'$, *i.e.* $\lim_{N\to\infty} \sup \{\sum_{n=N}^{\infty} |f(x_n)| : f \in X', \|f\| \le 1\} = 0$.

**Proof.** $(a) \Rightarrow (b)$. Suppose that $\{x_n : n \in \mathbb{N}\}$ is sumable; let $x = \sum_{k\in\mathbb{N}} x_k = \lim_{F\in\mathcal{F}} \sum_{k\in F} x_k$. Consider any permutation $\sigma$ of the set $\mathbb{N}$ and fix $\epsilon > 0$. By the assumption we can find $F \in \mathcal{F}$ with the property

$$F \in \mathcal{F}, \;\; F_0 \subseteq F \Longrightarrow \|x - \sum_{k\in F} x_k\| < \epsilon. \tag{5}$$

Let us now find $n_0$ such that $F_0 \subseteq \{\sigma(1), \sigma(2), \dots \sigma(n_0)\}$. Clearly, for each $n \ge n_0$ and the set $F = \{\sigma(1), \sigma(2), \dots \sigma(n_0) \dots \sigma(n)\}$ we have $F_0 \subseteq F$ and therefore (5) implies

$$\|x - \sum_{k\in F} x_k\| < \epsilon, \quad \text{i.e.} \quad \|x - \sum_{k=1}^{n} x_{\sigma(k)}\| < \epsilon.$$

$(b) \Rightarrow (a)$. Let $x$ denotes the sum in the identical permutation: $x = \sum_{n=1}^{\infty} x_n$. We claim that $x = \sum_{n\in\mathbb{N}} x_n$, that is, $x = \lim_{F\in\mathcal{F}} \sum_{n\in F} x_n$. Suppose the opposite. Then there exists $\epsilon > 0$ such that

$$\forall F_0 \in \mathcal{F} \; \exists F \in \mathcal{F}, \, F_0 \subseteq F \text{ such that } \left\| x - \sum_{n\in F} x_n \right\| \ge \epsilon. \tag{6}$$

On the other hand, we know that for the same $\epsilon$

$$\exists M_1 \in \mathbb{N} \text{ such that } N \ge M_1 \Longrightarrow \left\| x - \sum_{n=1}^{N} x_n \right\| < \frac{\epsilon}{2}. \tag{7}$$

Using (6) and (7) we shall construct a permutation $\sigma$ of $\mathbb{N}$ for which the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ diverges. Put $F_1 = \{1, 2, \dots, M_1\}$. By (6) there exists $G_1 \in \mathcal{F}$ such that

$$F_1 \subseteq G_1 \text{ and } \left\| x - \sum_{n\in G_1} x_n \right\| \ge \epsilon.$$

Let $M_2 = \max G_1$ and $F_2 = \{1, 2, \dots, M_1, \dots, M_2\}$. Again by (6) there exists $G_2 \in \mathcal{F}$ such that

$$F_2 \subseteq G_2 \text{ and } \left\| x - \sum_{n\in G_2} x_n \right\| \ge \epsilon.$$

We proceed by induction. In this way we obtain a sequence of sets in $\mathcal{F}$

$$F_1 \subseteq G_1 \subseteq F_2 \subseteq G_2 \subseteq \dots$$

for which we have

$$\left\| x - \sum_{n\in G_N} x_n \right\| \ge \epsilon \;\text{ and }\; \left\| x - \sum_{n\in F_N} x_n \right\| < \frac{\epsilon}{2}, \; \forall N \in \mathbb{N} \tag{8}$$

(the second inequality follows from (7) since each of the sets $F_N$ is of the form $F_n = \{1, 2 \ldots, M_N\}$ and $M_N \geq M_{N-1} \geq \ldots \geq M_1$).

From inequalities (8) we conclude

$$\begin{aligned}\left\|\sum_{n\in G_N\setminus F_N} x_n\right\| &= \left\|\sum_{n\in G_N} x_n - \sum_{n\in F_N} x_n\right\| \\ &\geq \left\|x - \sum_{n\in G_N} x_n\right\| - \left\|\sum_{n\in F_N} x_n - x\right\| \\ &\geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}.\end{aligned}$$

In particular, this shows that $F_N \neq G_N$, that is $\operatorname{card} F_N < \operatorname{card} G_N$.

Consider now the permutation $\sigma$ of $\mathbb{N}$ defined by enumerating in turn the elements of the sets $F_1, G_1 \setminus F_1, F_2 \setminus G_1, G_2 \setminus F_2, \ldots$ (and keeping the natural order in each of these sets). We now have for each $N \in \mathbb{N}$

$$\left\|\sum_{n=\operatorname{card} F_N+1}^{\operatorname{card} G_N} x_{\sigma(n)}\right\| = \left\|\sum_{n\in G_N\setminus F_N} x_n\right\| \geq \frac{\epsilon}{2}.$$

Since $\operatorname{card} F_N, \operatorname{card} G_N \to \infty$ as $N \to \infty$, this shows that the sequence of partial sums $(\sum_{n=1}^{M} x_{\sigma(n)})_M$ is not a Cauchy sequence. Hence $\sum_{n=1}^{\infty} x_{\sigma(n)}$ diverges - a contradiction.

$(a) \Rightarrow (f)$. Assume (a). By Proposition 1.1.10, we have (3). Then, for any $\epsilon > 0$, by applying Lemma 1.1.11, we can find a set $N(\epsilon)$ from condition (4).

For $L \geq K > N(\epsilon)$ and any $f \in X'$, $\|f\| \leq 1$, define

$$F^+ = \{n \in \mathbb{N} : K \leq n \leq L,\ \operatorname{Re} f(x_n) \geq 0\},$$
$$F^- = \{n \in \mathbb{N} : K \leq n \leq L,\ \operatorname{Re} f(x_n) < 0\}.$$

Note that $\min F^+ \geq K > N(\epsilon)$. Therefore

$$\begin{aligned}\sum_{n\in F^+} |\operatorname{Re} f(x_n)| &= \sum_{n\in F^+} \operatorname{Re} f(x_n) \\ &= \operatorname{Re} f\left(\sum_{n\in F^+} x_n\right) \\ &\leq \left|f\left(\sum_{n\in F^+} x_n\right)\right| \\ &\leq \|f\| \cdot \left\|\sum_{n\in F^+} x_n\right\| \overset{(4)}{<} \|f\|\epsilon \leq \epsilon\end{aligned}$$

By a similar computation we obtain analogous inequality for the set $F^-$, so we get

$$\sum_{n=K}^{L} |\operatorname{Re} f(x_n)| < 2\epsilon.$$

Working similarly with imaginary parts we obtain

$$\sum_{n=K}^{L} |f(x_n)| < 4\epsilon.$$

By letting $L \to \infty$ we conclude that

$$K > N(\epsilon) \Longrightarrow \sup \left\{ \sum_{n=K}^{\infty} |f(x_n)| : f \in X',\ \|f\| \leq 1 \right\} \leq 4\epsilon.$$

$(f) \Rightarrow (e)$. Assume (f) and take any sequence of scalars $(\lambda_n)_n$ such that $|\lambda_n| \leq 1$ for all $n$. For a given $\epsilon > 0$ there exists, by our hypothesis (f), an index $N_0$ with the property

$$N > N_0 \Longrightarrow \sup \left\{ \sum_{n=N}^{\infty} |f(x_n)| : f \in X',\ \|f\| \leq 1 \right\} \leq \epsilon. \tag{9}$$

Let us now take any $N, M$ such that $N_0 \leq M < N$. By the Hahn-Banach theorem ([10], Corollary 4.2.1), there exists $f \in X'$, $\|f\| = 1$, such that

$$f\left(\sum_{n=M+1}^{N} \lambda_n x_n\right) = \left\| \sum_{n=M+1}^{N} \lambda_n x_n \right\|.$$

This gives us

$$\begin{aligned} \left\| \sum_{n=M+1}^{N} \lambda_n x_n \right\| &= f\left(\sum_{n=M+1}^{N} \lambda_n x_n\right) \\ &= \left| f\left(\sum_{n=M+1}^{N} \lambda_n x_n\right) \right| \\ &\leq \sum_{n=M+1}^{N} |\lambda_n| \cdot |f(x_n)| \\ &\leq \sum_{n=M+1}^{N} |f(x_n)| \overset{(9)}{\leq} \epsilon. \end{aligned}$$

This shows that $\left(\sum_{n=1}^{N} \lambda_n x_n\right)_N$ is a Cauchy sequence, so $\sum_{n=1}^{\infty} \lambda_n x_n$ converges.

$(e) \Rightarrow (d)$. Each sequence of signs is bounded.

$(d) \Rightarrow (c)$. Choose any subsequence $(x_{p(n)})_n$ of $(x_n)_n$. Define two sequences of signs:

$$\varepsilon_n = 1,\ n \in \mathbb{N},$$

$$\eta_n = \begin{cases} 1, & \text{if } n = p(j) \text{ for some } j \\ -1, & \text{if } n \neq p(j) \text{ for all } j \end{cases}$$

By our hypothesis (d), both $\sum_{n=1}^{\infty} \epsilon_n x_n$ and $\sum_{n=1}^{\infty} \eta_n x_n$ converge, whence

$$\sum_{j=1}^{\infty} x_{p(j)} = \frac{1}{2}\left(\sum_{n=1}^{\infty} \epsilon_n x_n + \sum_{n=1}^{\infty} \eta_n x_n\right)$$

converges as well.

$(c) \Rightarrow (a)$. Assume (c). To prove (a), it suffices, by Proposition 1.1.10 and Lemma 1.1.11, to obtain condition (4). We prove by contradiction. Suppose that (4) does not hold. Then there exists $\epsilon > 0$ with the property

$$\forall N \in \mathbb{N}\ \exists F_N \in \mathcal{F} \text{ such that } \min F_N > N \quad \text{and} \quad \left\| \sum_{n \in F_N} x_n \right\| \geq \epsilon.$$

Put $G_1 = F_1$ and $N_1 = \max G_1$. Let us now take $F_{N_1}$ from the above condition and put $G_2 = F_{N_1}$, $N_2 = \max G_2$. Let $G_3 = F_{N_2}$. Continuing in this way, we obtain a sequence of finite sets $G_K$ such that for each $K$,

$$\max G_k < \min G_{k+1} \text{ and } \left\| \sum_{n \in G_k} x_n \right\| \geq \epsilon. \tag{10}$$

Consider $\cup_K G_K$. Let

$$p(1), p(2), \ldots, p(\operatorname{card} G_1), p(\operatorname{card} G_1 + 1), \ldots, p(\operatorname{card} G_1 + \operatorname{card} G_2), \ldots$$

be the complete enumeration of $\cup_K G_K$ (with the elements of each $G_K$ listed in their natural order and followed by the elements of $G_{K+1}$). We now claim that $\sum_{n=1}^{\infty} x_{p(n)}$ cannot converge which contradicts our hypothesis (c). Indeed, we see from (10) that the corresponding sequence of partial sums $\left(\sum_{n=1}^{N} x_{p(n)}\right)_N$ is not a Cauchy sequence. $\square$

**Remark 1.1.13.** We note that in the proof the equivalence $(a) \Leftrightarrow (b)$ from the preceding theorem we did not use completeness. So, this equivalence holds in general normed spaces.

Another consequence of the the equivalence $(a) \Leftrightarrow (b)$ and the proof of the implication $(a) \Rightarrow (b)$ is the following important corollary.

**Corollary 1.1.14.** *Let $(x_n)_n$ be a sequence in a normed space $X$. If the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally, then $\sum_{n=1}^{\infty} x_{\sigma(n)} = \sum_{n=1}^{\infty} x_n$, for each permutation $\sigma$ of $\mathbb{N}$.*

**Corollary 1.1.15.** *Let $(x_n)_n$ be a sequence in a finite-dimensional normed space $X$. Then the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally if and only if it converges absolutely.*

**Proof.** Having in mind Theorem 1.1.3, we only need to show that unconditional convergence implies absolute convergence. We first prove this implication for sequences of scalars.

Consider first a sequence $(x_n)_n$ of real numbers such that the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally. Consider the sequence of signs $(\varepsilon_n)_n$ defined by

$$\varepsilon_n = \begin{cases} 1 & \text{if } x_n \geq 0 \\ -1 & \text{if } x_n < 0 \end{cases}$$

By Theorem 1.1.12 $(a) \Rightarrow (d)$, we conclude that the series

$$\sum_{n=1}^{\infty} \varepsilon_n x_n = \sum_{n=1}^{\infty} |x_n|$$

converges.

Let us now take a sequence of complex numbers $(x_n)_n$ such that the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally. Write $x_n = y_n + iz_n$, $y_n, z_n \in \mathbb{R}$, $n \in \mathbb{N}$.

Choose any permutation $\sigma$. Let $\sum_{n=1}^{\infty} x_{\sigma(n)} = x = y + iz$. Then we have

$$\left| y - \sum_{n=1}^{N} y_{\sigma(n)} \right| = \left| \operatorname{Re} \left( x - \sum_{n=1}^{N} x_{\sigma(n)} \right) \right| \leq \left| x - \sum_{n=1}^{N} x_{\sigma(n)} \right|, \quad \forall N \in \mathbb{N}.$$

This shows us that $\sum_{n=1}^{\infty} y_{\sigma(n)} = y$. Since $\sigma$ was arbitrary, we conclude that the series $\sum_{n=1}^{\infty} y_n$ converges unconditionally. By the first part of the proof, it converges absolutely as well. In the same way we conclude that $\sum_{n=1}^{\infty} z_n$ also converges absolutely. Finally, we see from

$$\sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} |y_n + iz_n| \leq \sum_{n=1}^{\infty} |y_n| + \sum_{n=1}^{\infty} |z_n|$$

that the series $\sum_{n=1}^{\infty} |x_n|$ converges.

The desired conclusion in an arbitrary finite-dimensional space (it is enough to consider $\mathbb{R}^n$ and $\mathbb{C}^n$, $n \in \mathbb{N}$) now follows by component-wise reasoning using the preceding part of the proof. $\square$

**Remark 1.1.16.** Let $(x_n)_n$ be a sequence in a Banach space $X$. The preceding results show that

$$\sum_{n=1}^{\infty} x_n \text{ converges absolutely } \Longrightarrow \sum_{n=1}^{\infty} x_n \text{ converges unconditionally } \Longrightarrow \sum_{n=1}^{\infty} x_n \text{ converges.}$$

In general, the implications in the opposite direction are not true. We know from Corollary 1.1.15 that unconditional convergence is equivalent to absolute convergence if $X$ is finite-dimensional, but this is no longer true for infinite-dimensional spaces (as demonstrated in Remark 1.1.5). In fact, the Dvoretzky-Rogers theorem asserts that one can find in each infinite-dimensional Banach space an unconditionally convergent series that does not converge absolutely.

On the other hand, the second implication cannot be reversed even for sequences of scalars. Example: $x_n = \frac{(-1)^n}{n}$, $n \in \mathbb{N}$.

Motivated by conditions (b), (d), and (e) from Theorem 1.1.12 we now introduce some quantities that can be attached to any sequence in a Banach space.

**Definition 1.1.17.** *Let $(x_n)_n$ be a sequence in a normed space $X$. Denote by $\mathcal{F}$ the set of all finite subsets of $\mathbb{N}$. Define the numbers $R, R_{\mathcal{E}}, R_\Lambda \in [0, +\infty]$ by*

$$\begin{aligned} R &= \sup\left\{\left\|\sum_{n\in F} x_n\right\| : F \in \mathcal{F}\right\}, \\ R_{\mathcal{E}} &= \sup\left\{\left\|\sum_{n\in F} \varepsilon_n x_n\right\| : F \in \mathcal{F}, \varepsilon_n = \pm 1,\ \forall n\right\}, \\ R_\Lambda &= \sup\left\{\left\|\sum_{n\in F} \lambda_n x_n\right\| : F \in \mathcal{F}, \lambda_n \in \mathbb{F},\ |\lambda_n| \le 1,\ \forall n\right\}. \end{aligned}$$

*Notice that we always have $0 \le R \le R_{\mathcal{E}} \le R_\Lambda \le +\infty$.*

To proceed, we need the following classical result.

**Theorem 1.1.18.** *(Caratheodory) Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers such that $|\lambda_n| \le 1$, for all $n = 1, 2, \dots, N$. Then there exist real numbers $c_1, c_2, \dots, c_N, c_{N+1} \ge 0$ and signs $\varepsilon_k^n = \pm 1$, $n = 1, 2, \dots, N$, $k = 1, 2, \dots, N, N+1$, such that*

$$\sum_{k=1}^{N+1} c_k = 1 \quad \textit{and} \quad \lambda_n = \sum_{k=1}^{N+1} \varepsilon_k^n c_k,\ \forall n = 1, 2, \dots, N.$$

**Proposition 1.1.19.** *Let $(x_n)_n$ be a sequence in a normed space $X$. Then*

*(a) $R \le R_{\mathcal{E}} \le 2R$;*

*(b) $R_{\mathcal{E}} = R_\Lambda$, if $X$ is real;*

*(c) $R_{\mathcal{E}} \le R_\Lambda \le 2R_{\mathcal{E}}$, if $X$ is complex.*

*In particular, any one of $R, R_{\mathcal{E}}, R_\Lambda$ is finite if and only if the other two are.*

**Proof.** (a) For $F \in \mathcal{F}$ and any sequence of signs $\varepsilon_n = \pm 1$ define

$$F^+ = \{n \in F : \varepsilon_n = 1\} \quad \text{and} \quad F^- = \{n \in F : \varepsilon_n = -1\}.$$

Then

$$\left\|\sum_{n\in F} \varepsilon_n x_n\right\| = \left\|\sum_{n\in F^+} x_n - \sum_{n\in F^-} x_n\right\| \le \left\|\sum_{n\in F^+} x_n\right\| + \left\|\sum_{n\in F^-} x_n\right\| \le 2R.$$

Taking supremum on the left hand side, we obtain $R_{\mathcal{E}} \le 2R$. The first inequality, namely $R \le R_{\mathcal{E}}$, is evident.

(b) Choose any $F \in \mathcal{F}$ and any finite sequence $\Lambda = (\lambda_n)_{n\in F}$ of real scalars such that $|\lambda_n| \le 1$ for every $n$ in $F$. Let card $F = N$. By Caratheodory's theorem there exist $c_1, c_2, \dots, c_N, c_{N+1} \ge 0$ and signs $\varepsilon_k^n = \pm 1$, $n = 1, 2, \dots, N$, $k = 1, 2, \dots, N, N+1$, such that

$$\sum_{k=1}^{N+1} c_k = 1 \quad \text{and} \quad \lambda_n = \sum_{k=1}^{N+1} \varepsilon_k^n c_k,\ \forall n = 1, 2, \dots, N.$$

Then

$$\begin{aligned}
\left\|\sum_{n\in F}\lambda_n x_n\right\| &= \left\|\sum_{n\in F}\sum_{k=1}^{N+1}\varepsilon_k^n c_k x_n\right\| \\
&\le \sum_{k=1}^{N+1} c_k\left\|\sum_{n\in F}\varepsilon_k^n x_n\right\| \\
&\le \sum_{k=1}^{N+1} c_k R_{\mathcal{E}} = R_{\mathcal{E}}.
\end{aligned}$$

Taking supremum on the left hand side, we obtain $R_\Lambda \le R_{\mathcal{E}}$. The opposite inequality is obvious (and already noted in Definition 1.1.17).

(c) Choose any $F \in \mathcal{F}$ and any finite sequence $\Lambda = (\lambda_n)_{n\in F}$ of complex numbers such that $|\lambda_n| \le 1$ for every $n$ in $F$. Let $\lambda_n = \alpha_n + i\beta_n$, $\alpha_n, \beta_n \in \mathbb{R}$, $|\alpha_n|, |\beta_n| \le 1$, $n \in F$. Then, as in the proof of (b), we obtain

$$\left\|\sum_{n\in F}\alpha_n x_n\right\| \le R_{\mathcal{E}} \quad\text{and}\quad \left\|\sum_{n\in F}\beta_n x_n\right\| \le R_{\mathcal{E}}$$

whence

$$\left\|\sum_{n\in F}\lambda_n x_n\right\| \le 2R_{\mathcal{E}}$$

from which it follows that $R_\Lambda \le 2R_{\mathcal{E}}$. □

**Theorem 1.1.20.** *Let $(x_n)_n$ be a sequence in a normed space $X$. If $\sum_{n=1}^\infty x_n$ converges unconditionally, then $R, R_\Lambda$, and $2R_{\mathcal{E}}$ are all finite.*

**Proof.** By Proposition 1.1.19, it suffices to prove that $R < \infty$. If $\sum_{n=1}^\infty x_n$ converges unconditionally, we can find, using the implication $(b) \Rightarrow (a)$ from Theorem 1.1.12 and Remark 1.1.13 and then Proposition 1.1.10 and Lemma 1.1.11, an $N = N(1)$ such that

$$\forall G \in \mathcal{F},\ \min G > N \Longrightarrow \left\|\sum_{n\in G} x_n\right\| < 1.$$

Let $F_0 = \{1, 2, \ldots, N\}$ and $M = \max_{F\subseteq F_0}\left\|\sum_{n\in F} x_n\right\|$; observe that $M < \infty$.

Now choose any $F \in \mathcal{F}$. Notice that we can write $F = (F \cap F_0) \cup (F \setminus F_0)$. Then

$$\left\|\sum_{n\in F} x_n\right\| \le \left\|\sum_{n\in F\cap F_0} x_n\right\| + \left\|\sum_{n\in F\setminus F_0} x_n\right\| \le M + 1.$$

Taking supremum over all $F \in \mathcal{F}$, we obtain $R \le M + 1$, as desired. □

The converse of Theorem 1.1.20 is false in general; that is, finiteness of $R, R_\Lambda$, and $R_{\mathcal{E}}$ need not imply that the series under consideration converges, let alone converges unconditionally.

**Example 1.1.21.** Consider $X = \ell^\infty$ and $e_n = (\delta_{nk})_k$, $n \in \mathbb{N}$. For every $F \in \mathcal{F}$ we have $\left\|\sum_{n\in F} e_n\right\|_\infty = 1$ which obviously implies $R = 1$. (One also easily concludes that $R_\Lambda = R_\mathcal{E} = 1$.) However, the same argument shows that $\sum_{n=1}^\infty e_n$ cannot converge in $\ell^\infty$ simply because its sequence of partial sums is not a Cauchy sequence.

We end this section with Orlicz's theorem which provides a necessary condition for unconditional convergence of a series in a Hilbert space. Firts we need a lemma.

**Lemma 1.1.22.** *Let $H$ be a Hilbert space and $x_1, x_2, \ldots, x_N \in H$, $N \in \mathbb{N}$. Then there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that $|\lambda_n| \leq 1$, for all $n = 1, 2, \ldots, N$, and*

$$\sum_{n=1}^N \|x_n\|^2 \leq \left\|\sum_{n=1}^N \lambda_n x_n\right\|^2.$$

**Proof.** This is obvious for $N = 1$. For $N = 2$ take $\lambda_1 = 1$ and $\lambda_2 = e^{i\arg(\langle x_1, x_2\rangle)}$. Then

$$\begin{aligned} \|\lambda_1 x_1 + \lambda_2 x_2\|^2 &= \|x_1\|^2 + 2\mathrm{Re}\,\overline{\lambda_2}\langle x_1, x_2\rangle + \|x_2\|^2 \\ &= \|x_1\|^2 + 2|\langle x_1, x_2\rangle| + \|x_2\|^2 \geq \|x_1\|^2 + \|x_2\|^2. \end{aligned}$$

A general inductive step is established in the same way. □

**Proposition 1.1.23.** *If $(x_n)_n$ is a sequence in a Hilbert space, then $\sum_{n=1}^\infty \|x_n\|^2 \leq R_\Lambda^2$.*

**Proof.** Fix $N \in \mathbb{N}$. Then by the preceding lemma we can find scalars $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that $|\lambda_n| \leq 1$, for all $n = 1, 2, \ldots, N$ and

$$\sum_{n=1}^N \|x_n\|^2 \leq \left\|\sum_{n=1}^N \lambda_n x_n\right\|^2 \leq R_\Lambda^2.$$

Letting $N \to \infty$, we obtain the desired conclusion. □

**Theorem 1.1.24.** *(Orlicz) If $(x_n)_n$ is a sequence in a Hilbert space such that the series $\sum_{n=1}^\infty x_n$ converges unconditionally, then $\sum_{n=1}^\infty \|x_n\|^2 < \infty$.*

**Proof.** Immediate from Theorem 1.1.20 and the preceding proposition. □

*Concluding remarks.* (a) Orlicz's theorem is not true in general Banach spaces.

(b) In the second part of the section we have followed (in principle) Section II 2 from [81].

**Exercise 1.1.25.** Let $(e_n)_n$ be an ONB for a Hilbert space $H$, let $e_0 = \sum_{n=1}^\infty \frac{1}{n} e_n$. Consider $X = \mathrm{span}\,\{e_0, e_2, e_3, \ldots\}$ and observe that $X$ is not complete. Show that the sequence $(e_n)_{n\geq 2}$ is maximal in $X$, but is not an ONB for $X$. (Compare with Theorem 1.1.8.)

**Exercise 1.1.26.** Prove Caratheodory's theorem.

**Exercise 1.1.27.** Let $H$ be a Hilbert space. Fix any $x \in H$ such that $\|x\| = 1$. If $(c_n)_n$ is a sequence of scalars, show that

$$\sum_{n=1}^{\infty} c_n x \text{ converges in } H \Longleftrightarrow \sum_{n=1}^{\infty} c_n \text{ converges in } \mathbb{F}$$

and

$$\sum_{n=1}^{\infty} c_n x \text{ converges unconditionally } \Longleftrightarrow \sum_{n=1}^{\infty} c_n \text{ converges unconditionally.}$$

Show by taking an appropriate sequence $(c_n)_n$ that the converse of Orlicz's theorem fails.

## 1.2 Topological and Riesz bases

**Definition 1.2.1.** *A sequence $(x_n)_n$ is a topological basis (or simply a basis) for a normed space $X$ if for every $x$ in $X$ there exists a unique sequence of scalars $(a_n(x))_n$ such that*

$$x = \sum_{n=1}^{\infty} a_n(x) x_n. \tag{11}$$

**Remark 1.2.2.** (a) A normed space that possesses a basis is necessarily separable. (Clear.)

(b) An ONB for a unitary space $H$ is a basis for $H$ in the sense of the above definition. (See Remark 1.1.7 (c).)

(c) Let $(x_n)_n$ be a basis for a normed space $X$. For each $n \in \mathbb{N}$ consider a map $a_n : X \to \mathbb{F}$ defined by $x \mapsto a_n(x)$, where $a_n(x)$ is the $n$th coefficient in expansion (11). It is easy to see that $a_n$ are linear functions of $x$ (this follows from the uniqueness of the coefficients in (11)). We say that $a_n$ are coefficient functionals associated with the basis $(x_n)_n$.

We say that a basis $(x_n)_n$ is a Schauder basis if each coefficient functional $a_n$ is continuous.

Sometimes we write $((x_n)_n, (a_n)_n)$ to denote a basis together with the associated sequence of coefficient functionals.

(d) If $(x_n)_n$ is a basis for a normed space $X$, it follows immediately from the uniqueness of expansion (11) that $x_n \neq 0$ for every $n$.

(e) Suppose that $((x_n)_n, (a_n)_n)$ is a basis for a Banach space $X$. For each $m$ we have $x_m = \sum_{n=1}^{\infty} a_n(x_m) x_n$ and $x_m = \sum_{n=1}^{\infty} \delta_{mn} x_n$. From this we conclude that $a_n(x_m) = \delta_{mn}$ for all $n$ and $m$. In this sense we say that the sequences $(x_n)_n$ and $(a_n)_n$ are biorthogonal. In general, a sequence $(v_n)_n$ in $X$ can possess more biorthogonal sequences of functionals $(f_n)_n$. However, if $(x_n)_n$ is a basis we shall show that there is only one sequence of functionals biorthogonal with $(x_n)_n$ and, moreover, that these functionals are continuous.

**Definition 1.2.3.** *Let $(x_n)_n$ be a basis for a normed space $X$. We say that $(x_n)_n$ is*

*(a) an unconditional basis if the series (11) converges unconditionally for every $x$ in $X$,*

*(b) a bounded basis if $0 < \inf_n \|x_n\| \leq \sup_n \|x_n\| < \infty$.*

**Definition 1.2.4.** *Let $((x_n)_n, (a_n)_n)$ be a basis for a Banach space $X$. The associated partial sum operators are the mappings $S_N : X \to X$ defined by $S_N(x) = \sum_{n=1}^{N} a_n(x) x_n$, $N \in \mathbb{N}$.*

Clearly, the partial sum operators are linear. It turns out that all $S_N$ are bounded, in fact, uniformly bounded. The key technical result is the following proposition.

**Proposition 1.2.5.** *Let $(x_n)_n$ be a sequence in a Banach space $X$ such that $x_n \neq 0$ for each $n$. Consider the vector space of sequences of scalars defined by*

$$Y = \left\{ (c_n)_n : \sum_{n=1}^{\infty} c_n x_n \text{ converges in } X \right\}.$$

*For $(c_n)_n$ define*

$$\|(c_n)_n\|_Y = \sup_N \left\|\sum_{n=1}^N c_n x_n\right\|.$$

*Then $(Y, \|\cdot\|_Y)$ is a Banach space. Further, if $(x_n)_n$ is a basis for $X$ then*

$$S : Y \to X, \quad S((c_n)_n) = \sum_{n=1}^\infty c_n x_n$$

*defines a bounded linear bijection whose inverse $S^{-1} : X \to Y$ is bounded as well.*

**Proof.** Obviously, $Y$ contains all finite sequences, so $Y \neq \emptyset$. It is also clear that $Y$ is a vector space. If $(c_n)_n \in Y$ then there exists $\sum_{n=1}^\infty c_n x_n = \lim_{N\to\infty} \sum_{n=1}^N c_n x_n$. Since convergent sequences are bounded, $\|(c_n)_n\|_Y$ is well-defined. Clearly, $\|\cdot\|_Y$ is a semi-norm. Suppose that $\|(c_n)_n\|_Y = 0$. This implies $\sum_{n=1}^N c_n x_n = 0$ for all $N$. Taking $N = 1$ (recall that $x_n \neq 0$ for each $n$), we get $c_1 = 0$. Now taking $N = 2$, we conclude that $c_2 = 0$. Proceed by induction. This proves that $(Y, \|\cdot\|_Y)$ is a normed space.

Let $(C^N)_N$ be a Cauchy sequence in $(Y, \|\cdot\|_Y)$. Write $C^N = (c_n^N)_n$, $N \in \mathbb{N}$. Then for $n$ fixed we find for all natural numbers $M$ and $N$

$$\begin{aligned}
\left|c_n^M - c_n^N\right| \cdot \|x_n\| &= \left\|(c_n^M - c_n^N)x_n\right\| \\
&= \left\|\sum_{k=1}^n (c_k^M - c_k^N)x_k - \sum_{k=1}^{n-1}(c_k^M - c_k^N)x_k\right\| \\
&\leq \left\|\sum_{k=1}^n (c_k^M - c_k^N)x_k\right\| + \left\|\sum_{k=1}^{n-1}(c_k^M - c_k^N)x_k\right\| \\
&\leq 2\left\|C^M - C^N\right\|_Y.
\end{aligned}$$

Since $(C^N)_N$ is a Cauchy sequence and $x_n \neq 0$, this shows that $(c_n^N)_N$ is a Cauchy sequence of scalars; thus, there exists

$$c_n := \lim_{N\to\infty} c_n^N, \quad n \in \mathbb{N}.$$

Choose any $\epsilon > 0$. First, we can find $N_0 \in \mathbb{N}$ such that

$$N_0 \leq M, N \Longrightarrow \left\|C^M - C^N\right\|_Y = \sup_L \left\|\sum_{n=1}^L (c_n^M - c_n^N)x_n\right\| < \epsilon. \tag{12}$$

Let us now fix $N \geq N_0$ and $L \geq 1$, and put $y_M = \sum_{n=1}^L (c_n^M - c_n^N)x_n$, $M \in \mathbb{N}$. Then, by (12), $\|y_M\| < \epsilon$ for each $M \geq N_0$. Observe that $y_M \to y = \sum_{n=1}^L (c_n - c_n^N)x_n$ as $M \to \infty$. In particular, we have $\|y\| \leq \epsilon$. So we have shown that

$$N_0 \leq N \Longrightarrow \sup_L \left\|\sum_{n=1}^L (c_n - c_n^N)x_n\right\| \leq \epsilon. \tag{13}$$

Further, since $C^{N_0} = (c_n^{N_0})_n$ belongs to $Y$, $\sum_{n=1}^{\infty} c_n^{N_0} x_n$ converges. Hence, there exists an $M_0 \in \mathbb{N}$ with the property

$$M_0 \leq M < N \Longrightarrow \left\| \sum_{n=M+1}^{N} c_n^{N_0} x_n \right\| \leq \epsilon. \tag{14}$$

Thus, if $N > M \geq M_0, N_0$ we have

$$\begin{aligned}
\left\| \sum_{n=M+1}^{N} c_n x_n \right\| &= \left\| \sum_{n=1}^{N} (c_n - c_n^{N_0}) x_n - \sum_{n=1}^{M} (c_n - c_n^{N_0}) x_n + \sum_{n=M+1}^{N} c_n^{N_0} x_n \right\| \\
&\leq \left\| \sum_{n=1}^{N} (c_n - c_n^{N_0}) x_n \right\| + \left\| \sum_{n=1}^{M} (c_n - c_n^{N_0}) x_n \right\| + \left\| \sum_{n=M+1}^{N} c_n^{N_0} x_n \right\| \\
&\overset{(13),(14)}{\leq} \epsilon + \epsilon + \epsilon = 3\epsilon.
\end{aligned}$$

This shows that $\sum_{n=1}^{\infty} c_n x_n$ converges in $X$, i.e. $(c_n)_n \in Y$. Now (13) shows that $C^N \to (c_n)_n$ as $N \to \infty$, so $(Y, \|\cdot\|_Y)$ is complete.

To prove the second statement, suppose that $(x_n)_n$ is a basis for $X$. Clearly, the mapping $S$ is linear and, since $(x_n)_n$ is a basis, bijective. Finally,

$$\|S((c_n)_n)\| = \left\| \sum_{n=1}^{\infty} c_n x_n \right\| = \lim_{N\to\infty} \left\| \sum_{n=1}^{N} c_n x_n \right\| \leq \sup_N \left\| \sum_{n=1}^{N} c_n x_n \right\| = \|(c_n)_n\|_Y$$

shows that $S$ is bounded. The last assertion follows from the inverse mapping theorem. $\square$

**Corollary 1.2.6.** *Let $((x_n)_n, (a_n)_n)$ be a basis for a Banach space $X$. Then:*

*(a)* $\sup_N \|S_N(x)\| < \infty$ *for each $x$ in $X$;*

*(b)* $C := \sup_N \|S_N\| < \infty$*;*

*(c)* $|||x||| = \sup_N \|S_N(x)\|$ *is a norm on $X$ satisfying $\|x\| \leq |||x||| \leq C\|x\|$ for all $x$ in $X$; thus, equivalent to the original norm $\|\cdot\|$ on $X$.*

**Proof.** Let $S$ be as in the preceding proposition. Take any $x \in X$. Then we have $x = \sum_{n=1}^{\infty} a_n(x) x_n$. Since the scalars $a_n(x)$ in this decomposition are unique, we have $S^{-1}x = (a_n(x))_n$. Hence

$$\sup_N \|S_N(x)\| = \sup_N \left\| \sum_{n=1}^{N} a_n(x) x_n \right\| = \|(a_n(x))_n\|_Y = \|S^{-1}x\|_Y \leq \|S^{-1}\| \cdot \|x\|.$$

The same computation shows that $\sup_N \|S_N\| \leq \|S^{-1}\|$, so we have (b) with $C = \|S^{-1}\|$.

It is evident that $|||\cdot|||$ is a semi-norm on $X$. The rest follows from

$$|||x||| = \sup_N \|S_N(x)\| \leq \left( \sup_N \|S_N\| \right) \|x\| = C\|x\|$$

and

$$\|x\| = \lim_{N\to\infty} \|S_N(x)\| \le \sup_N \|S_N(x)\| = |||x|||.$$

□

**Definition 1.2.7.** *The constant $C$ from Corollary 1.2.6 is called the basis constant. The inequality $\|x\| \le C\|x\|$ for all $x$ in $X$ shows that $C \ge 1$. If the basis constant $C$ is equal to 1, the basis is said to be monotone.*

**Theorem 1.2.8.** *Every basis $((x_n)_n, (a_n)_n)$ for a Banach space $X$ is a Schauder basis for $X$, i.e. the coefficient functionals $a_n$, $n \in \mathbb{N}$, are continuous. In fact, the coefficient functionals satisfy the inequalities*

$$1 \le \|a_n\| \cdot \|x_n\| \le 2C, \quad \forall n \in \mathbb{N},$$

*where $C$ is the basis constant for $((x_n)_n, (a_n)_n)$.*

**Proof.** Fix $x \in X$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
|a_n(x)| \cdot \|x_n\| &= \|a_n(x)x_n\| \\
&= \left\| \sum_{k=1}^{n} a_k(x)x_k - \sum_{k=1}^{n-1} a_k(x)x_k \right\| \\
&\le \left\| \sum_{k=1}^{n} a_k(x)x_k \right\| + \left\| \sum_{k=1}^{n-1} a_k(x)x_k \right\| \\
&= \|S_n(x)\| + \|S_{n-1}(x)\| \le 2C\|x\|.
\end{aligned}$$

Since $x_n \neq 0$, it follows that $|a_n(x)| \le \frac{2C}{\|x_n\|}\|x\|$ and hence, by taking supremum over the unit ball in $X$, $\|a_n\| \le \frac{2C}{\|x_n\|}$. On the other hand, $1 = a_n(x_n) = |a_n(x_n)| \le \|a_n\| \cdot \|x_n\|$. □

**Lemma 1.2.9.** *Let $A \in \mathbb{B}(X, Y)$ be a bijection of Banach spaces $X$ and $Y$. If $(x_n)_n$ is a basis for $X$, then $(Ax_n)_n$ is a basis for $Y$.*

**Proof.** If $y$ is any element of $Y$ then $A^{-1}y \in X$, so there exist unique scalars $c_n$ such that $A^{-1}y = \sum_{n=1}^{\infty} c_n x_n$. Since $A$ is bounded, this gives us $y = \sum_{n=1}^{\infty} c_n A x_n$. The same computation shows that this is the only decomposition of $y$ in the form $y = \sum_{n=1}^{\infty} b_n A x_n$. □

**Definition 1.2.10.** *Let $X$ and $Y$ be Banach spaces. We say that a basis $(x_n)_n$ for $X$ is equivalent to a basis $(y_n)_n$ for $Y$ and write $(x_n)_n \sim (y_n)_n$ if there exists a bijective operator $A \in \mathbb{B}(X, Y)$ such that $y_n = Ax_n$ for all $n$ in $\mathbb{N}$.*

**Theorem 1.2.11.** *Let $(x_n)_n$ and $(y_n)_n$ be bases for Banach spaces $X$ and $Y$, respectively. The following statements are equivalent:*

*(a)* $(x_n)_n \sim (y_n)_n$;

*(b)* $\sum_{n=1}^{\infty} c_n x_n$ *converges in $X$ if and only if* $\sum_{n=1}^{\infty} c_n y_n$ *converges in $Y$.*

**Proof.** $(a) \Rightarrow (b)$. Let $A \in \mathbb{B}(X,Y)$ be a bijective operator such that $y_n = Ax_n$ for all $n$ in $\mathbb{N}$. Now (b) follows from the continuity of $A$ and $A^{-1}$.

$(b) \Rightarrow (a)$. Denote by $(a_n)_n$ and $(b_n)_n$ the corresponding sequences of coefficient functionals. Take any $x \in X$. We know that $x = \sum_{n=1}^{\infty} a_n(x)x_n$. Since, by (b), $\sum_{n=1}^{\infty} a_n(x)y_n$ converges in $Y$, we can define $Ax$ by $Ax = \sum_{n=1}^{\infty} a_n(x)y_n$. Since each $x$ has a unique expansion of the form $x = \sum_{n=1}^{\infty} c_n x_n$, this gives us a well-defined mapping $A : X \to Y$. Clearly, $A$ is linear and satisfies $Ax_n = y_n$ for every $n$. Suppose that $Ax = 0$, that is $\sum_{n=1}^{\infty} a_n(x)y_n = 0$. Since $0 = \sum_{n=1}^{\infty} 0 \cdot y_n$ is the unique expansion of 0 with the respect to $(y_n)_n$, we conclude that $a_n(x) = 0$ for each $n$ and therefore $x = 0$.

Consider now arbitrary $y \in Y$, $y = \sum_{n=1}^{\infty} b_n(y)y_n$. By our assumption (b), we now have a well-defined element $x = \sum_{n=1}^{\infty} b_n(y)x_n$ in $X$. Since $(x_n)_n$ is a basis, this forces $b_n(y) = a_n(x)$ for all $n$. Hence $Ax = y$.

It remains only to show that $A$ is bounded. For each $N \in \mathbb{N}$ define $A_N : X \to Y$ by $A_N(x) = \sum_{n=1}^{N} a_n(x)y_n$. Since all $a_n$ are continuous, we conclude that $A_N$ is continuous. In fact,

$$\|A_N(x)\| = \left\| \sum_{n=1}^{N} a_n(x)y_n \right\| \le \sum_{n=1}^{N} |a_n(x)| \cdot \|y_n\| \le \|x\| \sum_{n=1}^{N} \|a_n\| \cdot \|y_n\|, \ \ \forall x \in X.$$

Since $A_N(x) \to Ax$ as $N \to \infty$, the sequence $(A_N(x))_N$ is bounded and

$$\|Ax\| \le \sup_N \|A_N(x)\| < \infty, \ \ \forall x \in X.$$

By the uniform boundedness principle, it follows that $\sup_N \|A_N\| < \infty$. But then

$$\|Ax\| \le \sup_N \|A_N x\| \le \left( \sup_N \|A_N\| \right) \|x\|,$$

so $A$ is bounded. □

**Remark 1.2.12.** All ONB in a Hilbert space are equivalent. Indeed, if $(e_n)_n$ and $(f_n)_n$ are ONB for a Hilbert space $H$ then $U : e_n \mapsto f_n$, $n \in \mathbb{N}$, extends to a unitary operator $U \in \mathbb{B}(H)$.

Alternatively, the conclusion follows from Lemma 1.1.4 and Theorem 1.2.11.

In infinite-dimensional spaces there are several types of linear independence of sequences.

**Definition 1.2.13.** *A sequence $(x_n)_n$ in a Banach space is*

*(a) finitely independent, if $\sum_{n=1}^{N} c_n x_n = 0$, $N \in \mathbb{N}$, implies $c_1 = c_2 = \ldots = c_N = 0$;*

*(b) $\omega$-independent, if $\sum_{n=1}^{\infty} c_n x_n = 0$ implies $c_n = 0$ for every $n$;*

*(c) minimal, if $x_m \notin \overline{\text{span}}\{x_n : n \neq m\}$ for every $m$.*

Obviously, if $(x_n)_n$ is a basis then $(x_n)_n$ is $\omega$-independent (because the null-vector has a unique decomposition of the form $0 = \sum_{n=1}^{\infty} c_n x_n$). In fact, much more is true.

**Proposition 1.2.14.** *Let $(x_n)_n$ be a sequence in a Banach space $X$. Then:*

*(a) if $(x_n)_n$ is a basis then $(x_n)_n$ is minimal and fundamental;*

*(b) if $(x_n)_n$ is minimal then $(x_n)_n$ is $\omega$-independent;*

*(c) if $(x_n)_n$ is $\omega$-independent then $(x_n)_n$ is finitely independent.*

**Proof.** (a) Let $((x_n)_n, (a_n)_n)$ be a basis for $X$. It is evident that $(x_n)_n$ is fundamental. Fix $m \in \mathbb{N}$ and define $E = \operatorname{span}\{x_n : n \neq m\}$. Since $(x_n)_n$ and $(a_n)_n$ are biorthogonal, we have $a_m(x_n) = 0$ for each $n \neq m$. Using linearity and continuity of $a_m$, we conclude that $a_m(x) = 0$ for every $x$ in $\overline{E}$. Since $a_m(x_m) = 1$, it follows that $x_m \notin \overline{E}$.

(b) Suppose that $(x_n)_n$ is minimal and that $\sum_{n=1}^{\infty} c_n x_n$ converges and $\sum_{n=1}^{\infty} c_n x_n = 0$. Now assume that there exists $m$ such that $c_m \neq 0$. Then $x_m = -\frac{1}{c_m}\sum_{n\neq m} c_n x_n$; thus, $x_m \in \overline{\operatorname{span}}\{x_n : n \neq m\}$ - a contradiction.

(c) Obvious. $\square$

**Remark 1.2.15.** The implications in Proposition 1.2.14 are not reversible (see Exercises 1.2.31, 1.2.32, 1.2.33 ). In particular, if a sequence $(x_n)_n$ in a Banach space is minimal and fundamental it needs not be a basis even though (as we shall see in the following proposition) such a sequence possesses a unique biorthogonal sequence $(a_n)_n$ in $X'$.

**Proposition 1.2.16.** *Let $(x_n)_n$ be a sequence in a Banach space $X$.*

*(a) $(x_n)_n$ is minimal if and only if there exists a sequence $(a_n)_n$ in $X'$ biorthogonal to $(x_n)_n$.*

*(b) $(x_n)_n$ is minimal and fundamental if and only if there exists a unique sequence $(a_n)_n$ in $X'$ biorthogonal to $(x_n)_n$.*

**Proof.** (a) Suppose that there exists a sequence $(a_n)_n$ in $X'$ biorthogonal to $(x_n)_n$. Fix any $m$ and choose $x \notin \operatorname{span}\{x_n : n \neq m\}$; let $x = \sum_{j=1}^{N} c_{n_j} x_{n_j}$, $x_{n_j} \neq x_m$ for all $j = 1, 2, \ldots, N$. Clearly, $a_m(x) = 0$. Since $a_m$ is continuous, this implies $a_m(x) = 0$ for all $x$ in $\overline{\operatorname{span}}\{x_n : n \neq m\}$. Since $a_m(x_m) = 1$ this shows that $x_m \notin \overline{\operatorname{span}}\{x_n : n \neq m\}$. (Notice that this is the same argument as in the proof of Proposition 1.2.14 (a).)

Conversely, suppose that $(x_n)_n$ is minimal. Again, fix $m$ and put $E = \overline{\operatorname{span}}\{x_n : n \neq m\}$. By the Hahn-Banach theorem ([10], Theorem 4.2.3), there exists $a_m \in X'$ such that $a_m(x) = 0$ for every $x$ in $E$ and $a_m(x_m) = 1$.

(b) Suppose first that there is a unique sequence $(a_n)_n$ in $X'$ biorthogonal to $(x_n)_n$. We already know from (a) that $(x_n)_n$ is minimal. Assume now that there is $f \in X'$ such that $f(x_n) = 0$ for all $n$. Then, obviously, $(a_n + f)_n$ is a sequence in $X'$ biorthogonal to $(x_n)_n$. By our uniqueness hypothesis, we conclude that $f = 0$. Again the Hahn-Banach theorem implies that $(x_n)_n$ is a fundamental sequence.

Conversely, suppose that $(x_n)_n$ is minimal and fundamental. We already know by (a) that there exists a sequence $(a_n)_n$ in $X'$ biorthogonal to $(x_n)_n$. Suppose that $(b_n)_n$ is another such sequence. This implies $(a_m - b_m)(x_n) = 0$ for all $n$ and $m$. Since $(x_n)_n$ is fundamental, this gives us $a_m - b_m = 0$ for each $m$. $\square$

**Theorem 1.2.17.** *Let $(x_n)_n$ be a sequence in a Banach space $X$. The following statements are all equivalent:*

*(a) $(x_n)_n$ is a basis for $X$;*

*(b) there exists a sequence $(a_n)_n$ in $X'$ biorthogonal to $(x_n)_n$ such that $x = \sum_{n=1}^{\infty} a_n(x)x_n$ for every $x$ in $X$;*

*(c) $(x_n)_n$ is fundamental and there exists a sequence $(a_n)_n$ in $X'$ biorthogonal to $(x_n)_n$ such that $\sup_N \|S_N(x)\| < \infty$ for every $x$ in $X$, where $S_N(x) = \sum_{n=1}^{N} a_n(x)x_n$, $N \in \mathbb{N}$;*

*(d) $(x_n)_n$ is fundamental and there exists a sequence $(a_n)_n$ in $X'$ biorthogonal to $X$ such that $\sup_N \|S_N\| < \infty$.*

**Proof.** $(a) \Rightarrow (b)$. This follows from Remark 1.2.2 (e) and Theorem 1.2.8.

$(b) \Rightarrow (c)$. If we assume (b) then $(x_n)_n$ is necessarily fundamental and, for every $x$ in $X$, we have $\sup_N \|S_N(x)\| < \infty$ because each convergent sequence is bounded.

$(c) \Rightarrow (d)$. This follows from the uniform boundedness principle.

$(d) \Rightarrow (b)$. Choose any $x$ in span $\{x_n : n \in \mathbb{N}\}$; let $x = \sum_{n=1}^{M} c_n x_n$. Since $S_N$ is linear and $(x_n)_n$ and $(a_n)_n$ are biorthogonal, we have for each $N \geq M$ that

$$S_N(x) = S_N(\sum_{j=1}^{M} c_j x_j) = \sum_{j=1}^{M} c_j S_N(x_j) = \sum_{j=1}^{M} c_j \left( \sum_{n=1}^{N} a_n(x_j)x_n \right) = \sum_{j=1}^{M} c_j x_j = x.$$

This implies $x = \lim_{N\to\infty} S_n(x) = \sum_{n=1}^{\infty} a_n(x)x_n$ for all $x$ in span $\{x_n : n \in \mathbb{N}\}$. Let us now take any $x$ in $X$. Given $\epsilon > 0$, we can find an element $y \in$ span $\{x_n : n \in \mathbb{N}\}$ such that $\|x - y\| < \frac{\epsilon}{1+C}$ where $C = \sup_N \|S_N\|$. Put $y = \sum_{j=1}^{M} c_j x_j$. Then we have for each $N \geq M$

$$\|x - S_N(x)\| \leq \|x - y\| + \|y - S_N(y)\| + \|S_N(y) - S_N(x)\| \leq \|x - y\| + 0 + \|S_N\| \cdot \|x - y\| < \epsilon.$$

$(b) \Rightarrow (a)$. Assume (b). To prove (a), we only have to show, for each $x$ in $X$ that $x = \sum_{n=1}^{\infty} a_n(x)x_n$ is the only decomposition of $x$ with respect to $(x_n)_n$. Choose any $x$ and suppose that we also have $x = \sum_{n=1}^{\infty} c_n x_n$. Since $(a_n)_n$ is biorthogonal to $(x_n)_n$ and each $a_m$ is continuous, we conclude that $a_m(x) = c_m$ for all $m$. □

Next we prove a useful property of unconditional bases.

**Proposition 1.2.18.** *Let $(x_n)_n$ be a sequence in a Banach space $X$. The following statements are equivalent:*

*(a) $(x_n)_n$ is an unconditional basis for $X$;*

*(b) $(x_{\sigma(n)})_n$ is a basis for every permutation $\sigma$ of $\mathbb{N}$.*

**Proof.** $(a) \Rightarrow (b)$. Assume (a) and choose any permutation $\sigma$ and $x \in X$. Then, since the series $x = \sum_{n=1}^{\infty} a_n(x)x_n$ converges unconditionally, we have by Corollary 1.1.14 that

$x = \sum_{n=1}^{\infty} a_{\sigma(n)}(x) x_{\sigma(n)}$. We must show that this is the unique representation of $x$ in terms of $x_{\sigma(n)}$. Suppose that we also have $x = \sum_{n=1}^{\infty} c_n x_{\sigma(n)}$ for some scalars $c_n$. Then

$$a_{\sigma(m)}(x) = a_{\sigma(m)}\left(\sum_{n=1}^{\infty} c_n x_{\sigma(n)}\right) = \sum_{n=1}^{\infty} c_n a_{\sigma(m)}(x_{\sigma(n)}) = c_m$$

since $(x_n)_n$ and $(a_n)_n$ are biorthogonal.

$(b) \Rightarrow (a)$. Now we assume that $(x_{\sigma(n)})_n$ is a basis for $X$ for every permutation $\sigma$. Let $(a_n)_n$ be the sequence of coefficient functionals associated with the basis $(x_n)_n$. We must show that for each $x \in X$ the representation $x = \sum_{n=1}^{\infty} a_n(x) x_n$ converges unconditionally. Fix any permutation $\sigma$. Since $(x_{\sigma(n)})_n$ is a basis, there exist unique scalars $c_n$ such that $x = \sum_{n=1}^{\infty} c_n x_{\sigma(n)}$. By applying $a_{\sigma(m)}$ to this equality, we get $a_{\sigma(m)}(x) = c_m$. Therefore $x = \sum_{n=1}^{\infty} c_n x_n = \sum_{n=1}^{\infty} a_{\sigma(n)}(x) x_{\sigma(n)}$ converges for every permutation $\sigma$, so $x = \sum_{n=1}^{\infty} a_n(x) x_n$ converges unconditionally. $\square$

Let $X$ be a Banach space. Recall that, for each $x$ in $X$, we have $\hat{x} \in X''$ that is defined by $\hat{x}(f) = f(x)$, $f \in X'$ and satisfies $\|\hat{x}\| = \|x\|$.

**Theorem 1.2.19.** *Let $((x_n)_n, (a_n)_n)$ be a basis for a Banach space $X$. Then $((a_n)_n, (\hat{x_n})_n)$ is a basis for $\overline{\text{span}}\,\{a_n : n \in \mathbb{N}\} \leq X'$. If $(x_n)_n$ is an unconditional basis for $X$, then $(a_n)_n$ is an unconditional basis for $\overline{\text{span}}\,\{a_n : n \in \mathbb{N}\}$. If $(x_n)_n$ is a bounded basis for $X$, then $(a_n)_n$ is a bounded basis for $\overline{\text{span}}\,\{a_n : n \in \mathbb{N}\}$.*

**Proof.** By definition, $(a_n)_n$ is fundamental in $\overline{\text{span}}\,\{a_n : n \in \mathbb{N}\}$. Further, $((a_n)_n, (\hat{x_n})_n)$ is a biorthogonal system since $\hat{x_n}(a_m) = a_m(x_n) = \delta_{nm}$. By Proposition 1.2.17 (here we use the implication $(d) \Rightarrow (a)$) we only need to show that $\sup_N \|T_N\| < \infty$ where $T_N(f) = \sum_{n=1}^{N} \hat{x_n}(f)$, $f \in \overline{\text{span}}\,\{a_n : n \in \mathbb{N}\}$. As before, we denote by $S_N$ the partial sum operators associated with the basis $((x_n)_n, (a_n)_n)$; $S_N(x) = \sum_{n=1}^{N} a_n(x) x_n$. We know that $S_N$ are bounded and that $\sup_N \|S_N\| =: C < \infty$. We claim that $S_N^* = T_N$ for each $N$ which obviously gives us the desired conclusion. Indeed, we have for all $f \in X'$ and $x \in X$

$$\begin{aligned}
S_N^*(f)(x) &= f(S_N(x)) \\
&= f\left(\sum_{n=1}^{N} a_n(x) x_n\right) \\
&= \sum_{n=1}^{N} a_n(x) f(x_n) \\
&= \sum_{n=1}^{N} a_n(x) \hat{x_n}(f) \\
&= \left(\sum_{n=1}^{N} \hat{x_n}(f) a_n\right)(x).
\end{aligned}$$

The second statement follows from the first one combined with Proposition 1.2.18.

To prove the last statement, suppose that we have $0 < \inf_n \|x_n\| \le \sup_n \|x_n\| < \infty$. Recall that $1 \le \|a_n\| \cdot \|x_n\| \le 2C$ where $C$ is the basis constant for $((x_n)_n, (a_n)_n)$. This is enough to conclude that $0 < \inf_n \|a_n\| \le \sup_n \|a_n\| < \infty$. $\square$

**Corollary 1.2.20.** *If $((x_n)_n, (a_n)_n)$ is a basis, unconditional basis, or bounded basis for a reflexive Banach space $X$ then $((a_n)_n, (\hat{x_n})_n)$ is a basis, unconditional basis, or bounded basis for $X'$.*

**Proof.** We only need to show that $(a_n)_n$ is fundamental in $X'$, since we already know that $(a_n)_n$ is a basis for $\overline{\text{span}}\,\{a_n : n \in \mathbb{N}\}$. Suppose that $\varphi \in X''$ satisfies $\varphi(a_n) = 0$ for all $n$. Since $X$ is reflexive, $\varphi$ is of the form $\varphi = \hat{x}$ for some $x$ in $X$. But then we have $0 = \varphi(a_n) = \hat{x}(a_n) = a_n(x)$ for all $n$. This implies $x = \sum_{n=1}^{\infty} a_n(x)x_n = 0$, i.e. $\varphi = 0$. By the Hahn-Banach theorem, $(a_n)_n$ is fundamental in $X'$. $\square$

Suppose that $H$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. and that $(x_n)_n$ is a basis for $H$. If $a_n$ are associated coefficient functionals, then by the Riesz representation theorem we can understand $a_n$'s as vectors from $H$. Then the expansion of any $x \in H$ with respect to the basis $(x_n)_n$ can be written in the form $x = \sum_{n=1}^{\infty} \langle x, a_n \rangle x_n$.

**Corollary 1.2.21.** *Let $H$ be a Hilbert space. Then $((x_n)_n, (y_n)_n)$ is a basis, unconditional basis or bounded basis for $H$ if and only if the same is true for $((y_n)_n, (x_n)_n)$.*

**Remark 1.2.22.** Every ONB for a Hilbert space $H$ is unconditional and bounded. This follows from Remark 1.1.5 and the definition of an unconditional basis (boundedness is here trivial).

**Corollary 1.2.23.** *Let $(x_n)_n$ and $(y_n)$ be bases for Banach spaces $X$ and $Y$, respectively, such that $(x_n)_n \sim (y_n)_n$. Then $(x_n)_n$ is unconditional if and only if $(y_n)_n$ is unconditional and $(x_n)_n$ is bounded if and only if $(y_n)_n$ is bounded.*

**Proof.** The first part follows from Proposition 1.2.18. The second part is trivial (notice that a bijective bounded operator of Banach spaces is necessarily bounded from below). $\square$

**Definition 1.2.24.** *A sequence $(x_n)_n$ in a Hilbert space $H$ is a Riesz basis for $H$ if there exist an ONB $(e_n)_n$ for $H$ and a bijection $T \in \mathbb{B}(H)$ such that $x_n = Te_n$ for every $n$.*

**Remark 1.2.25.** Each Riesz basis is a basis. This follows from Lemma 1.2.9.

**Proposition 1.2.26.** *Let $H$ be a Hilbert space.*

*(a) Each Riesz basis for $H$ is an unconditional bounded basis.*

*(b) All Riesz bases for $H$ are equivalent.*

*(c) If $(x_n)_n$ is a Riesz basis for $H$ and if $S \in \mathbb{B}(H, K)$ is a bijection, then $(Sx_n)_n$ is a Riesz basis for $K$.*

**Proof.** (a) This follows from the definition of a Riesz basis, Remark 1.2.22, and Corollary 1.2.23.

(b) Let $(x_n)_n$ and $(y_n)_n$ be Riesz bases for $H$. Find ONB's $(e_n)_n$ and $(f_n)_n$ such that $(x_n)_n \sim (e_n)_n$ and $(y_n)_n \sim (f_n)_n$; denote by $T$ and $S$ the corresponding bounded invertible operators on $H$. In addition, let $U \in \mathbb{B}(H)$ be the unitary operator defined by $Ue_n = f_n$, $n \in \mathbb{N}$. Then $SUT^{-1}$ is an invertible bounded operator on $H$ for which we have $y_n = SUT^{-1}x_n$ for all $n$.

(c) By assumption, $H$ is separable. Since there exists a bijective bounded linear operator $S : H \to K$, $K$ is separable as well and $\dim H = \dim K$. In particular, there exists a unitary operator $U \in \mathbb{B}(H, K)$. Let $(e_n)_n$ be an ONB for $H$ such that there exists a bijection $T \in \mathbb{B}(H)$ for which we have $x_n = Te_n$ for all $n$. Observe that $(Ue_n)_n$ is an ONB for $K$ and that $Sx_n = (STU^*)(Ue_n)$ for every $n$. □

**Lemma 1.2.27.** *Let $((x_n)_n, (a_n)_n)_n$ and $((y_n)_n, (b_n)_n)_n$ be bases for a Hilbert space $H$. If $(x_n)_n \sim (y_n)_n$ then $(a_n)_n \sim (b_n)_n$.*

**Proof.** By Corollary 1.2.21 both $((a_n)_n, (x_n)_n)_n$ and $((b_n)_n, (y_n)_n)_n$ are bases for $H$. Suppose that there is a bijection $S \in \mathbb{B}(H)$ such that $Sx_n = y_n$ for all $n$. The adjoint operator $S^*$ is also a bijection and we have for all $m$ and $n$

$$\langle x_m, S^*b_n\rangle = \langle Sx_m, b_n\rangle = \langle y_m, b_n\rangle = \delta_{mn} = \langle x_m, a_n\rangle.$$

Since $(x_n)_n$ is fundamental, it follows that $S^*b_n = a_n$ for every $n$; thus, $(b_n)_n \sim (a_n)_n$. □

**Corollary 1.2.28.** *Let $((x_n)_n, (y_n)_n)_n$ be a basis for a Hilbert space $H$. Then the following statements are equivalent:*

*(a)* $(x_n)_n$ *is a Riesz basis;*

*(b)* $(y_n)_n$ *is a Riesz basis;*

*(c)* $(x_n)_n \sim (y_n)_n$.

**Proof.** Observe that each ONB is biorthogonal to itself. Suppose that $(x_n)_n$ is a Riesz basis; thus, $(x_n)_n \sim (e_n)_n$ for some ONB $(e_n)_n$ for $H$. Then the preceding lemma gives us $(y_n)_n \sim (e_n)_n$. This proves $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$.

Assume now (b). By Corollary 1.2.21 $((x_n)_n, (y_n)_n)_n$ is a basis for $H$. Hence, arguing as above, we conclude that $(b) \Rightarrow (a)$ and $(b) \Rightarrow (c)$.

To end the proof, assume (c). Let $S \in \mathbb{B}(H)$ be a bijection for which we have $Sx_n = y_n$ for every $n$. Since $((x_n)_n, (y_n)_n)_n$ is a basis, it follows that for each $x$ in $H$ we have

$$x = \sum_{n=1}^{\infty} \langle x, y_n\rangle x_n = \sum_{n=1}^{\infty} \langle x, Sx_n\rangle x_n.$$

This implies

$$Sx = \sum_{n=1}^{\infty} \langle x, Sx_n\rangle Sx_n \quad \text{and} \quad \langle Sx, x\rangle = \sum_{n=1}^{\infty} |\langle x, Sx_n\rangle|^2 \geq 0.$$

Then (see Exercise 1.2.35) $S$ is a positive operator. Hence $S^{-1}$ is a positive operator as well. Moreover, we have $S^{-\frac{1}{2}} = (S^{\frac{1}{2}})^{-1} \geq 0$. Further, since $S^{\frac{1}{2}}$ is self-adjoint, we have for all $m$ and $n$

$$\langle S^{\frac{1}{2}} x_m, S^{\frac{1}{2}} x_n \rangle = \langle x_m, S x_n \rangle = \langle x_m, y_n \rangle = \delta_{mn}.$$

This shows that $(S^{\frac{1}{2}} x_n)_n$ is an ON sequence. In addition, since $S^{\frac{1}{2}}$ is a bijection and since $(x_n)_n$ is fundamental in $H$, it follows from Theorem 1.1.8 that $(S^{\frac{1}{2}} x_n)_n$ is an ONB for $H$. Hence $(x_n)_n = (S^{-\frac{1}{2}}(S^{\frac{1}{2}} x_n))_n$ is a Riesz basis. By symmetry (or by applying $(a) \Rightarrow (b)$), we conclude that $(y_n)_n$ is a Riesz basis as well. □

**Theorem 1.2.29.** *Let $(x_n)_n$ be a sequence in a Hilbert space $H$. The following statements are all equivalent:*

*(a) $(x_n)_n$ is a Riesz basis for $H$.*

*(b) $(x_n)_n$ is a basis for $H$ and, if $(c_n)_n$ is a sequence of scalars,*

$$\sum_{n=1}^{\infty} c_n x_n \ \text{ converges } \iff (c_n)_{\in} \ell^2.$$

*(c) $(x_n)_n$ is fundamental in $H$ and there exist constants $A$ and $B$ such that*

$$\forall N \in \mathbb{N},\ \forall c_1, c_2, \ldots, c_N \in \mathbb{F},\ \ A \sum_{n=1}^{N} |c_n|^2 \leq \left\| \sum_{n=1}^{N} c_n x_n \right\|^2 \leq B \sum_{n=1}^{N} |c_n|^2.$$

*(d) There exists an equivalent inner product $(\cdot|\cdot)$ on $H$ such that $(x_n)_n$ is an ONB for $(H, (\cdot|\cdot))$.*

**Proof.** $(a) \Rightarrow (b)$. Suppose (a) and find an ONB $(e_n)_n$ and an invertible operator $T \in \mathbb{B}(H)$ such that $Te_n = x_n$ for all $n$. Let $(c_n)_n$ be a sequence of scalars. By Theorem 1.2.11,

$$\sum_{n=1}^{\infty} c_n x_n \ \text{ converges } \iff \sum_{n=1}^{\infty} c_n e_n \ \text{ converges.}$$

On the other hand, we know from Lemma 1.1.4 that

$$\sum_{n=1}^{\infty} c_n e_n \ \text{ converges } \iff (c_n)_n \in \ell^2.$$

$(b) \Rightarrow (a)$. Suppose (b) and take any ONB $(e_n)_n$ for $H$. Then again by Lemma 1.1.4 and Theorem 1.2.11 we conclude that $(x_n)_n \sim (e_n)_n$. So, by definition, $(x_n)_n$ is a Riesz basis.

$(a) \Rightarrow (d)$. Suppose (a) and find an ONB $(e_n)_n$ and an invertible operator $T \in \mathbb{B}(H)$ such that $Te_n = x_n$ for all $n$. Define a new inner product on $H$ by

$$(x|y) = \langle T^{-1}x, T^{-1}x \rangle,\ \ x, y \in H$$

(it is easy to verify that this is indeed an inner product; we omit the details). Note that the resulting norm is given by

$$|||x||| = \|T^{-1}x\|, \;\; x \in H.$$

Since $T$ is bounded and bounded from below, $|||\cdot|||$ is obviously equivalent to the original norm $\|\cdot\|$ on $H$. Clearly, this implies that $(x_n)_n$ is also fundamental with respect to this new norm and

$$(x_n|x_m) = \langle T^{-1}x_n, T^{-1}x_m\rangle = \langle e_n, e_m\rangle = \delta_{mn}.$$

By Theorem 1.2.11, $(x_n)_n$ is an ONB for $(H, (\cdot|\cdot))$.

$(d) \Rightarrow (c)$. Assume (d). Let $A$ and $B$ be the constants for which we have

$$A|||x|||^2 \leq \|x\|^2 \leq B|||x|||^2, \;\; \forall x \in H. \tag{15}$$

(where $|||\cdot|||$ denotes the norm arising from the new inner product $(\cdot|\cdot)$ from our hypothesis (d)). Since $(x_n)_n$ is an ONB with respect to $(\cdot|\cdot)$, the second inequality in (15) shows that $(x_n)_n$ is fundamental also with respect to the original norm $\|\cdot\|$. Choose $N \in \mathbb{N}$ and arbitrary scalars $c_1, c_2, \ldots, c_N$. Then

$$|||\sum_{n=1}^{N} c_n x_n|||^2 = \left(\sum_{n=1}^{N} c_n x_n | \sum_{n=1}^{N} c_n x_n\right) = \sum_{n=1}^{N} |c_n|^2.$$

This together with (15) gives us the desired conclusion.

$(c) \Rightarrow (a)$. Assume (c) and take any ONB $(e_n)_n$ for $H$. Choose any $x \in H$. Then

$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n \quad \text{and} \quad \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2.$$

Let $M > N$. Define $c_1 = c_2 = \ldots = c_M = 0$ and $c_n = \langle x, e_n\rangle$ for $n = M+1, M+2, \ldots, N$. Then by hypothesis (c),

$$\left\|\sum_{n=M+1}^{N} \langle x, e_n\rangle x_n\right\| = \left\|\sum_{n=1}^{N} c_n x_n\right\| \leq B\sum_{n=1}^{N} |c_n|^2 = B\sum_{n=M+1}^{N} |\langle x, e_n\rangle|^2.$$

From this we conclude that the series $\sum_{n=1}^{\infty} \langle x, e_n\rangle x_n$ is convergent, so we can define $Sx = \sum_{n=1}^{\infty} \langle x, e_n\rangle x_n$ for every $x \in H$. Clearly, $S$ is a linear map. We claim that $S$ is bounded and bijective.

By applying our hypothesis (c) and letting $N \to \infty$ we obtain

$$A\|x\|^2 = A\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|Sx\|^2 \leq B\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = B\|x\|^2.$$

This tells us that $S$ is bounded and bounded from below. In particular, the range of $S$, $\mathrm{R}(S)$, is closed. On the other hand, we have $Se_m = x_m$ for every $m$; hence, $\mathrm{R}(S)$ is dense in $H$. Therefore, $S$ is a bijection. By definition, $(x_n)_n = (Se_n)_n$ is a Riesz basis for $H$. $\square$

**Corollary 1.2.30.** *If $(x_n)_n$ is a Riesz basis for a Hilbert space $H$, there exist constants $A$ and $B$ such that*

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \ \ \forall x \in H. \tag{16}$$

**Proof.** Find an ONB $(e_n)_n$ and an invertible bounded operator $T$ such that $x_n = Te_n$ for all $n$. Then we have

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle x, Te_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle T^*x, e_n\rangle|^2 = \|T^*x\|^2. \tag{17}$$

Since $T^*$ is also an invertible operator, it is bounded and bounded from below. This together with (17) implies (16) with $B = \|T\|^2$ and $A = \frac{1}{\|T^{-1}\|^2}$. To verify this last assertion observe that

$$\|x\| = \|(T^*)^{-1}T^*x\| \leq \|(T^*)^{-1}\| \cdot \|T^*x\| = \|T^{-1}\| \cdot \|T^*x\|.$$

□

*Concluding remarks.* (a) The question of whether every separable Banach space possesses a basis was a longstanding problem. It was shown by Enflo in 1973 that the answer is negative.

(b) A classical reference for the material covered in this section (and incomparably more) is [107].

(c) The material in this section is organized by following (partly) Chapters III and IV from [81].

**Exercise 1.2.31.** (A minimal and fundamental system that is not a basis.) Consider the Banach space $X = C(\mathbb{T}) = \{f \in C(\mathbb{R}) : f(t) = f(t+1)\}$ of all continuous 1-periodic complex functions with the supremum norm $\|\cdot\|_\infty$. For $n \in \mathbb{Z}$ let $e_n(t) = e^{2\pi int}$. Observe that $e_n$ belong to $X$, but also define elements $\varphi_n$ of the dual space $X'$ by $\varphi_n(f) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)e^{-2\pi int}dt$. Conclude that $(\varphi_n)_n$ is a system biorthogonal to $(e_n)_n$, so that, by Proposition 1.2.16 (a), $(e_n)_n$ is minimal. Use the Stone-Weierstrass theorem to conclude that $(e_n)_n$ is fundamental in $X$. Show that $(e_n)_n$ is not a basis for $X$.

**Exercise 1.2.32.** (An $\omega$-independent and fundamental system that is not minimal.) Let $(e_n)_n$ be an ONB for a Hilbert space $H$. Define $f_1 = e_1$ and $f_n = e_1 + \frac{e_n}{n}$ for $n \geq 2$. Show that $(f_n)_n$ is an $\omega$-independent and fundamental system in $H$ that is not minimal.

**Exercise 1.2.33.** (A finitely independent fundamental system that is not $\omega$-independent.) Let $((x_n)_n, (a_n)_n)$ be a basis for a Banach space $H$. Show that there exists $x_0 \in X$ such that $a_n(x) \neq 0$ for all $n$ and consider a new sequence $(x_n)_{n=0}^{\infty}$. Show that $(x_n)_{n=0}^{\infty}$ is fundamental and finitely independent, but not $\omega$-independent.

**Exercise 1.2.34.** A basis $((x_n)_n, (a_n)_n)$ for a Banach space $X$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} a_n(x)x_n$ is absolutely convergent for each $x$ in $X$. Show that the canonical basis in $\ell^1$ is absolutely convergent. Prove that each Banach space that possesses an absolutely convergent basis is topologically isomorphic to $\ell^1$ (i.e. that there exists a bounded bijective linear operator $T : X \to \ell^1$.) Using this, show that a separable Hilbert space does not possess any absolutely convergent basis. (Hint: topologically isomorphic spaces have topologically isomorphic duals.)

**Exercise 1.2.35.** Let $S$ be a bounded operator on a Hilbert space $H$ such that $\langle Sx, x\rangle \geq 0$ for all $x$ in $H$. Show that $S$ is a positive operator. (Hint. We need to show that $S$ is self-adjoint. Consider a sesquilinear functional $(x, y) \mapsto \langle Sx, y\rangle$ and use polarization.)

Further, if $S$ is a positive invertible bounded operator on $H$, show that $S^{-1}$ is also positive and that $(S^{-1})^{\frac{1}{2}} = (S^{\frac{1}{2}})^{-1}$

**Exercise 1.2.36.** Are there bases for separable Hilbert spaces that are not Riesz bases?

## 1.3 Bessel sequences

**Definition 1.3.1.** *A sequence $(x_n)_n$ in a Hilbert space $H$ is said to be a Bessel sequence if*

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 < \infty, \ \ \forall x \in H. \tag{18}$$

**Lemma 1.3.2.** *If $(x_n)_n$ is a Bessel sequence in a Hilbert space $H$, the mapping $U : H \to \ell^2$ defined by $Ux = (\langle x, x_n\rangle)_n$ is a bounded linear operator. In particular, there exists a constant $B > 0$ such that*

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \ \ \forall x \in H. \tag{19}$$

**Proof.** It is clear from (18) that $U$ is well-defined. Obviously, $U$ is linear. We will show that the graph of $U$ is closed to prove that $U$ is bounded.

Suppose that $y_N \to y \in H$ and $Uy_N \to (c_n)_n \in \ell^2$. Then for each fixed $m$ we have

$$|c_m - \langle y_N, x_m\rangle|^2 \leq \sum_{n=1}^{\infty} |c_n - \langle y_N, x_n\rangle|^2 = \|(c_n)_n - Uy_N\|^2 \to 0 \ \text{ as } \ N \to \infty.$$

Therefore $c_m = \lim_{N\to\infty}\langle y_N, x_m\rangle = \langle y, x_m\rangle$ for every $m$. Hence $(c_n)_n = (\langle y, x_n\rangle)_n$, so $U$ has a closed graph. $\square$

**Definition 1.3.3.** *The operator $U$ from Lemma 1.3.2 is called the analysis operator associated with $(x_n)_n$. Its adjoint $U^* \in \mathbb{B}(\ell^2, H)$ is called the synthesis operator.*

*The constant $B$ from (19) is called a Bessel bound of the sequence $(x_n)_n$.*

Note that a Bessel bound is not unique and that the optimal (i.e. minimal) Bessel bound is equal to $\|U\|^2$.

**Proposition 1.3.4.** *Let $(x_n)_n$ be a Bessel sequence in a Hilbert space $H$ with the analysis operator $U$. Then for each sequence $(c_n)_n$ in $\ell^2$ the series $\sum_{n=1}^{\infty} c_n x_n$ converges unconditionally and the synthesis operator $U^*$ is given by $U^*(c_n)_n = \sum_{n=1}^{\infty} c_n x_n$. In particular, if $(e_n)_n$ is the canonical basis for $\ell^2$, we have $U^* e_n = x_n$ and, consequently, $\|x_n\| \leq \|U\|$ for each $n$.*

**Proof.** Let $B$ be a Bessel bound for $(x_n)_n$. Choose any $(c_n)_n$ in $\ell^2$. Since by Theorem 1.1.12 unconditional convergence of the series $\sum_{n=1}^{\infty} c_n x_n$ is equivalent to sumability of the family $\{c_n x_n : n \in \mathbb{N}\}$, it suffices to show that the net $\left(\sum_{n\in F} c_n x_n\right)_{F\in\mathcal{F}}$ converges. We shall show that $\left(\sum_{n\in F} c_n x_n\right)_{F\in\mathcal{F}}$ is in fact a Cauchy net.

In the computation that follows we will use a well-known trick based on the Riesz representation theorem for bounded functionals on a Hilbert space: the norm of a vector is equal to the norm of the induced bounded functional.

Let $F$ be an arbitrary finite subset of $\mathbb{N}$. Let $\operatorname{card} F = N$. Then

$$\begin{aligned}
\left\|\sum_{n\in F} c_n x_n\right\|^2 &= \sup\left\{\left|\left\langle\sum_{n\in F} c_n x_n, y\right\rangle\right|^2 : \|y\| = 1\right\} \\
&= \sup\left\{\left|\sum_{n\in F} c_n \langle x_n, y\rangle\right|^2 : \|y\| = 1\right\} \quad \text{(by the Cauchy-Schwarz inequality in } \mathbb{F}^N\text{)} \\
&\le \sup\left\{\left(\sum_{n\in F} |c_n|^2\right)\left(\sum_{n\in F} |\langle x_n, y\rangle|^2\right) : \|y\| = 1\right\} \\
&\le \sup\left\{B\|y\|^2 \sum_{n\in F} |c_n|^2 : \|y\| = 1\right\} = B\sum_{n\in F} |c_n|^2.
\end{aligned}$$

Since the series $\sum_{n=1}^{\infty} |c_n|^2$ converges absolutely and unconditionally, it follows by Theorem 1.1.12 that $\left(\sum_{n\in F} |c_n|^2\right)_{F\in\mathcal{F}}$ is a Cauchy net. This, together with the above computation, shows us that the net $\left(\sum_{n\in F} c_n x_n\right)_{F\in\mathcal{F}}$ is Cauchy as well.

We are now in position to obtain a formula for the action of $U^*$: for all $x \in H$ and $(c_n)_n$ in $\ell^2$ we have

$$\langle x, U^*(c_n)_n\rangle = \langle Ux, (c_n)_n\rangle = \sum_{n=1}^{\infty} \langle x, x_n\rangle \overline{c_n} = \left\langle x, \sum_{n=1}^{\infty} c_n x_n\right\rangle.$$

□

A result related to the preceding proposition provides a sufficient condition for the Bessel property of a sequence.

**Proposition 1.3.5.** *Let $(x_n)_n$ be a sequence in a Hilbert space $H$ such that the series $\sum_{n=1}^{\infty} c_n x_n$ converges for each $(c_n)_n$ in $\ell^2$. Then $(x_n)_n$ is a Bessel sequence.*

**Proof.** Define the mapping $T : \ell^2 \to H$ by $T(c_n)_n = \sum_{n=1}^{\infty} c_n x_n$. Clearly, $T$ is linear. Consider also, for each $N$, the operator $T_N \in \mathbb{B}(\ell^2, H)$ defined by $T_N(c_n)_n = \sum_{n=1}^{N} c_n x_n$. Then, obviously, $T$ is the strong limit of the sequence $(T_N)_N$. By the uniform boundedness principle ([10], Proposition 5.4.10), it follows that $T$ is a bounded operator. Let $\|T\| = \sqrt{B}$.

Consider $T^*$ and observe that $\|T^*\| = \sqrt{B}$. We also have (we denote again the canonical basis for $\ell^2$ by $(e_n)_n$)

$$\langle T^*x, e_n\rangle = \langle x, Te_n\rangle = \langle x, x_n\rangle, \ \ \forall x \in H, \forall n \in \mathbb{N}.$$

This tells us that $T^*x = (\langle x, x_n\rangle)_n$ for every $x \in H$. Hence, for each $x$ in $H$ we conclude: $(\langle x, x_n\rangle)_n$, being a sequence that belongs to $\ell^2$, satisfies $\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 < \infty$. □

**Remark 1.3.6.** The preceding results show us: a sequence $(x_n)_n$ in a Hilbert space $H$ is Bessel if and only if there exists a bounded operator $T \in \mathbb{B}(\ell^2, H)$ such that $Te_n = x_n$, for all

$n \in \mathbb{N}$, where $(e_n)_n$ is the canonical ONB for $\ell^2$. Whenever this is the case, $T$ coincides with the synthesis operator $U^*$ of $(x_n)_n$.

Moreover, since all ONB's for separable Hilbert spaces are equivalent (via unitary operators), we conclude: a sequence $(x_n)_n$ in a Hilbert space $H$ is Bessel if and only if there exist a Hilbert space $K$, an ONB $(f_n)_n$ for $K$, and a bounded operator $T \in \mathbb{B}(K, H)$ such that $Tf_n = x_n$ for each $n$ in $\mathbb{N}$.

Another useful tool for studying Bessel sequences is the Gram matrix.

Suppose that $(x_n)_n$ is a Bessel sequence in a Hilbert space $H$. Consider the operator $UU^* \in \mathbb{B}(\ell^2)$. Let $(e_n)_n$ be the canonical basis for $\ell^2$. Observe that for every $k$ in $\mathbb{N}$ we have

$$UU^* e_k = Ux_k = \sum_{n=1}^{\infty} \langle x_k, x_n \rangle e_n.$$

This shows us that, if we denote by $[UU^*]$ the (infinite) matrix of $UU^*$ with respect to $(e_n)_n$, we have $[UU^*]_{(n,k)} = \langle x_k, x_n \rangle$. Thus, $[UU^*]$ is the Gram matrix of the sequence $(x_n)_n$.

In general, we can consider the Gram matrix $G(x_n)_n$ of any sequence $(x_n)_n$, but this matrix need not to represent a bounded operator on $\ell^2$.

**Theorem 1.3.7.** *Let $(x_n)_n$ be a sequence in a Hilbert space $H$. The following statements are equivalent:*

*(a) $(x_n)_n$ is a Bessel sequence with a Bessel bound $B$;*

*(b) the Gram matrix $G(x_n)_n$ defines a bounded operator $G$ on $\ell^2$ such that $\|G\| \leq B$.*

**Proof.** $(a) \Rightarrow (b)$. We have already noted that, if $B$ is a Bessel bound of a Bessel sequence $(x_n)_n$, then the corresponding analysis operator $U$ satisfies $\|U\| \leq \sqrt{B}$. This implies that $\|UU^*\| \leq B$. On the other hand, the preceding discussion shows that the matrix representation of $UU^*$ with respect to the canonical basis $(e_n)_n$ is precisely $G(x_n)_n$.

$(b) \Rightarrow (a)$. Assume (b) and choose any sequence $(c_k)_k$ from $\ell^2$. Then we have

$$\|G(c_k)_k\|^2 = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} \langle x_k, x_n \rangle c_k \right|^2 \leq B^2 \sum_{k=1}^{\infty} |c_k|^2. \tag{20}$$

Let us now take arbitrary $N$ and $M$ such that $N > M$. Then we have

$$\begin{aligned}
\left\| \sum_{k=1}^{N} c_k x_k - \sum_{k=1}^{M} c_k x_k \right\|^4 &= \left| \left\langle \sum_{k=M+1}^{N} c_k x_k, \sum_{n=M+1}^{N} c_n x_n \right\rangle \right|^2 \\
&= \left| \sum_{n=M+1}^{N} \overline{c_n} \sum_{k=M+1}^{N} c_k \langle x_k, x_n \rangle \right|^2 \quad \text{(by the Cauchy-Schwarz inequality)} \\
&\leq \left( \sum_{n=M+1}^{N} |c_n|^2 \right) \left( \sum_{n=M+1}^{N} \left| \sum_{k=M+1}^{N} c_k \langle x_k, x_n \rangle \right|^2 \right).
\end{aligned}$$

Let us now apply (20) to the sequence $(0, \ldots, 0, c_{M+1}, c_{M+2}, \ldots, c_N, 0, \ldots)$. This gives us

$$\sum_{n=M+1}^{N} \left| \sum_{k=M+1}^{N} c_k \langle x_k, x_n \rangle \right|^2 \le \sum_{n=1}^{\infty} \left| \sum_{k=M+1}^{N} c_k \langle x_k, x_n \rangle \right|^2 \le B^2 \sum_{k=M+1}^{N} |c_k|^2.$$

This, together with the result of the preceding computation, shows us that

$$\left\| \sum_{k=1}^{N} c_k x_k - \sum_{k=1}^{M} c_k x_k \right\|^4 = \left\| \sum_{k=M+1}^{N} c_k x_k \right\|^4 \le B^2 \left( \sum_{k=M+1}^{N} |c_k|^2 \right)^2.$$

This implies that the series $\sum_{k=1}^{\infty} c_k x_k$ converges. By repeating the argument, we conclude

$$\left\| \sum_{k=1}^{\infty} c_k x_k \right\| \le \sqrt{B} \left( \sum_{k=1}^{\infty} |c_k|^2 \right)^{\frac{1}{2}}.$$

This proves that $T : (c_k)_k \mapsto \sum_{k=1}^{\infty} c_k x_k$ is a well-defined bounded linear operator from $\ell^2$ into $H$ such that $\|T\| \le \sqrt{B}$. Since

$$\langle T^* x, e_n \rangle = \langle x, T e_n \rangle = \langle x, x_n \rangle, \ \ \forall n \in \mathbb{N},$$

it follows that $T^*$ coincides with the analysis operator of the sequence $(x_n)$. Since $\|T^*\| \le \sqrt{B}$, it follows that $(x_n)_n$ is a Bessel sequence with a Bessel bound $B$. $\square$

**Lemma 1.3.8.** *(Schur) Let $(\alpha_{ij})$ be an infinite matrix. Suppose that there exist a sequence $(p_i)_i$ of positive numbers and constants $r, s > 0$ such that*

$$\sum_{j=1}^{\infty} |\alpha_{ij}| p_j \le r p_i, \ \forall i, \quad \text{and} \quad \sum_{i=1}^{\infty} |\alpha_{ij}| p_i \le s p_j, \ \forall j. \tag{21}$$

*Let $(e_n)_n$ be an orthonormal basis for a Hilbert space $H$. Then there exists a bounded operator $A$ on $H$ such that $\langle A e_j, e_i \rangle = \alpha_{ij}$ for all $i$ and $j$ and $\|A\|^2 \le rs$.*

**Proof.** For $x = \sum_{j=1}^{\infty} c_j e_j = (c_1, c_2, c_3, \ldots)$ put

$$Ax = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \alpha_{ij} c_j \right) e_i. \tag{22}$$

Observe that, in particular, we have

$$A e_n = \sum_{i=1}^{\infty} \alpha_{in} e_i, \ \ \forall n.$$

We must show that (22) gives us a well-defined operator $A$. More precisely, we must show that $Ax \in \ell^2$ and $\|Ax\|^2 \le rs \|x\|^2$ for every $x$. It suffices to obtain these conclusions for all $x \in \operatorname{span}\{e_n : n \in \mathbb{N}\}$.

Choose any $x \in \text{span}\,\{e_n : n \in \mathbb{N}\}$, $x = \sum_{j=1}^{N} c_j e_j$, where $N$ is a natural number depending on $x$. Now we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \left| \sum_{j=1}^{N} \alpha_{ij} c_j \right|^2 &= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{N} \left(\sqrt{\alpha_{ij}}\sqrt{p_j}\right) \left(\frac{\sqrt{\alpha_{ij}}c_j}{\sqrt{p_j}}\right) \right|^2 \text{ (by the Cauchy-Schwarz inequality in } \mathbb{F}^N\text{)} \\
&\leq \sum_{i=1}^{\infty} \left( \sum_{j=1}^{N} |\alpha_{ij}| p_j \right) \left( \sum_{j=1}^{N} \frac{|\alpha_{ij}| \cdot |c_j|^2}{p_j} \right) \\
&\leq \sum_{i=1}^{\infty} r p_i \sum_{j=1}^{N} \frac{|\alpha_{ij}| \cdot |c_j|^2}{p_j} \\
&= r \sum_{j=1}^{N} \frac{|c_j|^2}{p_j} \sum_{i=1}^{\infty} |\alpha_{ij}| p_i \\
&\leq r \sum_{j=1}^{N} \frac{|c_j|^2}{p_j} s p_j = rs\|x\|^2
\end{aligned}$$

This shows us that the sequence $\left(\sum_{j=1}^{N} \alpha_{ij} c_j\right)_i$ belongs to $\ell^2$. Hence, if $x = \sum_{j=1}^{N} c_j e_j$, $Ax$ is well-defined by (22). In the same time we have obtained the desired estimate for the norm of $A$. $\square$

**Corollary 1.3.9.** *Let $(\alpha_{ij})$ be a symmetric infinite matrix. Suppose that there exists a constant $B$ such that*

$$\sum_{j=1}^{\infty} |\alpha_{ij}| \leq B, \ \forall i. \tag{23}$$

*Let $(e_n)_n$ be an orthonormal basis for a Hilbert space $H$. Then there exists a bounded operator $A$ on $H$ such that $\langle Ae_j, e_i \rangle = \alpha_{ij}$ for all $i$ and $j$ and $\|A\| \leq B$.*

**Proof.** We only need to observe that the matrix $(\alpha_{ij})$ satisfies the conditions of the preceding lemma with $r = s = B$ and $p_i = 1$ for all $i$. $\square$

**Corollary 1.3.10.** *Let $(x_n)_n$ be a sequence in a Hilbert space $H$ such that there exists a constant $B$ with the property*

$$\sum_{j=1}^{\infty} |\langle x_i, x_j \rangle| \leq B, \ \forall i.$$

*Then $(x_n)_n$ is a Bessel sequence with a Bessel bound $B$.*

**Proof.** Immediate from Theorem 1.3.7 and Corollary 1.3.9. $\square$

We end the section with another result that characterizes Riesz bases. Observe that, by Corollary 1.2.30, each Riesz basis for a Hilbert space $H$ is a Bessel sequence. The converse is not true. Example: choose any ONB $(e_n)_n$ for $H$ and put $x_n = \frac{1}{n} e_n$, $n \in \mathbb{N}$. Notice that a Bessel sequence even need not be fundamental in $H$.

**Theorem 1.3.11.** *Let $(x_n)_n$ be a sequence in a Hilbert space $H$. The following statements are equivalent:*

*(a) $(x_n)_n$ is a Riesz basis for $H$.*

*(b) $(x_n)_n$ is an unconditional bounded basis for $H$.*

*(c) $(x_n)_n$ is a fundamental Bessel sequence and possesses a biorthogonal system $(y_n)_n$ that is also a fundamental Bessel sequence.*

**Proof.** $(a) \Rightarrow (b)$. This is Proposition 1.2.26 (a).

$(b) \Rightarrow (c)$. Assume (b). Denote the associated biorthogonal sequence by $(y_n)_n$. Then, by Corollary 1.2.21, $((y_n)_n, (x_n)_n)$ is also an unconditional bounded basis for $H$. Therefore, if $x \in H$, then $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle y_n$ and this series converges unconditionally. By Orlicz's theorem, we now have $\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \|y_n\|^2 < \infty$. Further, since $(y_n)_n$ is a bounded basis, there exist constants $C_1, C_2 > 0$ such that $0 < C_1 \leq \|y_n\| \leq C_2 < \infty$ for all $n$. Thus,

$$\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 C_1^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \|y_n\|^2 < \infty$$

whence

$$\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 < \infty, \;\; \forall x \in H,$$

which means that $(x_n)_n$ is Bessel. Clearly, since it is a basis, it must be fundamental. The same conclusions for $(y_n)_n$ follow by symmetry.

$(c) \Rightarrow (a)$. We shall prove that (c) implies (b) from Theorem 1.2.29 (and then apply the implication $(b) \Rightarrow (a)$ from Theorem 1.2.29).

Suppose (c). Since $(x_n)_n$ and $(y_n)_n$ are both Bessel sequences, Lema 1.3.2 tells us that there exist constants $C$ and $D$ such that

$$\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq C\|x\|^2 \;\text{ and }\; \sum_{n=1}^{\infty} |\langle x, y_n \rangle|^2 \leq D\|x\|^2, \;\; \forall x \in H.$$

Since by assumption $(x_n)_n$ is fundamental and $(y_n)_n$ is biorthogonal to $(x_n)_n$, to show that $((x_n)_n, (y_n)_n)$ is a basis for $H$, it suffices by Theorem 1.2.17 (d), to show that $\sup_N \|S_N\| < \infty$ where $S_N(x) = \sum_{n=1}^{N} \langle x, y_n \rangle x_n$, $N \in \mathbb{N}$, $x \in H$.

$$\begin{aligned}
\|S_N(x)\|^2 &= \sup_{\|y\|=1} |\langle S_N(x), y \rangle|^2 \\
&= \sup_{\|y\|=1} \left| \sum_{n=1}^{N} \langle x, y_n \rangle \langle x_n, y \rangle \right|^2 \;\; \text{(by the Cauchy-Schwarz inequality in } \mathbb{F}^N\text{)} \\
&\leq \sup_{\|y\|=1} \left( \sum_{n=1}^{N} |\langle x, y_n \rangle|^2 \right) \left( \sum_{n=1}^{N} |\langle x_n, y \rangle|^2 \right) \\
&\leq \sup_{\|y\|=1} D\|x\|^2 C\|y\|^2 = CD\|x\|^2.
\end{aligned}$$

This proves that $((x_n)_n, (y_n)_n)$ is a basis for $H$. To finish the proof, it remains to show: if $(c_n)_n$ is a sequence of scalars, then $\sum_{n=1}^{\infty} c_n x_n$ converges if and only if $(c_n)_n$ belongs to $\ell^2$. Suppose first that $\sum_{n=1}^{\infty} c_n x_n$ converges to $x$. Then we must have $c_n = \langle x, y_n \rangle$, $\forall n \in \mathbb{N}$, since $((x_n)_n, (y_n)_n)$ is a basis for $H$. This implies

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |\langle x, y_n \rangle|^2 \leq D\|x\|^2;$$

hence, $(c_n)_n \in \ell^2$. Conversely, if $(c_n)_n \in \ell^2$ then $\sum_{n=1}^{\infty} c_n x_n$ converges by Proposition 1.3.4. $\square$

*Concluding remarks.* (a) This section contains the standard facts concerning (infinite) Bessel sequences. (Observe that each finite sequence is obviously Bessel.)

(b) Both the results and the proofs are combinations of those from [51] and [81] (with the exception of Lemma 1.3.8 which is borrowed from [76]).

**Exercise 1.3.12.** Suppose that $(x_n)_n$ is a sequence in a Hilbert space $H$ for which there exists a constant $B$ such that

$$\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B\|x\|^2$$

for all $x$ from a dense subset $S$ of $H$. Show that $(x_n)_n$ is a Bessel sequence with a Bessel bound $B$.

**Exercise 1.3.13.** Consider the infinite matrix $(\alpha_{ij})_{i,j=0}^{\infty}$ where $\alpha_{ij} = \frac{1}{i+j+1}$ for all integers $i, j \geq 0$. Show that this matrix defines a bounded operator $A$ on $\ell^2$ such that $\|A\| \leq \pi$ *Hint:* apply Lemma 1.3.8 with $p_i = \frac{1}{\sqrt{i+\frac{1}{2}}}$, $i = 0, 1, 2, \ldots$, and $r = s = \pi$.

**Exercise 1.3.14.** Show that the sequence of monomials $(x^n)_{n=0}^{\infty}$ is Bessel in the Hilbert space $L^2([0, 1])$.

# 2 General theory of frames

## 2.1 Fundamental properties of frames

Bases, in particular Riesz and orthonormal bases, exist in all separable Hilbert spaces. However, the conditions for being a basis are so strong that it is often impossible to construct a basis with special (prediscribed) properties. Also, even a slight modification of a basis might destroy the basis property.

On the other hand, there are reproducing systems for Hilbert spaces, more general than bases, which show much more flexibility. In fact, such systems exist in abundance and appear quite naturally. To demonstrate an easy example, consider an ONB $(e_n)_n$ for a Hilbert space $H$. Then we have $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ for all $x$ from $H$. Let us now take a closed subspace $M$ of $H$ and the orthogonal projection $P$ to $M$. Then we have $Px = x$ for each $x$ in $M$, so the preceding equality, when applied to elements from $M$ becomes

$$x = Px = \sum_{n=1}^{\infty}\langle Px, e_n\rangle Pe_n = \sum_{n=1}^{\infty}\langle x, Pe_n\rangle Pe_n, \ \ \forall x \in M.$$

This shows us that the sequence $(Pe_n)_n$ serves as a reconstructing system for the Hilbert space $M$; moreover, the above formula is completely analogous to the Fourier expansion in an ONB, although $Pe_n$'s need not be independent in any sense.

We shall see soon that the sequence $(Pe_n)_n$ is in fact a typical example of a (Parseval) frame.

**Definition 2.1.1.** *A sequence $(x_n)_n$ in a Hilbert space $H$ is a* frame *for $H$ if there exist positive constants $A$ and $B$, that are called frame bounds, such that*

$$A\|x\|^2 \leq \sum_{n=1}^{\infty}|\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad \forall x \in H. \tag{1}$$

*A frame is said to be tight if $A = B$. In particular, if $A = B = 1$ so that*

$$\sum_{n=1}^{\infty}|\langle x, x_n\rangle|^2 = \|x\|^2, \quad \forall x \in H, \tag{2}$$

*we say that $(x_n)_n$ is a* Parseval frame*.*

*The frame is* exact *if it ceases to be a frame whenever any single element is deleted from the sequence.*

**Remark 2.1.2.** (a) The frame bounds are not unique. The maximal $A$ and the minimal $B$ are called the optimal frame bounds and will be denoted by $A_{\mathrm{opt}}$ and $B_{\mathrm{opt}}$.

(b) If $(x_n)_n$ is a frame then the series $\sum_{n=1}^{\infty}|\langle x, x_n\rangle|^2$ is an absolutely convergent series of non-negative real numbers. It therefore converges unconditionally. As a consequence, every rearrangement of a frame is also a frame, and therefore we can use any countable set to index a frame. In these general considerations we will always use the set of natural numbers as the index set.

**Example 2.1.3.** Let $(e_n)_n$ be an ONB for a Hilbert space $H$. Then the sequence

(a) $e_1, e_2, e_3, \ldots$ is a Parseval exact frame for $H$;

(b) $e_1, 0, e_2, 0, e_3, \ldots$ ia a Parseval non-exact frame for $H$;

(c) $e_1, e_1, e_2, e_2, \ldots$ is a tight (with $A = B = 2$) non-exact frame for H;

(d) $2e_1, e_2, e_3, e_4, \ldots$ is an exact frame ($A = 1$, $B = 2$) for $H$;

(e) $e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots$ is a Parseval non-exact frame for $H$;

(f) $e_1, \frac{1}{2}e_2, \frac{1}{3}e_3, \ldots$ is orthogonal and fundamental, but not a frame for $H$.

**Remark 2.1.4.** (a) Each frame $(x_n)_n$ for a Hilbert space $H$ is fundamental in $H$. To see this, it suffices to show that $(x_n)_n$ is maximal, and this is immediate from the first inequality in (1).

The converse is not true as it is demonstrated by the last sequence in the preceding example.

(b) For this reason here and in the sequel we restrict ourselves to separable Hilbert spaces.

(c) We also conclude from (a) that there are no finite frames for infinite-dimensional Hilbert spaces. Finite frames do exist (see the following remark) when $\dim H < \infty$.

**Remark 2.1.5.** A finite sequence $(x_n)_{n=1}^M$ is a frame for a finite-dimensional Hilbert space $H$ if and only if $\{x_n : 1 \leq n \leq M\}$ is a spanning set for $H$.

In one direction this is proved precisely as in the preceding remark: it is evident that each frame for $H$ is a maximal system in $H$.

To prove the converse, suppose that a finite sequence $(x_n)_{n=1}^M$ generates $H$ and define the operator

$$U : H \to \mathbb{F}^M, \;\; Ux = (\langle x, x_1\rangle, \langle x, x_2\rangle, \ldots, \langle x, x_M\rangle).$$

Obviously, $U$ is linear and injective. Therefore, the operator $U_0 : H \to \mathrm{R}(U)$ defined by $U_0 x = Ux$ is invertible and its inverse $V : \mathrm{R}(U) \to H$ is bounded. Choose a constant $C > 0$ such that

$$\|V(Ux)\|^2 \leq C\|Ux\|^2, \;\; \forall x \in H.$$

If we put $A = \frac{1}{C}$ this can be rewritten (notice also that $VUx = x$) as

$$A\|x\|^2 \leq \sum_{n=1}^{M} |\langle x, x_n\rangle|^2, \;\; \forall x \in H.$$

On the other hand, we have for all $x$ in $H$

$$\sum_{n=1}^{M} |\langle x, x_n\rangle|^2 \leq \sum_{n=1}^{M} \|x\|^2\|x_n\|^2 \leq B\|x\|^2$$

where $B = \sum_{n=1}^{M} \|x_n\|^2$.

It is clear from the definition of a frame that each frame is a Bessel sequence. Thus, if $(x_n)_n$ is a frame for $H$, its analysis operator $U : H \to \ell^2$ is well defined and bounded. Here we make the following convention: by writing $(x_n)_n$ we admit the possibility that $(x_n)_n$ is a finite sequence (consisting of, say, $M$ elements) and when this is the case, we understand that the analysis operator $U$ takes values in $\mathbb{F}^M$.

In particular, we know from Proposition 1.3.4, that the synthesis operator $U^*$ is given by $U^*(c_n)_n = \sum_{n=1}^{\infty} c_n x_n$ where this series converges unconditionally for each $(c_n)_n \in \ell^2$.

In addition to these properties, it is clear from Definition 2.1.1 that the analysis operator $U$ of each frame is also bounded from below. To proceed, we need a general result on bounded Hilbert space operators. First we prove the following lemma.

**Lemma 2.1.6.** *Let $H$ and $K$ be Hilbert spaces. If $T \in \mathbb{B}(H, K)$ is a surjection, $T^*$ is bounded from below.*

**Proof.** Suppose that $T$ is a surjection and consider the set $S = \{y \in K : \|T^*y\| = 1\}$. Notice that $S$ is weakly bounded. Indeed, for any $z$ in $K$ we first find $x$ in $H$ such that $Tx = z$ and then we have, for every $y$ in $S$,

$$|\langle y, z\rangle| = |\langle y, Tx\rangle| = |\langle T^*y, x\rangle| \leq \|T^*y\| \cdot \|x\| = \|x\|.$$

By the uniform boundedness principle we conclude that $S$ is bounded. Thus, there exists a constant $C > 0$ such that $\|y\| \leq C$ for all $y$ from $S$.

Now observe that the equality $K = \overline{\mathrm{R}(T)} \oplus \mathrm{N}(T^*)$ and surjectivity of $T$ imply that $T^*$ is an injection. Hence $T^*v \neq 0$ for all $v \neq 0$. Therefore, if $v \neq 0$ then $\frac{v}{\|T^*v\|}$ is a well defined vector in $S$. By the conclusion of the first part of the proof, this implies $\left\|\frac{v}{\|T^*v\|}\right\| \leq C$, i.e. $\|T^*v\| \geq \frac{1}{C}\|v\|$. □

**Proposition 2.1.7.** *Let $H$ and $K$ be Hilbert spaces and $T \in \mathbb{B}(H, K)$.*

*(a) $R(T)$ is closed if and only if $R(T^*)$ is closed.*

*(b) $T$ is a surjection if and only if $T^*$ is bounded from below.*

*(c) If $R(T)$ is closed, $TT^*$ is invertible on $R(T)$.*

**Proof.** Suppose that $\mathrm{R}(T)$ is closed. If, additionally, $T$ is a surjection, then by the preceding lemma $T^*$ is bounded from below and this immediately implies that $\mathrm{R}(T^*)$ is closed. If $T$ is not a surjection denote $\mathrm{R}(T)$ by $K_0$ and consider $T_0 : H \to K_0$, $T_0x = Tx$. By the preceding conclusion we now know that the operator $(T_0)^*$ has the closed range. It only remains to observe that $\mathrm{R}((T_0)^*) = \mathrm{R}(T^*)$. Indeed: the equality $T^*_{|K_0} = (T_0)^*$ gives us $\mathrm{R}((T_0)^*) \subseteq \mathrm{R}(T^*)$, while the reverse inclusion follows from $\mathrm{R}(T^*) \subseteq (\mathrm{N}(T))^\perp = (\mathrm{N}(T_0))^\perp = \mathrm{R}((T_0)^*)$.

Thus, we have proved: $\mathrm{R}(T^*)$ is closed whenever $\mathrm{R}(T)$ is closed. The converse follows by applying this conclusion to the operator $T^*$. This finishes the proof of (a).

Let us prove (b). In one direction, this is the statement of the preceding lemma. To prove the converse, suppose that $T^*$ is bounded from below. Then $N(T^*) = \{0\}$ and $\mathrm{R}(T^*)$ is a closed subspace. By the first part of the lemma, we know that $\mathrm{R}(T)$ closed is well. Hence $\mathrm{R}(T) = N(T^*)^\perp = K$; i.e. $T$ is a surjection.

Finally, to prove (c), suppose that $\mathrm{R}(T)$ is a closed subspace of $K$. Let $TT^*y = 0$ for some $y \in \mathrm{R}(T)$. Observe that $\mathrm{N}(TT^*) = \mathrm{N}(T^*)$ (this is true for all operators). So, we have $y \in \mathrm{N}(T^*) \cap \mathrm{R}(T)$ and this obviously implies $y = 0$. Thus, $TT^*$ is injective on $\mathrm{R}(T)$. On the other hand, by (a), the range of $T^*$ is also closed; hence, $H = \mathrm{N}(T) \oplus \mathrm{R}(T^*)$. This implies that each $x$ in $H$ has the form $x = y + T^*v$ for some $y \in \mathrm{N}(T)$ and $v \in K$. By applying $T$ we obtain $Tx = TT^*v$ which shows that $\mathrm{R}(T) \subseteq \mathrm{R}(TT^*)$. Since the opposite inclusion is obvious, we conclude that $\mathrm{R}(T) = \mathrm{R}(TT^*)$. Finally, we then have

$$\mathrm{R}(T) = \mathrm{R}(TT^*) = TT^*(K) = TT^*(\mathrm{N}(T^*) \oplus \mathrm{R}(T)) = TT^*(\mathrm{R}(T)),$$

which shows us that $TT^*$ is a surjection on $\mathrm{R}(T)$. □

**Theorem 2.1.8.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$. Then its analysis operator $U$ is bounded and bounded from below and the synthesis operator $U^*$ is a surjection. Conversely, if $T \in \mathbb{B}(\ell^2, H)$ is a surjection, then the sequence $(x_n)_n$, $x_n = Te_n$, $n \in \mathbb{N}$, where $(e_n)_n$ is the canonical basis for $\ell^2$, is a frame for $H$ whose analysis operator coincides with $T^*$.*

**Proof.** Suppose that $(x_n)_n$ is a frame. We already know that $U$ is bounded and bounded from below. The preceding proposition implies that $U^*$ is surjective.

Suppose now we have a surjection $T \in \mathbb{B}(\ell^2, H)$. From the preceding proposition we conclude that $T^*$ is bounded from below. Thus, there are constants $A, B > 0$ such that

$$A\|x\|^2 \leq \|T^*x\|^2 \leq B\|x\|^2, \ \ \forall x \in H.$$

On the other hand, we have for each $x$ in $H$

$$T^*x = \sum_{n=1}^{\infty} \langle T^*x, e_n\rangle e_n = \sum_{n=1}^{\infty} \langle x, Te_n\rangle e_n = \sum_{n=1}^{\infty} \langle x, x_n\rangle e_n.$$

whence

$$T^*x = (\langle x, x_n\rangle)_n \ \text{ and } \ \|T^*x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2.$$

□

**Corollary 2.1.9.** *A sequence $(x_n)_n$ in a Hilbert space $H$ is a frame for $H$ if and only if there exist a separable Hilbert space $L$, a surjective operator $T \in \mathbb{B}(L, H)$ and an ONB $(f_n)_n$ for $L$ such that $x_n = Tf_n$ for all $n$ in $\mathbb{N}$.*

**Proof.** In one direction this is the first statement of Theorem 2.1.8 (and Proposition 1.3.4).

The converse follows from the second statement of Theorem 2.1.8 and the following well known fact: if $(f_n)_n$ is an ONB for a separable Hilbert space $L$, then there exists a unitary operator $V : \ell^2 \to L$ such that $Ve_n = f_n$ for all $n$ in $N$, where $(e_n)_n$ is the canonical basis for $\ell^2$. □

**Remark 2.1.10.** The analysis operator of a Parseval frame is an isometry. Consequently, the corresponding synthesis operator is a co-isometry (a surjective partial isometry). It is easy to conclude from the preceding results and their proofs that Parseval frames are those sequences that are co-isometrical images of orthonormal bases.

Suppose now we have a frame $(x_n)_n$ for a Hilbert space $H$ with the analysis operator $U$. By Proposition 2.1.7 applied to $U^*$, the composition $U^*U$ (that is often called the frame operator) is an invertible operator on $\mathrm{R}(U^*) = H$. Using Proposition 1.3.4 we obtain

$$U^*Ux = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n, \ \ \forall x \in H. \tag{3}$$

Taking the inner product by $x$ we get

$$\langle U^*Ux, x \rangle = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2, \ \ \forall x \in H. \tag{4}$$

From this we conclude that

$$A_{\text{opt}} I \leq U^*U \leq B_{\text{opt}} I \tag{5}$$

which immediately implies

$$B_{\text{opt}} = \|U^*U\| = \|U\|^2.$$

On the other hand, we have

$$A_{\text{opt}} I \leq U^*U \iff (U^*U)^{-1} \leq \frac{1}{A_{\text{opt}}} I. \tag{6}$$

We now claim that

$$\frac{1}{A_{\text{opt}}} = \|(U^*U)^{-1}\|.$$

Indeed, (6) shows us that $\|(U^*U)^{-1}\| \leq \frac{1}{A_{\text{opt}}}$. To prove the opposite inequality, suppose that $\|(U^*U)^{-1}\| = C < \frac{1}{A_{\text{opt}}}$. This implies $(U^*U)^{-1} \leq C \cdot I$ and hence $\frac{1}{C} I \leq U^*U$. But this contradicts to the fact that $A_{\text{opt}}$ is the maximal lower frame bound since $\frac{1}{C} > A_{\text{opt}}$.

In this way we have proved

**Proposition 2.1.11.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. Then the frame operator $U^*U$ is invertible and the optimal frame bounds are given by*

$$A_{opt} = \frac{1}{\|(U^*U)^{-1}\|} = \min\{\lambda : \lambda \in \sigma(U^*U)\}, \ \ B_{opt} = \|U^*U\| = \max\{\lambda : \lambda \in \sigma(U^*U)\}. \tag{7}$$

As a direct consequence of the preceding results we also get the following useful fact.

**Corollary 2.1.12.** *Let $H$ and $K$ be Hilbert spaces. Suppose that $(x_n)_n$ is a frame for $H$ and that $T \in \mathbb{B}(H, K)$ is a surjection. Let $y_n = Tx_n$, $n \in \mathbb{N}$. Then $(y_n)_n$ is a frame for $K$. If $A$ and $B$ are frame bounds for $(x_n)_n$ then the frame bounds for $(y_n)_n$ are $\frac{A}{\|(TT^*)^{-1}\|}$ and $B\|T\|^2$.*

**Proof.** The first statement is immediate from Corollary 2.1.9.

To prove the other one, denote by $U$ the analysis operator of $(x_n)_n$. Then it is easy to see that the analysis operator $V$ of $(y_n)_n$ is given by $V = UT^*$. Since $A \le A_{\text{opt}}$ and $B_{\text{opt}} \le B$, we conclude from (5) that

$$AI \le U^*U \le BI. \tag{8}$$

Since $T$ is surjective, $TT^*$ is invertible by the third statement of Proposition 2.1.7. Hence,

$$(TT^*)^{-1} \le \|(TT^*)^{-1}\| \cdot I$$

which we rewrite as

$$TT^* \ge \frac{1}{\|(TT^*)^{-1}\|} I. \tag{9}$$

We now have

$$V^*V = (TU^*)(UT^*) = T(U^*U)T^* \overset{(8)}{\le} B \cdot TT^* \le B\|TT^*\|I = B\|T\|^2 I$$

and

$$V^*V = (TU^*)(UT^*) = T(U^*U)T^* \overset{(8)}{\ge} A \cdot TT^* \overset{(9)}{\ge} \frac{A}{\|(TT^*)^{-1}\|} I.$$

Thus, we have obtained

$$\frac{A}{\|(TT^*)^{-1}\|} I \le V^*V \le B\|T\|^2 I$$

which is precisely what we needed to prove. $\square$

Consider again an arbitrary frame $(x_n)_n$ for $H$ with the analysis operator $U$. Let us turn back to equation (3) that describes the action of the frame operator $U^*U$:

$$U^*Ux = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n, \quad \forall x \in H.$$

After applying $(U^*U)^{-1}$ this gives us

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle (U^*U)^{-1} x_n, \quad \forall x \in H.$$

Put $y_n = (U^*U)^{-1} x_n$ for all $n \in \mathbb{N}$. Then the preceding equality reads

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle y_n, \quad \forall x \in H. \tag{10}$$

Moreover, Corollary 2.1.12 tells us that $(y_n)_n$ is also a frame for $H$. If we denote its analysis operator by $V$, (10) can be rewritten in the form

$$V^*U = I. \tag{11}$$

By taking adjoints, we get

$$U^*V = I \tag{12}$$

which is the same as

$$x = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n, \ \ \forall x \in H. \tag{13}$$

In fact, it is evident that (10) and (13) are equivalent.

**Definition 2.1.13.** *Let $(x_n)_n$ be a frame with the analysis operator $U$. The frame $(y_n)_n$ defined by $y_n = (U^*U)^{-1}x_n$, $n \in \mathbb{N}$, is called the canonical dual of $(x_n)_n$.*

Observe that the analysis operator of the canonical dual is $U(U^*U)^{-1}$.

Notice also that a Parseval frame coincides with its canonical dual; moreover, only Parseval frames have this property. This is simply because an operator $U$ is an isometry if and only if $U^*U = I$.

In the following theorem we summarize the preceding conclusions:

**Theorem 2.1.14.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. Then $U^*U$ is an invertible operator on $H$ and the sequence $(y_n)_n$ defined by $y_n = (U^*U)^{-1}x_n$, $n \in \mathbb{N}$, is also a frame for $H$ which satisfies*

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle y_n = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n, \ \ \forall x \in H. \tag{14}$$

*In particular, if $(x_n)_n$ is a Parseval frame, equalities (14) reduce to*

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n, \ \ \forall x \in H. \tag{15}$$

One often refers to equalities (14) resp. (15) as to the reconstruction property of frames.

Since, in general, frames are linearly dependent systems, it is intuitively clear that the canonical dual of a given frame is not the only sequence which can be used in analyzing or synthesizing vectors (signals) in order to obtain equalities analogous to (14). We postpone a general discussion on dual frame sequences to the following section.

We end this section with a couple of observations concerning Parseval frames.

**Proposition 2.1.15.** *Let $(x_n)_n$ be a frame for $H$ with the analysis operator $U$. Put $u_n = (U^*U)^{-\frac{1}{2}}x_n$, $n \in \mathbb{N}$. Then $(u_n)_n$ is a Parseval frame for $H$.*

**Proof.** Since $(U^*U)^{-\frac{1}{2}}$ is an invertible operator (in particular, a surjection), Corollary 2.1.12 (see also the proof) tells us that $(u_n)_n$ is a frame whose analysis operator is equal to $U(U^*U)^{-\frac{1}{2}}$. Since $(U(U^*U)^{-\frac{1}{2}})^*U(U^*U)^{-\frac{1}{2}} = I$, this frame is Parseval. $\square$

**Proposition 2.1.16.** *Let $(x_n)_n$ be a sequence in a Hilbert space $H$. Then $(x_n)_n$ is a Parseval frame for $H$ if and only if*

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n, \quad \forall x \in H. \tag{16}$$

*In particular, if $(f_n)_n$ is an ONB for a Hilbert space $H$ and if $M$ is a closed subspace of $H$, then the sequence $(Pf_n)_n$ is a Parseval frame for $M$, where $P$ denotes the orthogonal projection to $M$.*

**Proof.** In one direction this is already observed in Theorem 2.1.14.

Conversely, assume (16). Taking the inner product on both sides by $x$, we obtain $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$.

To prove the second statement observe that the equality $x = \sum_{n=1}^{\infty} \langle x, f_n \rangle f_n$ can be rewritten for $x \in M$ in the form

$$x = Px = \sum_{n=1}^{\infty} \langle Px, f_n \rangle Pf_n = \sum_{n=1}^{\infty} \langle x, Pf_n \rangle Pf_n.$$

$\square$

The preceding proposition shows that a sequence constructed in the example from the beginning of this section is a Parseval frame. In fact, each Parseval frame arises in that way.

**Proposition 2.1.17.** *Let $(x_n)_n$ be a Parseval frame for a Hilbert space $H$. Then there exist a Hilbert space $H_0$ which contains $H$ as a closed subspace and an ONB $(f_n)_n$ for $H_0$ such that $x_n = Pf_n$ for all $n$, where $P \in \mathbb{B}(H_0)$ is the orthogonal projection to $H$.*

**Proof.** Denote by $U$ the analysis operator of $(x_n)_n$. We know that $U$ is an isometry and that $M = \mathrm{R}(U)$ is a closed subspace of $\ell^2$. Let $Q \in \mathbb{B}(\ell^2)$ be the orthogonal projection to $M$. Let $H_0 = H \oplus M^{\perp}$. Obviously, we can identify $H$ with $H \oplus \{0\} \leq H_0$. Denote by $P \in \mathbb{B}(H_0)$ the orthogonal projection to $H \oplus \{0\}$.

Consider the sequence $(f_n)_n$ in $H_0$ defined by $f_n = (x_n, (I - Q)e_n)$, $n \in \mathbb{N}$, where $(e_n)_n$ is the canonical basis for $\ell^2$. Obviously, we have $Pf_n = (x_n, 0)$ for all $n$. We now claim that $(f_n)_n$ is an ONB for $H_0$. To prove our claim we will construct a unitary operator $W : \ell^2 \to H_0$ such that $We_n = f_n$ for all $n$.

To do that, first observe that $U^*|_M : M \to H$ is a unitary operator. Also, since $\ell^2 = M \oplus \mathrm{N}(U^*)$, we have $x_n = U^* e_n = U^* Q e_n = (U^*|_M) Q e_n$ for all $n$.

Let $W = U^*|_M \oplus I_{M^{\perp}} : \ell^2 \to H_0$, where $I_{M^{\perp}}$ is the identity operator on $M^{\perp}$. Then, clearly, $W$ is a unitary operator and

$$We_n = (U^*|_M \oplus I_{M^{\perp}}) (Qe_n + (I - Q)e_n) = (x_n, (I - Q)e_n) = f_n, \quad \forall n \in \mathbb{N}.$$

$\square$

**Example 2.1.18.** Take any $b > 0$ and consider the sequence $(e^{2\pi ibnt})_{n\in\mathbb{Z}}$ where each of the functions $e^{2\pi ibnt}$ is regarded as a function defined on the interval $[0, 1)$ and then extended 1-peridically to $\mathbb{R}$. In this way we assume that our system $(e^{2\pi ibnt})_{n\in\mathbb{Z}}$ belongs to $L^2(\mathbb{T})$ (where $\mathbb{T}$ denotes the torus) which we identify with $L^2([0, 1])$.

If $b = 1$ we already know that $(e^{2\pi ibnt})_{n\in\mathbb{Z}}$ is an ONB for $L^2([0, 1])$.

If $b > 1$, our functions $e^{2\pi ibnt}$, considered on all of $\mathbb{R}$, are $\frac{1}{b}$-periodic. Observe that the interval $[0, \frac{1}{b})$ is strictly contained in $[0, 1)$. For those $t$ such that $t$ and $t+\frac{1}{b}$ belong to $[0, 1)$ we have $f(t) = f(t + \frac{1}{b})$ for all functions $f$ from $\text{span}\left\{e^{2\pi ibnt} : n \in \mathbb{Z}\right\}$. It is now easy to conclude that $\overline{\text{span}}\left\{e^{2\pi ibnt} : n \in \mathbb{Z}\right\}$ is a proper subspace of $L^2([0, 1])$. This means that the sequence $(e^{2\pi ibnt})_{n\in\mathbb{Z}}$ is not fundamental in $L^2([0, 1])$ and therefore cannot be a frame for $L^2([0, 1])$.

Finally, consider the case $b < 1$. For example, if $b = \frac{1}{2}$ we see that

$$\begin{aligned} (e^{2\pi int\frac{1}{2}})_{n\in\mathbb{Z}} &= (e^{2\pi int})_{n\in\mathbb{Z}} \cup (e^{2\pi i(n+\frac{1}{2})t})_{n\in\mathbb{Z}} \\ &= (e^{2\pi int})_{n\in\mathbb{Z}} \cup (e^{\pi it}e^{2\pi int})_{n\in\mathbb{Z}} \end{aligned}$$

which is the union of two ONB's and hence a tight frame with the frame bound equal to 2.

Similarly, for $b = \frac{1}{M}$, $M \in \mathbb{N}$, it turns out that $(e^{2\pi ibnt})_{n\in\mathbb{Z}}$ is the union of $M$ ONB's; thus a tight frame fith the frame bound $M$.

In general, we can argue in the following way. Again, our functions $e^{2\pi ibnt}$, considered on all of $\mathbb{R}$, are $\frac{1}{b}$-periodic, but now we have $[0, 1) \subset [0, \frac{1}{b})$. Since the operator $D : L^2([0, 1]) \to L^2([0, \frac{1}{b}])$ defined by $Df(t) = \sqrt{b}f(bt)$ is unitary, the sequence $(\sqrt{b}e^{2\pi ibnt})_{n\in\mathbb{Z}}$ is an ONB for $L^2([0, \frac{1}{b}])$.

We can understand $L^2([0, 1])$ as a closed subspace of $L^2([0, \frac{1}{b}])$ by extending each function $f \in L^2([0, 1])$ by zero on $(1, \frac{1}{b}]$. Clearly, the operator $P$ defined on $L^2([0, \frac{1}{b}])$ by $Pf = f\chi_{[0,1]}$ is the orthogonal projection to $L^2([0, 1])$. Thus, by Proposition 2.1.16, the sequence $(P\sqrt{b}e^{2\pi ibnt})_{n\in\mathbb{Z}}$ is a Parseval frame for $L^2([0, 1])$. In other words, $(e^{2\pi ibnt})_{n\in\mathbb{Z}}$ is a tight frame for $L^2([0, 1])$ with the frame bound $\frac{1}{b}$.

*Concluding remarks.* Frames first appeared in the literature in 1952 in a paper of R. J. Duffin and A. C. Schaeffer ([64]). In 1980's frames begun to play an important role in Gabor analysis and wavelet theory (see [59]). Since then the theory has grown rapidly. The standard references include [51], [82], [90]. The material in this section is well known. Some of the results appeared already in the pioneering work [64]. The concluding Example 2.1.18 is a combination of Example 8.7 and Remark 8.8 from [81].

**Exercise 2.1.19.** Let $H$ and $K$ be Hilbert spaces. Suppose that for $A \in \mathbb{B}(H, K)$ there exists a constant $m$ such that $\|Ax\| \geq m\|x\|$ for all $x$ in $H$. Show that each operator in the open ball $K(A, m) \subset \mathbb{B}(H, K)$ is bounded from below.

**Exercise 2.1.20.** Let $H$ and $K$ be Hilbert spaces. Show that the set of all surjective operators in $\mathbb{B}(H, K)$ is open (in the norm-topology).

**Exercise 2.1.21.** Let $(e_n)_n$ be an ONB for a Hilbert space $H$. Consider the sequence $(x_n)_n$ where $x_n = e_n + e_{n+1}$, $n \in \mathbb{N}$. Show that $(x_n)_n$ is minimal and fundamental and find its biorthogonal sequence (see Proposition 1.2.16). Further, show that $(x_n)_n$ is a Bessel sequence, but not a frame.

**Exercise 2.1.22.** Let $(e_n)_n$ be an ONB for a Hilbert space $H$. Consider the sequence $(x_n)_n$, $x_n = Re_n$, $n \in \mathbb{N}$, where $R \in \mathbb{B}(H)$ is some non-surjective operator. Show that (a) $(x_n)_n$ is a Bessel sequence, (b) $(x_n)_n$ is fundamental if and only if $S$ has a dense range, (c) $(x_n)_n$ is not a frame for $H$.

**Exercise 2.1.23.** Suppose that $(x_n)_n$ is a Bessel sequence that is a basis for a Hilbert space $H$ with a Bessel bound $B$. Let $(y_n)_n$ be the biorthogonal sequence. Show that

$$\frac{1}{B}\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, y_n\rangle|^2, \ \ \forall x \in H,$$

and

$$\frac{1}{B}\sum_{n=1}^{N} |c_n|^2 \leq \left\| \sum_{n=1}^{N} c_n y_n \right\|^2, \ \ \forall c_1, \ldots, c_n \in \mathbb{F}, \, \forall N \in \mathbb{N}.$$

(Observe that $(y_n)_n$ need not be a frame since, unless $(x_n)_n$ is a Schauder basis, $(y_n)_n$ is not Bessel; see Theorem 1.3.11.)

**Exercise 2.1.24.** Let $(x_n)_n$ be a frame with the canonical dual $(y_n)_n$. Show that the canonical dual of $(y_n)_n$ coincides with $(x_n)_n$.

**Exercise 2.1.25.** Let $(x_n)_n$ be a Parseval frame for a Hilbert space $H$. Prove:

(a) $\|x_n\| \leq 1$ for all $n$.

(b) If $\|x_m\| = 1$ then $x_m \perp x_n$ for all $n \neq m$.

(c) If $\|x_m\| < 1$ then $(x_n)_{n\neq m}$ is a frame for $H$ whose optimal lower frame bound is $1-\|x_m\|^2$.

**Exercise 2.1.26.** Let $(f_n)_n$ be a sequence in a Hilbert space $H$ with the property

$$f_k = \sum_{n=1}^{\infty} \langle f_k, f_n\rangle f_n, \ \ \forall k \in \mathbb{N}.$$

Denote the sum $\sum_{n=1}^{\infty} \|f_n\|^2$ by $d$ (we allow the possibility $d = \infty$). Show that

$$d = \dim(\overline{\operatorname{span}}\,\{f_n : n \in \mathbb{N}\}).$$

**Exercise 2.1.27.** Let $\{e_1, e_2, \ldots, e_k\}$, $k \in \mathbb{N}$, be an ONB for a Hilbert space $H$. Let the sequence $(x_n)_{n=1}^{k+1}$ be defined by

$$x_n = e_n - \frac{1}{k}\sum_{i=1}^{k} e_i, \ \ n = 1, 2, \ldots, k \ \text{ and } x_{k+1} = \frac{1}{\sqrt{k}}\sum_{i=1}^{k} e_i.$$

Show that $(x_n)_{n=1}^{k+1}$ is a Parseval frame for $H$.

**Exercise 2.1.28.** Let $(e_n)_n$ be an ONB for a Hilbert space $H$. Consider the subspaces

$$H_1 = \operatorname{span}\{e_1\},\ H_2 = \operatorname{span}\{e_2, e_3\},\ H_3 = \operatorname{span}\{e_4, e_5, e_6\},\ \ldots$$

Observe that we have for each $k \in \mathbb{N}$

$$H_k = \operatorname{span}\{e_{n(k)+1}, e_{n(k)+2}, \ldots, e_{n(k)+k}\},\ \ n(k) = 1 + 2 + \ldots + (k-1) = \frac{1}{2}(k-1)k.$$

Let $(x_n^{(k)})_{n=1}^{k+1}$ be the Parseval frame for $H_k$, $k \in \mathbb{N}$, from Exercise 2.1.27. Conclude that

$$\bigcup_{k=1}^{\infty} (x_n^{(k)})_{n=1}^{k+1}$$

is a Parseval frame for $H$.

## 2.2 Dual frames

Consider an arbitrary frame $(x_n)_n$ for a Hilbert space $H$ with the analysis operator $U$. Recall that the canonical dual $(y_n)_n$ is the frame for $H$ defined by $y_n = (U^*U)^{-1}x_n$, $n \in \mathbb{N}$, that, by Theorem 2.1.14, satisfies

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle y_n = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n, \ \ \forall x \in H.$$

However, as we already suggested, since frames are (in general) linearly dependent systems, the canonical dual is not the only sequence that can be used in combination with $(x_n)_n$ to reconstruct every vector from $H$.

**Definition 2.2.1.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$. Each sequence $(z_n)_n$ in $H$ with the property*

$$x = \sum_{n=1}^{\infty} \langle x, z_n \rangle x_n, \ \ \forall x \in H$$

*is said to be a dual of $(x_n)_n$.*

**Example 2.2.2.** Take any ONB $(e_n)_n$ of a Hilbert space $H$ and consider the tight frame $e_1, e_1, e_2, e_2, \ldots$ for $H$. Here we have $U^*U = 2I$, so the canonical dual is the sequence $\frac{1}{2}e_1, \frac{1}{2}e_1, \frac{1}{2}e_2, \frac{1}{2}e_2, \ldots$. On the other hand, if we denote by $(v_n)_n$ the sequence $e_1, 0, e_2, 0, \ldots$, it is easy to verify that $x = \sum_{n=1}^{\infty} \langle x, v_n \rangle x_n$, $\forall x \in H$; so the frame $(v_n)_n$ is a dual of $(x_n)_n$. In fact, here is easy to construct infinitely many dual frames.

The situation can be even more complicated: a sequence that is dual to a frame $(x_n)_n$ need not be a frame. To see this, consider the Parseval frame $e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots$ (here again, $(e_n)_n$ is an ONB). One of its duals is the sequence $e_1, \sqrt{2}e_2, 0, \sqrt{3}e_3, 0, 0, \ldots$ which, since it is unbounded, cannot be a frame.

In the light of the last example, the following proposition is useful. Basically, it tells us that a sequence that is Bessel and dual to a frame is necessarily a frame itself. Even more is true.

**Proposition 2.2.3.** *Suppose that $(v_n)_n$ and $(w_n)_n$ are Bessel sequences in a Hilbert space $H$ such that*

$$x = \sum_{n=1}^{\infty} \langle x, v_n \rangle w_n, \ \ \forall x \in H.$$

*Then both $(v_n)_n$ and $(w_n)_n$ are frames for $H$ that are dual to each other. In particular, if a Bessel sequence is a dual to some frame, then this sequence is necessarily a frame.*

**Proof.** Denote the corresponding analysis operators by $V$ and $W$, respectively. Then our assumption can be written as $W^*V = I$. This implies that $W^*$ is a surjection, so $(w_n)_n$ is a frame by Corollary 2.1.9. Since $W^*V = I$ implies $V^*W = I$, the same argument applies to $(v_n)_n$. □

The following proposition provides us with two specific properties of the canonical dual.

**Proposition 2.2.4.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$ and let $(y_n)_n$ be its canonical dual.*

*(a) If, for some $x$ in $H$, a sequence of scalars $(c_n)_n$ satisfies $x = \sum_{n=1}^{\infty} c_n x_n$, then*

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |\langle x, y_n\rangle|^2 + \sum_{n=1}^{\infty} |\langle x, y_n\rangle - c_n|^2.$$

*(In other words, the sequence $(\langle x, y_n\rangle)_n$ has the minimal $\ell^2$-norm among all sequences that synthesize $x$ in terms of $x_n$'s.)*

*(b) If $(z_n)_n$ is a dual of $(x_n)_n$ for which there exists an operator $D \in \mathbb{B}(H)$ such that $z_n = Dx_n$ for every $n$, then $D = (U^*U)^{-1}$ and $z_n = y_n$ for all $n$. (The canonical dual is the only dual of $(x_n)_n$ that arises as an image of $(x_n)_n$ under the action of a bounded operator.)*

**Proof.** (a) We know that $(\langle x, y_n\rangle)_n \in \ell^2$. Suppose that $(c_n)_n$ is also an $\ell^2$-sequence (if not, there is nothing to prove). Denote for simplicity $\langle x, y_n\rangle$ by $a_n$, $n \in \mathbb{N}$. Then

$$\langle x, (U^*U)^{-1}x\rangle = \left\langle \sum_{n=1}^{\infty} a_n x_n, (U^*U)^{-1}x \right\rangle = \sum_{n=1}^{\infty} a_n \langle (U^*U)^{-1}x_n, x\rangle = \sum_{n=1}^{\infty} a_n \langle y_n, x\rangle = \langle (a_n)_n, (a_n)_n\rangle$$

and

$$\langle x, (U^*U)^{-1}x\rangle = \left\langle \sum_{n=1}^{\infty} c_n x_n, (U^*U)^{-1}x \right\rangle = \sum_{n=1}^{\infty} c_n \langle (U^*U)^{-1}x_n, x\rangle = \sum_{n=1}^{\infty} c_n \langle y_n, x\rangle = \langle (c_n)_n, (a_n)_n\rangle.$$

By comparing the final expressions we conclude that $((c_n)_n - (a_n)_n) \perp (a_n)_n$. Hence

$$\|(c_n)_n\|^2 = \|(c_n - a_n)_n + (a_n)_n\|^2 = \|(c_n - a_n)_n\|^2 + \|(a_n)_n\|^2.$$

(b) By assumption, we have $x = \sum_{n=1}^{\infty} \langle x, Dx_n\rangle x_n$ and $x = \sum_{n=1}^{\infty} \langle x, (U^*U)^{-1}x_n\rangle x_n$ for every $x$. If we take any $x$ in $H$ and apply the second equality to $(U^*U)D^*x$, we get

$$(U^*U)D^*x = \sum_{n=1}^{\infty} \langle (U^*U)D^*x, (U^*U)^{-1}x_n\rangle x_n = \sum_{n=1}^{\infty} \langle x, Dx_n\rangle x_n = x.$$

This shows that $(U^*U)D^* = I$ which implies $D^* = (U^*U)^{-1}$ and, by taking adjoints, $D = (U^*U)^{-1}$. $\square$

Consider now again an arbitrary frame $(x_n)_n$ for a Hilbert space $H$ and denote by $U$ the corresponding analysis operator. We now restrict our discussion to frames dual to $(x_n)_n$. A natural question arises: can we describe all frames that are dual to $(x_n)_n$?

Suppose we have a frame $(z_n)_n$ for $H$ which satisfies

$$x = \sum_{n=1}^{\infty} \langle x, z_n\rangle x_n, \quad \forall x \in H. \tag{17}$$

If we denote by $V$ the analysis operator of $(z_n)_n$, then, obviously, (17) can be rewritten as

$$U^*V = I. \tag{18}$$

Clearly, then there are two more equivalent equalities:

$$V^*U = I, \tag{19}$$

and

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle z_n, \ \ \forall x \in H. \tag{20}$$

So, in this situation we can say (and we will) that $(x_n)_n$ and $(z_n)_n$ are dual to each other.

Since frames for $H$ are in a bijective correspondence with their analysis/synthesis operators, in order to obtain all frames $(z_n)_n$ dual to $(x_n)_n$, it suffices to describe all operators $V \in \mathbb{B}(H, \ell^2)$ which satisfy (19). In other words, if $U \in \mathbb{B}(H, \ell^2)$ is the analysis operator of a frame $(x_n)_n$, we want to find all left inverses of $U$.

We start with a brief general discussion on left inverses of bounded operators.

**Lemma 2.2.5.** *Let $H$ and $K$ be Hilbert spaces and $T \in \mathbb{B}(H, K)$. Suppose that there exists $S \in \mathbb{B}(K, H)$ such that $ST = I$. Then $T$ is bounded from below and its range is closed.*

**Proof.** The first statement is clear and the second statement is an immediate consequence of the first one. □

Suppose that a Hilbert space (or, more generally, a normed space) is decomposed into a direct sum of closed subspaces $H = X \dot{+} Y$. Then each $h \in H$ can be written in a unique way in the form $h = x + y$, $x \in X$, $y \in Y$, and the operator $F$ on $H$ defined by $Fh = x$ is called the oblique projection onto $X$ parallel to $Y$. Notice that $F$ is idempotent and bounded. Conversely, each bounded idempotent on $H$ is the oblique projection onto its range parallel to its null-space (see Exercise 2.2.23).

**Lemma 2.2.6.** *Let $H$ and $K$ be Hilbert spaces. Suppose that $T \in \mathbb{B}(H, K)$ and $S \in \mathbb{B}(K, H)$ satisfy $ST = I$. Then*

*(a) $N(S) = (I - TS)(N(T^*))$,*

*(b) $K = R(T) \dot{+} N(S)$,*

*(c) $TS$ is the oblique projection onto $R(T)$ parallel to $N(S)$.*

**Proof.** We first claim that

$$\mathrm{N}(S) = \mathrm{R}(I - TS). \tag{21}$$

Indeed, from $ST = I$ we get $STS = S$ and $S(I - TS) = 0$. This immediately implies $\mathrm{R}(I - TS) \subseteq \mathrm{N}(S)$. Conversely, for each $y \in \mathrm{N}(S)$ we have $(I - TS)y = y$, which gives us $y \in \mathrm{R}(I - TS)$.

Next we claim

$$\mathrm{R}(I - TS) = (I - TS)(\mathrm{N}(T^*)). \tag{22}$$

To see this, we first note that the assumed equality $ST = I$ implies, by Lemma 2.2.5, that $T$ has closed range. Now take any $(I - TS)y \in \mathrm{R}(I - TS)$. Since the range of $T$ is closed, we can write $y$ in the form $y = Tx + z$ for some $Tx \in \mathrm{R}(T)$ and $z \in \mathrm{N}(T^*)$. Next we observe that

$$(I - TS)(Tx) = Tx - T(ST)x = Tx - Tx = 0.$$

This implies that

$$(I - TS)y = (I - TS)(Tx + z) = (I - TS)z \in (I - TS)(\mathrm{N}(T^*)).$$

Thus, $\mathrm{R}(I - TS) \subseteq (I - TS)(\mathrm{N}(T^*))$. Since the reverse inclusion is trivial, this completes the proof of (22). At the same time, we have also proved (a), since (a) follows directly from (21) and (22).

To prove (b), take any $y \in \mathrm{R}(T) \cap \mathrm{N}(S)$. This means that $y = Tx$ for some $x$ and $Sy = 0$. Thus, $0 = STx = x$, so $y = Tx = 0$. Next, take arbitrary $y \in K$. As in the first part of the proof we have $y = Tx + z$ for some $Tx \in \mathrm{R}(T)$ and $z \in \mathrm{N}(T^*)$, and

$$(I - TS)y = (I - TS)z. \tag{23}$$

Put $u = TSy \in \mathrm{R}(T)$ and $v = (I - TS)z$. Since $v \in (I - TS)(\mathrm{N}(T^*))$, (a) implies that $v \in \mathrm{N}(S)$. Therefore, we can rewrite (23) in the form

$$y = TSy + (I - TS)z = u + v \in \mathrm{R}(T) \dotplus \mathrm{N}(S),$$

which completes the proof of (b).

To prove (c), first observe that $(TS)^2 = T(ST)S = TS$ which shows that $TS$ is an oblique projection. Obviously, $TS$ acts as the identity on $\mathrm{R}(T)$, and trivially on $\mathrm{N}(S)$. $\square$

**Proposition 2.2.7.** *Let $H$ and $K$ be Hilbert spaces and $T \in \mathbb{B}(H, K)$. Then $T$ possesses a left inverse if and only if it is bounded from below.*

**Proof.** In one direction, the statement is a part of the content of Lemma 2.2.5. To prove the converse, suppose that $T$ is bounded from below. By Proposition 2.1.7, $T^*$ is then a surjection. Now Proposition 2.1.7 (c) applied to $T^*$ implies that $T^*T$ is invertible on $\mathrm{R}(T^*) = H$. Thus, $S := (T^*T)^{-1}T^*$ is a well defined bounded operator from $K$ to $H$. Obviously, we now have $ST = I$. $\square$

**Corollary 2.2.8.** *Let $H$ and $K$ be Hilbert spaces and let $T \in \mathbb{B}(H, K)$ be bounded from below. Then $T(T^*T)^{-1}T^*$ is the orthogonal projection to $R(T)$.*

**Proof.** We know from (the proof of) the preceding proposition that $S = (T^*T)^{-1}T^*$ is a left inverse of $T$. Then, by Lemma 2.2.6 (c), $TS = T(T^*T)^{-1}T^*$ is the oblique projection to $\mathrm{R}(T)$ parallel to $\mathrm{N}(S)$. But here, in this situation, since $(T^*T)^{-1}$ is an invertible operator on $H$, we have

$$\mathrm{N}(S) = \mathrm{N}((T^*T)^{-1}T^*) = \mathrm{N}(T^*) = \mathrm{R}(T)^{\perp}.$$

From this we conclude: $T(T^*T)^{-1}T^*$ is the oblique projection to $\mathrm{R}(T)$ parallel to $\mathrm{R}(T)^{\perp}$; in other words, $T(T^*T)^{-1}T^*$ is the orthogonal projection to $\mathrm{R}(T)$. $\square$

**Corollary 2.2.9.** *Let $H$ and $K$ be Hilbert spaces and let $T \in \mathbb{B}(H,K)$ be bounded from below. Then every left inverse $S \in \mathbb{B}(K,H)$ of $T$ is of the form $S = (T^*T)^{-1}T^*F$, where $F \in \mathbb{B}(K)$ is the oblique projection to $R(T)$ parallel to some closed direct complement of $R(T)$ in $K$.*

**Proof.** Clearly, if $S = (T^*T)^{-1}T^*F$, where $F \in \mathbb{B}(K)$ is the oblique projection to $\mathrm{R}(T)$ parallel to some closed direct complement of $\mathrm{R}(T)$, then $ST = I$.

Conversely, suppose we have $S \in \mathbb{B}(K,H)$ such that $ST = I$. Then, by Lemma 2.2.6 (c), $F = TS \in \mathbb{B}(K)$ is an oblique projection to $\mathrm{R}(T)$ for which we have $(T^*T)^{-1}T^*F = (T^*T)^{-1}T^*TS = S$. $\square$

**Remark 2.2.10.** Suppose that an operator $T \in \mathbb{B}(H,K)$ is bounded from below. Then the preceding corollary tells us that all left inverses of $T$ are in a bijective correspondence with all bounded oblique projections to $\mathrm{R}(T)$ and, equivalently, with all closed direct complements of $\mathrm{R}(T)$ in $K$. In this light, we may say that the canonical left inverse of $T$ is the left inverse that corresponds to the orthogonal complement of $\mathrm{R}(T)$ in $K$. Recall from Corollary 2.2.8 that the orthogonal projection to $\mathrm{R}(T)$ is given by $P = T(T^*T)^{-1}T^*$. Substituting $P$ for $F$ in our general formula for a left inverse $S = (T^*T)^{-1}T^*F$ from Corollary 2.2.9, we get $S = (T^*T)^{-1}T^*T(T^*T)^{-1}T^* = (T^*T)^{-1}T^*$. This shows that the left inverse of $T$ constructed in Proposition 2.2.7 is in fact the canonical left inverse of $T$.

**Corollary 2.2.11.** *Let $H$ and $K$ be Hilbert spaces and let $T \in \mathbb{B}(H,K)$ be bounded from below. Then $S \in \mathbb{B}(K,H)$ is a left inverse of $T$ if and only if $S$ is of the form $S = (T^*T)^{-1}T^* + W(I - T(T^*T)^{-1}T^*)$ for some $W \in \mathbb{B}(K,H)$.*

**Proof.** If $S = (T^*T)^{-1}T^* + W(I - T(T^*T)^{-1}T^*)$, then, obviously, $ST = I$.

Conversely, if $ST = I$, we can take $W = S$. Then we obtain

$$(T^*T)^{-1}T^* + W(I - T(T^*T)^{-1}T^*) = (T^*T)^{-1}T^* + S(I - T(T^*T)^{-1}T^*) = S.$$

$\square$

Taking into account our discussion preceding Lemma 2.2.5 together with Corollary 2.2.9 and Corollary 2.2.11 we obtain the following conclusion:

**Corollary 2.2.12.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. Suppose that $(v_n)_n$ is a frame for $H$ with the analysis operator $V$. The following conditions are equivalent:*

*(a) $(v_n)_n$ is dual to $(x_n)_n$.*

*(b) $V^*$ is of the form $V^* = (U^*U)^{-1}U^*F$, where $F \in \mathbb{B}(\ell^2)$ is the oblique projection to $R(U)$ parallel to some closed direct complement of $R(U)$ in $\ell^2$.*

*(c) $V^*$ is of the form $V^* = (U^*U)^{-1}U^* + W(I - U(U^*U)^{-1}U^*)$, for some $W \in \mathbb{B}(\ell^2, H)$.*

**Corollary 2.2.13.** *Let $(x_n)_n$ and $(v_n)_n$ be frames for a Hilbert space $H$ dual to each other with the analysis operators $U$ and $V$, respectively. Then*

*(a)* $\ell^2 = R(U) \dot{+} N(V^*)$,

*(b)* $UV^*$ *is the oblique projection to* $R(U)$ *parallel to* $N(V^*)$,

*(c)* $\ell^2 = R(V) \dot{+} N(U^*)$,

*(d)* $VU^*$ *is the oblique projection to* $R(V)$ *parallel to* $N(U^*)$.

**Proof.** (a) and (b) are immediate from Lemma 2.2.6 with $S = V^*$ and $T = U$. Since $V^*U = I$ is equivalent to $U^*V = I$, (c) and (d) follow from (a) and (b) by symmetry. □

**Corollary 2.2.14.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$. Then $(x_n)_n$ possesses a unique dual frame if and only if $(x_n)_n$ is a Riesz basis for $H$.*

**Proof.** Denote by $U$ the analysis operator of $(x_n)_n$. Let $(e_n)_n$ be the canonical basis for $\ell^2$.

If $(x_n)_n$ is a Riesz basis, then, since all ONB's in all separable Hilbert spaces are equivalent, there is an invertible operator $T \in \mathbb{B}(\ell^2, H)$ such that $Te_n = x_n$ for all $n$. In particular, $T$ coincides with the synthesis operator $U^*$. Thus, $U^*$ is injective. This implies that $\ell^2 = \mathrm{N}(U^*)^\perp = \mathrm{R}(U)$, so by Corollary 2.2.12, the canonical dual is the unique frame dual to $(x_n)_n$.

Conversely, if $(x_n)_n$ possesses a unique dual frame, then again by the preceding corollary $\mathrm{R}(U)$ has a unique closed direct complement in $\ell^2$ (which is necessarily the null-space); hence, $\mathrm{R}(U) = \ell^2$. This tells us that $U$ is a bijection; thus, $U^*$ is a bijection, and therefore $(x_n)_n$ is a Riesz basis. □

The preceding discussion on left inverses of operators bounded from below is in fact a special case of a more general considerations concerned with operators with closed ranges and their pseudo-inverses. Here we include basic facts about pseudo-inverses.

**Definition 2.2.15.** *Let $H$ and $K$ be Hilbert spaces and let $T \in \mathbb{B}(H, K)$ be an operator with closed range. The pseudo-inverse of $T$ is the operator $T^\dagger \in \mathbb{B}(K, H)$ defined by $T^\dagger|_{R(T)} = T_0^{-1}$ and $T^\dagger|_{R(T)^\perp} = 0$, where $T_0 : N(T)^\perp \to R(T)$ is a restriction of $T$ to $N(T)^\perp$ regarded as an operator which takes values in $R(T)$.*

Notice that $T_0$ is a bijection and, since by the assumption $\mathrm{R}(T)$ is closed in $K$, the inverse mapping theorem ensures that $T_0^{-1}$ is a bounded operator from $\mathrm{R}(T)$ to $\mathrm{N}(T)^\perp$. Hence, $T^\dagger$ is well-defined and bounded.

**Remark 2.2.16.** If the range of $T \in \mathbb{B}(H, K)$ is closed, then, by Proposition 2.1.7, $T^*$ also has closed range. It is now clear that the pseudo-inverse $T^\dagger$ satisfies

$$\mathrm{N}(T^\dagger) = \mathrm{R}(T)^\perp, \tag{24a}$$

$$\mathrm{R}(T^\dagger) = \mathrm{N}(T)^\perp = \mathrm{R}(T^*), \tag{24b}$$

$$TT^\dagger x = x, \ \forall x \in \mathrm{R}(T). \tag{24c}$$

It is easy to verify that $T^\dagger$ is the only operator in $\mathbb{B}(K, H)$ that satisfies equalities (24a), (24b) and (24c).

It is also evident from the definition that $TT^\dagger$ and $T^\dagger T$ are the orthogonal projections to $\mathrm{R}(T)$ and $\mathrm{R}(T^*)$, respectively.

**Proposition 2.2.17.** *Let $H$ and $K$ be Hilbert spaces and let $T \in \mathbb{B}(H, K)$ be an operator with closed range. Then:*

*(a)* $(T^*)^\dagger = (T^\dagger)^*$.

*(b)* $T^\dagger$ *is given on $R(T)$ by $T^\dagger = T^*(TT^*)^{-1}$. In particular, if $T$ is a surjection, $T^\dagger = T^*(TT^*)^{-1}$.*

*(c)* $T^*T$ *has closed range and* $(T^*T)^\dagger = T^\dagger(T^*)^\dagger$.

**Proof.** (a) It suffices to show that the operator $(T^\dagger)^*$ satisfies equalities (24a), (24b), (24c) with respect to $T^*$. First,

$$\mathrm{N}((T^\dagger)^*) = \mathrm{R}(T^\dagger)^\perp \overset{(24b)}{=} \mathrm{N}(T) = \mathrm{R}(T^*)^\perp.$$

Secondly,

$$\mathrm{R}((T^\dagger)^*) = \mathrm{N}(T^\dagger)^\perp \overset{(24a)}{=} \mathrm{R}(T).$$

Finally, we see from the last part of Remark 2.2.16 that $T^\dagger T$ is a Hermitian operator. This implies that $T^*(T^\dagger)^* = (T^\dagger T)^* = T^\dagger T$. On the other hand, this operator is by the last part of Remark 2.2.16 the orthogonal projection to $\mathrm{R}(T^*)$ which is precisely the remaining property (24c) of the pseudo-inverse $(T^*)^\dagger$. Thus, $(T^\dagger)^*$ coincides with $(T^*)^\dagger$.

(b) First recall from Proposition 2.1.7 (c) that $TT^*$ is invertible on $\mathrm{R}(T)$. It is evident that the operator that acts as $T^*(TT^*)^{-1}$ on $\mathrm{R}(T)$ and trivially on $\mathrm{R}(T)^\perp$ satisfies equalities (24a), (24b), (24c) from the first part of Remark 2.2.16.

(c) We leave this part of the proof as an exercise. □

**Remark 2.2.18.** Suppose that $H$ and $K$ are Hilbert spaces and that $T \in \mathbb{B}(H, K)$ is bounded from below. By the discussion from the beginning of this section we know that $T$ has a left inverse. In particular, the canonical left inverse of $T$ is given by $(T^*T)^{-1}T^*$.

On the other hand, we know by Proposition 2.1.7 that $T^*$ is a surjection. By Proposition 2.2.17 (b), the pseudo-inverse of $T^*$ is given by $(T^*)^\dagger = T(T^*T)^{-1}$. Moreover, by the last part of Remark 2.2.16, we know that $T^*(T^*)^\dagger$ is the orthogonal projection to $\mathrm{R}(T^*)$. Since $T^*$ is a surjection, this gives us

$$T^*T(T^*T)^{-1} = I.$$

By taking adjoints, we conclude

$$(T^*T)^{-1}T^*T = I$$

which is the equality we already know from (the proof of) Proposition 2.2.7 (see also Corollary 2.2.8). In other words, the canonical left inverse of $T$ is in fact the adjoint of the pseudo-inverse of $T^*$.

In the frame context operators with closed ranges and their pseudoinverses naturally arise in connection with frame sequences. Roughly speaking a frame sequence in a Hilbert space $H$ is a frame for a closed subspace of $H$.

**Definition 2.2.19.** *A sequence $(x_n)_n$ in a Hilbert space $H$ is said to be a frame sequence if it is a frame for $\overline{\text{span}}\,\{x_n : n \in \mathbb{N}\}$.*

Suppose that $(x_n)_n$ is a frame sequence in $H$ and denote $\overline{\text{span}}\,\{x_n : n \in \mathbb{N}\}$ by $M$. If $U \in \mathbb{B}(M, \ell^2)$ is the corresponding analysis operator, we naturally understand it as an operator on $H$ extending $U$ trivially on $M^\perp$. Then $U$ is bounded below on $M$ and the corresponding synthesis operator $U^*$ is an operator with closed range: $\mathrm{R}(U^*) = M$. Conversely, if $T \in \mathbb{B}(\ell^2, H)$ is an operator with closed range it is easy to see that $(Te_n)_n$ is a frame sequence. The following proposition tells us that frame sequences arise in the same way from frames.

**Proposition 2.2.20.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with frame bounds $A$ and $B$, and let $T \in \mathbb{B}(H)$ be an operator with closed range. Then $(Tx_n)_n$ is a frame sequence with frame bounds $\frac{A}{\|T^\dagger\|^2}$ and $B\|T\|^2$.*

**Proof.** Clearly, $(Tx_n)_n$ is a Bessel sequence with $B\|T\|^2$ as a Bessel (i.e. upper frame) bound. Take any $y \in \text{span}\,\{Tx_n : n \in \mathbb{N}\}$. We first find $x \in \text{span}\,\{x_n : n \in \mathbb{N}\}$ such that $y = Tx$. By Remark 2.2.16 $TT^\dagger$ is the orthogonal projection to $\mathrm{R}(T)$ and therefore self-adjoint. Hence,

$$y = Tx = (TT^\dagger)Tx = (TT^\dagger)^*Tx = (T^\dagger)^*T^*Tx.$$

From this we obtain

$$\begin{aligned} \|y\|^2 &\leq \|(T^\dagger)^*\|^2\|T^*Tx\|^2 \\ &\leq \frac{\|(T^\dagger)^*\|^2}{A}\sum_{n=1}^{\infty}|\langle T^*Tx, x_n\rangle|^2 \\ &= \frac{\|T^\dagger\|^2}{A}\sum_{n=1}^{\infty}|\langle y, Tx_n\rangle|^2. \end{aligned}$$

By Exercise 2.2.29 we now conclude that the lower frame condition is satisfied on $\overline{\text{span}}\,\{Tx_n : n \in \mathbb{N}\}$. $\square$

**Remark 2.2.21.** (a) The conclusion of the preceding proposition might fail if $T$ does not have close range. As an example we may take an ONB $(e_n)_n$ and the operator $T = S + I$, where $S$ is the unilateral shift.

(b) Even if $T$ has closed range it is not enough to take a frame sequence instead of a frame. To see this, consider again an ONB $(e_n)_n$ for $H$ and the operator $T$ defined by $Te_{2k+1} = \frac{1}{k}e_{2k}$, $Te_{2k} = e_{2k}$, for al $k$. It is now easy to conclude that $T$ is bounded and has closed range and that $(e_{2k+1})_k$ is a frame sequence, but $(e_{2k+1})_k$ is not.

(c) Observe that the preceding proposition is in accordance with Corollary 2.1.12 when $T$ is a surjection. Namely, It $T$ is a surjection then we know from Proposition 2.2.17 (b) that $T^\dagger = T^*(TT^*)^{-1}$. But then we have $\|T^\dagger\|^2 = \|(T^\dagger)^*T^\dagger\| = \|(TT^*)^{-1}TT^*(TT^*)^{-1}\| = \|(TT^*)^{-1}\|$.

*Concluding remarks.* The material in this section is standard and well known. Proposition 2.2.4 appeared already in [64]. Corollary 2.2.9 and the corresponding equivalence $(a) \Leftrightarrow (b)$ in Corollary 2.2.12 are first observed in [13]. Proposition 2.2.20 and examples from Remark 2.2.20 are borrowed from [51].

**Exercise 2.2.22.** Let $M$ be a closed subspace of a Banach space $X$, let $N$ be a subspace of $X$ such that $X = M \dot{+} N$, and let $F$ be the oblique projection to $M$ along $N$. Prove that $F$ is bounded if and only if $N$ is closed.

**Exercise 2.2.23.** Let $X$ be a normed space and let $F \in \mathbb{B}(X)$ be an idempotent ($F^2 = F$). Prove that $\mathrm{R}(F)$ is closed, $X = \mathrm{R}(F) \dot{+} \mathrm{N}(F)$ and that $F$ is the oblique projection to $\mathrm{R}(F)$ along $\mathrm{N}(F)$.

**Exercise 2.2.24.** Let $H$ be an infinite-dimensional separable Hilbert space. Prove that there exists an unbounded linear idempotent on $H$ with closed range (cf. [30]).

**Exercise 2.2.25.** Let $M$ and $L$ be closed subspaces of a Hilbert space $H$ such that $H = M \dot{+} L$. Let $F$ be the oblique projection to $M$ along $L$. Prove that $F^*$ is also an oblique projection and find $\mathrm{R}(F^*)$ and $\mathrm{N}(F^*)$.

**Exercise 2.2.26.** Let $F$ be a bounded idempotent on a Hilbert space $H$. Prove that $\|F\| = \|I - F\|$ (see [3]).

**Exercise 2.2.27.** Let $M$ be a non-trivial closed subspace of a Hilbert space $H$. Prove that $M$ has infinitely many closed direct complements in $H$.

**Exercise 2.2.28.** Prove Proposition 2.2.17 (c): if $H$ and $K$ are Hilbert spaces and the range of $T \in \mathbb{B}(H, K)$ is closed, then $T^*T$ has also closed range and $(T^*T)^\dagger = T^\dagger (T^*)^\dagger$.

**Exercise 2.2.29.** Suppose that $(x_n)_n$ is a sequence in a Hilbert space $H$ for which there exist constants $A, B > 0$ such that

$$A\|y\|^2 \leq \sum_{n=1}^{\infty} |\langle y, x_n \rangle|^2 \leq B\|y\|^2$$

for all $y$ from a dense set $Y$ in $H$. Show that $(x_n)_n$ is a frame for $H$. *Remark.* Observe that the proof that the lower frame condition extends from $Y$ to $H$ uses the fact that that the upper frame condition is satisfied on $H$.

## 2.3 Characterizations of frames

By Corollary 2.1.9, a sequence of elements in a Hilbert space $H$ is a frame if and only if it is the image of an orthonormal basis under the action of a bounded surjective operator. On the other hand, a sequence in $H$ is a Riesz basis for $H$ if, by definition, it is the image of an ONB under an invertible bounded operator. Thus, each Riesz basis is a frame. In this light it would be useful to find another descriptions of those frames that are in fact Riesz bases. We start with a natural question: if a Riesz basis is regarded as a frame, what is its canonical dual?

**Proposition 2.3.1.** *Let $((x_n)_n, (y_n)_n)$ be a Riesz basis for a Hilbert space $H$. Then the canonical dual of the frame $(x_n)_n$ coincides with $(y_n)_n$.*

**Proof.** Since we have, for all $x \in H$, $x = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n$, the sequence $(y_n)_n$ is a dual of $(x_n)_n$. Since $(x_n)_n$ is a Riesz basis, there exist an ONB $(e_n)_n$ for $H$ and an invertible operator $T \in \mathbb{B}(H)$ such that $x_n = Te_n$, for all $n$. On the other hand, by Corollary 1.2.28, $(y_n)_n$ is also a Riesz basis for $H$. Thus, there exists an invertible bounded operator $S \in \mathbb{B}(H)$ with the property $Se_n = y_n$, $n \in \mathbb{N}$. From this we conclude that $y_n = ST^{-1}x_n$ for all $n$. Proposition 2.2.4 (b) now implies that $(y_n)_n$ is the canonical dual of $(x_n)_n$. □

Recall that a basis for a Hilbert space need not be a Riesz basis. However, if a basis is also a frame, the Riesz property follows.

**Proposition 2.3.2.** *If a frame $(x_n)_n$ for a Hilbert space $H$ is a basis for $H$, then it is a Riesz basis.*

**Proof.** Suppose that a frame $(x_n)_n$ is a basis. Denote by $U$ its analysis operator. We know that $x_n = U^*e_n$, for all $n$, where $(e_n)_n$ is the canonical basis for $\ell^2$. It suffices to show that $U^*$ is a bijection. Take any $(c_n)_n \in \mathrm{N}(U^*)$. Then we have $U^*(c_n)_n = \sum_{n=1}^{\infty} c_n x_n = 0$. Since $(x_n)_n$ is a basis, this implies $c_n = 0$ for all $n$. Thus, $\mathrm{N}(U^*) = \{0\}$. □

**Remark 2.3.3.** It is interesting to note the following consequence. If $(x_n)_n$ is a frame for $H$ then either each $x \in H$ has a unique expansion of the form $x = \sum_{n=1}^{\infty} c_n x_n$ (so that $(x_n)_n$ is a basis for $H$), or each $x \in H$ has infinitely many expansions $x = \sum_{n=1}^{\infty} d_n x_n$ with $(d_n)_n \in \ell^2$ (where one can choose $(d_n)_n$ as any element of the linear manifold $\langle x, (U^*U)^{-1}x_n \rangle + \mathrm{N}(U^*)$).

**Corollary 2.3.4.** *If a Parseval frame for a Hilbert space $H$ is a basis for $H$, then it is an ONB for $H$.*

**Proof.** Suppose that a Parseval frame $(x_n)_n$ is a basis for $H$. Then its analysis operator $U$ is an isometry and, as the preceding proof shows, a surjection. Thus, $U$ is unitary. Since we have $x_n = U^*e_n$, $n \in \mathbb{N}$, where $(e_n)_n$ is the canonical basis for $\ell^2$, it follows that $(x_n)_n$ is an ONB for $H$. □

Recall that a frame $(x_n)_n$ is said to be exact if removal of any of its elements destroys the frame property. We shall see that exact frames are precisely Riesz bases. Let us start with a technical result concerning the inner products of elements of a frame with the corresponding members of its canonical dual $(y_n)_n$, $y_n = (U^*U)^{-1}x_n$, $n \in \mathbb{N}$.

**Proposition 2.3.5.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$.*

*(a) For each $m$,*

$$\sum_{n\neq m} |\langle x_m, (U^*U)^{-1}x_n\rangle|^2 = \frac{1}{2}\left(1 - |\langle x_m, (U^*U)^{-1}x_m\rangle|^2 - |1 - \langle x_m, (U^*U)^{-1}x_m\rangle|^2\right).$$

*(b) If there exists $m$ such that $\langle x_m, (U^*U)^{-1}x_m\rangle = 1$, then $\langle x_m, (U^*U)^{-1}x_n\rangle = 0$ for all $n \neq m$.*

*(c) If there exists $m$ such that $\langle x_m, (U^*U)^{-1}x_m\rangle = 1$, then the sequence $(x_n)_{n\neq m}$ is not fundamental.*

*(d) For each $m$ such that $\langle x_m, (U^*U)^{-1}x_m\rangle \neq 1$, the sequence $(x_n)_{n\neq m}$ is a frame for $H$.*

**Proof.** (a) Choose and fix any $m$ and write $a_n = \langle x_m, (U^*U)^{-1}x_n\rangle$, $n \in \mathbb{N}$. Observe that we have $x_m = \sum_{n=1}^{\infty} a_n x_n$ and $x_m = \sum_{n=1}^{\infty} \delta_{mn} x_n$. Now Proposition 2.2.4 (a) implies

$$1 = \sum_{n=1}^{\infty} |\delta_{mn}|^2 = \sum_{n=1}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} |a_n - \delta_{mn}|^2 = |a_m|^2 + \sum_{n\neq m} |a_n|^2 + |a_m - 1|^2 + \sum_{n\neq m} |a_n|^2$$

whence

$$2\sum_{n\neq m} |a_n|^2 = 1 - |a_m|^2 - |a_m - 1|^2.$$

This proves (a).

(b) is evident from (a).

(c) Suppose that $\langle x_m, (U^*U)^{-1}x_m\rangle = 1$. Then, clearly, $(U^*U)^{-1}x_m \neq 0$ and, by (b), $(U^*U)^{-1}x_m \perp x_n$ for all $n \neq m$.

(d) Let $\langle x_m, (U^*U)^{-1}x_m\rangle \neq 1$. Write again $a_n = \langle x_m, (U^*U)^{-1}x_n\rangle$, $n \in \mathbb{N}$. From $x_m = \sum_{n=1}^{\infty} a_n x_n$ we conclude that $x_m = \frac{1}{1-a_m}\sum_{n\neq m} a_n x_n$. This, together with the Cauchy-Schwarz inequality in $\ell^2$, implies

$$|\langle x, x_m\rangle|^2 = \left|\frac{1}{1-a_m}\sum_{n\neq m} \overline{a_n}\langle x, x_n\rangle\right|^2 \leq C\sum_{n\neq m} |\langle x, x_n\rangle|^2,$$

where $C = \frac{1}{|1-a_m|^2}\sum_{n\neq m} |a_n|^2$. Therefore,

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 = |\langle x, x_m\rangle|^2 + \sum_{n\neq m} |\langle x, x_n\rangle|^2 \leq (1 + C)\sum_{n\neq m} |\langle x, x_n\rangle|^2.$$

Finally, if $A$ and $B$ are frame bounds of the original frame $(x_n)_n$, this gives us

$$\frac{A}{1+C}\|x\|^2 \leq \frac{A}{1+C}\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq \sum_{n\neq m} |\langle x, x_n\rangle|^2 \leq B\|x\|.$$

Hence, $(x_n)_{n\neq m}$ is a frame with frame bounds $\frac{A}{1+C}$ and $B$. $\square$

**Remark 2.3.6.** Here we provide an alternative proof of (b), (c), and (d) from the preceding proposition.

Suppose first that $(x_n)_n$ is a Parseval frame. Take any $m \in \mathbb{N}$. Then we have

$$\|x_m\|^2 = \sum_{n=1}^{\infty} |\langle x_m, x_n\rangle|^2 = \|x_m\|^4 + \sum_{n \neq m} |\langle x_m, x_n\rangle|^2.$$

When $\|x_m\| = 1$ this implies $\sum_{n\neq m} |\langle x_m, x_n\rangle|^2 = 0$ and hence $\langle x_m, x_n\rangle = 0$ for all $n \neq m$. This proves (b) and (c).

To prove (d), suppose that $\|x_m\| \neq 1$ which means that $\|x_m\| < 1$ since all elements of a Parseval frame belong to the closed unit ball. Now we have, for each $x$ in $H$,

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 = |\langle x, x_m\rangle|^2 + \sum_{n \neq m} |\langle x, x_n\rangle|^2;$$

thus,

$$\|x\|^2 - |\langle x, x_m\rangle|^2 = \sum_{n \neq m} |\langle x, x_n\rangle|^2. \tag{25}$$

Let $A' = 1 - \|x_m\|^2 > 0$. Since $|\langle x, x_m\rangle|^2 \leq \|x\|^2\|x_m\|^2$, we now have

$$A'\|x\|^2 = (1 - \|x_m\|^2)\|x\|^2 \leq \|x\|^2 - |\langle x, x_m\rangle|^2 \overset{(25)}{\leq} \sum_{n \neq m} |\langle x, x_n\rangle|^2, \ \ \forall x \in H.$$

Therefore, $(x_n)_{n\neq m}$ is a frame.

Take now arbitrary frame $(x_n)_n$ for $H$ and denote its analysis operator by $U$. Recall from Proposition 2.1.15 that $((U^*U)^{-\frac{1}{2}}x_n)_n$ is a Parseval frame and observe that

$$\langle x_m, (U^*U)^{-1}x_m\rangle = \langle (U^*U)^{-\frac{1}{2}}x_m, (U^*U)^{-\frac{1}{2}}x_m\rangle = \|(U^*U)^{-\frac{1}{2}}x_m\|^2.$$

It is now clear that the desired conclusions (that is, (b), (c), and (d) from Proposition 2.3.5) follow from the corresponding statements which we just have proved for Parseval frames.

**Corollary 2.3.7.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. The following conditions are equivalent:*

*(a) $(x_n)_n$ is exact;*

*(b) $(x_n)_n$ and $((U^*U)^{-1}x_n)_n$ are biorthogonal sequences;*

*(c) $\langle x_n, (U^*U)^{-1}x_n\rangle = 1$, $\forall n \in \mathbb{N}$.*

**Proof.** If $(x_n)_n$ is exact, the last statement of the preceding proposition implies (c). From the second statement of the preceding proposition we see that (c) implies (b). Observe that (b) obviously implies (c). Finally, if we have (c), the third statement from the preceding proposition gives us (a). □

**Corollary 2.3.8.** *Let $(x_n)_n$ be a Parseval frame for a Hilbert space $H$. The following conditions are equivalent:*

*(a)* $(x_n)_n$ *is exact;*

*(b)* $(x_n)_n$ *is an orthonormal sequence;*

*(c)* $\|x_n\| = 1$, $\forall n \in \mathbb{N}$.

Each frame is a bounded sequence; if $B$ is an upper bound of a frame $(x_n)_n$, we know that $\|x_n\| \leq \sqrt{B}$ for every $n$. However, norms of frame elements need not be bounded from below by a positive constant. As an example we may take $e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots$, where $(e_n)_n$ is an orthonormal basis. Notice that this frame is inexact.

**Proposition 2.3.9.** *Let $(x_n)_n$ be an exact frame for a Hilbert space $H$ with a lower frame bound $A$. Then $\|x_n\| \geq \sqrt{A}$ for every $n$.*

**Proof.** Suppose that $(x_n)_n$ is an exact frame and denote by $U$ its analysis operator. Using (b) from Corollary 2.3.7 we obtain for any $m$

$$A\left\|(U^*U)^{-1}x_m\right\|^2 \leq \sum_{n=1}^{\infty}\left|\langle (U^*U)^{-1}x_m, x_n\rangle\right|^2 = \left|\langle (U^*U)^{-1}x_m, x_m\rangle\right|^2 \leq \left\|(U^*U)^{-1}x_m\right\|^2 \|x_m\|^2.$$

Since $(x_n)_n$ is exact, we have $x_m \neq 0$ and, consequently, $(U^*U)^{-1}x_m \neq 0$ for every $m$. □

**Proposition 2.3.10.** *A frame $(x_n)_n$ for a Hilbert space $H$ is a basis for $H$ if and only if $(x_n)_n$ is an exact frame.*

**Proof.** Suppose that a frame $(x_n)_n$ is a basis. By Proposition 2.3.2 $(x_n)_n$ is then a Riesz basis. By definition, there exist an ONB $(e_n)_n$ for $H$ and an invertible operator $T \in \mathbb{B}(H)$ such that $x_n = Te_n$ for all $n$. Since each ONB is an exact frame and invertible operators map exact frames into exact frames (obvious), this implies that $(x_n)_n$ is exact.

Suppose now that $(x_n)_n$ is an exact frame for $H$. We must show that each $x$ admits a unique expansion of the form $x = \sum_{n=1}^{\infty}\lambda_n x_n$. The reconstruction formula (recall that $((U^*U)^{-1}x_n)_n$ is the canonical dual of $(x_n)_n$) gives us $x = \sum_{n=1}^{\infty}\langle x, (U^*U)^{-1}x_n\rangle x_n$. Suppose now that we have a sequence of scalars $(\lambda_n)_n$ such that $x = \sum_{n=1}^{\infty}\lambda_n x_n$. This implies, for each $m$,

$$\langle x, (U^*U)^{-1}x_m\rangle = \left\langle \sum_{n=1}^{\infty}\lambda_n x_n, (U^*U)^{-1}x_m\right\rangle = \sum_{n=1}^{\infty}\lambda_n\langle x_n, (U^*U)^{-1}x_m\rangle = \lambda_m,$$

where the last equality follows from Corollary 2.3.7 (b). □

**Theorem 2.3.11.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. The following conditions are equivalent:*

*(a)* $(x_n)_n$ *is a Riesz basis;*

*(b)* $(x_n)_n$ *is an exact frame;*

*(c)* $(x_n)_n$ *and* $((U^*U)^{-1}x_n)_n$ *are biorthogonal;*

*(d)* $(x_n)_n$ *has a biorthogonal sequence;*

*(e)* $(x_n)_n$ *is minimal;*

*(f)* $(x_n)_n$ *is* $\omega$*-independent;*

*(g) If* $\sum_{n=1}^{\infty} c_n x_n = 0$ *for some* $(c_n)_n \in \ell^2$ *then* $c_n = 0$ *for all* $n$*.*

**Proof.** $(a) \Rightarrow (b)$ is proved in Proposition 2.3.10.

$(b) \Rightarrow (c)$ is proved in Corollary 2.3.7.

$(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (e)$ is proved in Proposition 1.2.16 (a).

$(e) \Rightarrow (f)$ is the statement of Proposition 1.2.14 (b).

$(f) \Rightarrow (g)$ is obvious.

$(g) \Rightarrow (a)$ Our assumption (g) implies that $U^*$ is injective. Thus, $U^*$ is in fact a bijection which means, since $U^*$ maps the canonical basis of $\ell^2$ to $(x_n)_n$, that $(x_n)_n$ is a Riesz basis. $\square$

We already know that frames for a Hilbert space $H$ are in a bijective correspondence with surjective bounded operators from $\ell^2$ to $H$. In the rest of this section we provide another description of all frames on $H$. We will show that all frames on $H$ can also be described in terms of a given frame and a class of bounded operators on $\ell^2$.

Suppose we are given a frame $(x_n)_n$ for a Hilbert space $H$ with the analysis operator $U$. Take any $T \in \mathbb{B}(\ell^2)$ and denote by $[T]_{(e_n)_n}$ the (infinite) matrix of $T$ with respect to the canonical basis $(e_n)_n$ of $\ell^2$:

$$[T]_{(e_n)_n} = (t_{ij}),\ t_{ij} = \langle Te_j, e_i \rangle,\ i, j \in \mathbb{N}. \tag{26}$$

Consider the composition $TU \in \mathbb{B}(H, \ell^2)$. If $T$ is bounded from below on $\mathrm{R}(U)$, then $TU$ is bounded from below on $H$ and hence $(TU)^* = U^*T^*$ is a surjection. Hence, $TU$ is the analysis operator of a frame $(f_n)_n$ for $H$ that is given by

$$f_n = U^*T^*e_n = U^*\left(\sum_{j=1}^{\infty} \langle T^*e_n, e_j \rangle e_j\right) = \sum_{j=1}^{\infty} \langle e_n, Te_j \rangle U^*e_j,\ n \in \mathbb{N}. \tag{27}$$

Recalling that $U^*e_j = x_j$, $j \in \mathbb{N}$, and using (26), we get

$$f_n = \sum_{j=1}^{\infty} \overline{t_{nj}} x_j,\ n \in \mathbb{N}. \tag{28}$$

Conversely, if we define a sequence $(f_n)_n$ using (28) with coefficients $t_{ij}$ arising from an operator $T \in \mathbb{B}(\ell^2)$ as in (26), then (27) shows that $(f_n)_n$ is a Bessel sequence with the analysis operator $TU$. If, moreover, $(f_n)_n$ is a frame, $TU$ is bounded from below and hence $T$ must be bounded from below on $\mathrm{R}(U)$.

In the sequel we shall write $(f_n)_n = [T]_{(e_n)_n}(x_n)_n$ if $(f_n)_n$ is a frame that is obtained from a frame $(x_n)_n$ and an operator $T$ by the procedure described above.

**Theorem 2.3.12.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. For every frame $(f_n)_n$ for $H$ there exists an operator $T \in \mathbb{B}(\ell^2)$ bounded from below on $R(U)$ such that $(f_n)_n = [T]_{(e_n)_n}(x_n)_n$.*

**Proof.** Take any frame $(f_n)_n$ for $H$. Denote by $V$ its analysis operator and by $D$ its upper frame bound. Using the reconstruction formula with respect to $(x_n)_n$ and its canonical dual we can write

$$f_n = \sum_{j=1}^{\infty} \langle f_n, (U^*U)^{-1}x_j \rangle x_j, \ \ \forall n \in \mathbb{N}. \tag{29}$$

Put

$$t_{jn} = \langle (U^*U)^{-1}x_j, f_n \rangle, \ \ j, n \in \mathbb{N}. \tag{30}$$

We must show that the map $T$ defined by

$$Te_n = \sum_{j=1}^{\infty} t_{jn} e_j, \ \ n \in \mathbb{N}, \tag{31}$$

(end extended by linearity) is a bounded operator on $\ell^2$, where $(e_n)_n$ is the canonical basis for $\ell^2$. By our considerations preceding Theorem 2.3.12, this will then imply that $T$ is bounded from below on $\mathrm{R}(U)$. Let $y = \sum_{n=1}^{N} c_n e_n$ be any finite sequence in $\ell^2$. Then, if denote by $B'$ an upper frame bound of $((U^*U)^{-1}x_n)_n$, we have

$$\begin{aligned}
\|Ty\|^2 &= \left\| T\left(\sum_{n=1}^{N} c_n e_n\right)\right\|^2 = \left\|\sum_{n=1}^{N} c_n T e_n\right\|^2 \overset{(31)}{=} \left\|\sum_{n=1}^{N} c_n \sum_{j=1}^{\infty} t_{jn} e_j\right\|^2 \\
&= \left\|\sum_{j=1}^{\infty}\left(\sum_{n=1}^{N} t_{jn} c_n\right) e_j\right\|^2 \\
&= \sum_{j=1}^{\infty}\left|\sum_{n=1}^{N} t_{jn} c_n\right|^2 \\
&\overset{(30)}{=} \sum_{j=1}^{\infty}\left|\sum_{n=1}^{N} \langle (U^*U)^{-1}x_j, f_n\rangle c_n\right|^2 \\
&= \sum_{j=1}^{\infty}\left|\left\langle (U^*U)^{-1}x_j, \sum_{n=1}^{N} \overline{c_n} f_n\right\rangle\right|^2 \\
&\leq B' \left\|\sum_{n=1}^{N} \overline{c_n} f_n\right\|^2 \\
&= B' \left\| V^*\left(\sum_{n=1}^{N} \overline{c_n} e_n\right)\right\|^2 \leq B' D \|y\|^2.
\end{aligned}$$

This shows that $T$ is bounded on the subspace $c_{00}$ of all finite sequences which is dense in $\ell^2$. Therefore, it extends to a bounded operator on $\ell^2$. $\square$

*Concluding remarks.* A major part of the material in this section (up to Theorem 2.3.11) first appeared in [64]. In the exposition we have followed [81]. Theorem 2.3.12 is proved in [1].

**Exercise 2.3.13.** Let $(x_n)_n$ be a Riesz basis for a Hilbert space $H$. Show that the sequence $(f_n)_n$ defined by $f_n = x_n + x_{n+1}$, $n \in \mathbb{N}$ is not a frame for $H$.

## 2.4 Near-Riesz bases

**Definition 2.4.1.** *A frame $(x_n)_n$ for a Hilbert space $H$ is said to be a near-Riesz basis for $H$ if there exists a finite set of indices $S$ such that $(x_n)_{n\notin S}$ is a Riesz basis for $H$.*

*We say that a frame $(x_n)_n$ is Besselian if convergence of the series $\sum_{n=1}^{\infty} c_n x_n$, where $(c_n)_n$ is some sequence of scalars, implies that $(c_n)_n \in \ell^2$.*

*Finally, a frame $(x_n)_n$ is said to be an unconditional frame if, whenever the series $\sum_{n=1}^{\infty} c_n x_n$ converges for some sequence $(c_n)_n$ of scalars, this convergence is unconditional.*

If $(x_n)_n$ is a Riesz basis for $H$ we know from Theorem 1.2.29 that, if $(c_n)_n$ is a sequence of scalars, then

$$\sum_{n=1}^{\infty} c_n x_n \ \text{ converges } \iff (c_n)_n \in \ell^2.$$

It is easy to conclude that the same holds for near-Riesz bases.

Furthermore, if $(x_n)_n$ is any frame (in fact, merely a Bessel sequence), then we know from Proposition 1.3.4 that the series $\sum_{n=1}^{\infty} c_n x_n$ converges unconditionally for all $\ell^2$-sequences $(c_n)_n$.

By combining these two observations we conclude:

**Remark 2.4.2.** Let $(x_n)_n$ be a near-Riesz basis for a Hilbert space $H$. Then, if $(c_n)_n$ is a sequence of scalars,

$$\sum_{n=1}^{\infty} c_n x_n \ \text{ converges } \iff (c_n)_n \in \ell^2 \implies \sum_{n=1}^{\infty} c_n x_n \ \text{ converges unconditionally.}$$

This gives us immediately the following conclusions::

(a) each near-Riesz basis is a Besselian frame,

(b) each Besselian frame is an unconditional frame.

However, a frame need not be unconditional. To see this, consider an ONB $(e_n)_n$ for $H$ and a frame $(e_1, e_1, e_2, e_2, \ldots)$. It is clear that the series

$$e_1 - e_1 + \frac{1}{\sqrt{2}}e_2 - \frac{1}{\sqrt{2}}e_2 + \frac{1}{\sqrt{3}}e_3 - \frac{1}{\sqrt{3}}e_3 + \ldots$$

converges, but not unconditionally.

It turns out that the Besselian property of frames has deep implications to their analysis operators. We first introduce the notion of similar frames.

**Definition 2.4.3.** *Let $(x_n)_n$ and $(y_n)_n$ be frames for Hilbert spaces $H$ and $K$, respectively. We say that $(x_n)_n$ and $(y_n)_n$ are similar frames if there exists an invertible operator $T \in \mathbb{B}(H, K)$ such that $y_n = Tx_n$ for all $n$.*

Notice that each frame is similar to its canonical dual.

Clearly, a frame $(y_n)_n$ that is similar to a near-Riesz basis/Besselian frame/unconditional frame is also a near-Riesz basis/Besselian frame/unconditional frame.

The following lemma should be compared to Proposition 2.1.17.

**Lemma 2.4.4.** *Each frame is similar to the frame of the form $(Pe_n)_n$ for a closed subspace $M$ of $\ell^2$, where $(e_n)_n$ is the canonical basis for $\ell^2$ and $P$ is the orthogonal projection to $M$.*

**Proof.** Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. Since $U$ is bounded from below, its range $M = \mathrm{R}(U)$ is a closed in $\ell^2$. Denote by $P \in \mathbb{B}(\ell^2)$ the orthogonal projection to $M$. We know from Proposition 2.1.16 that $(Pe_n)_n$ is a Parseval frame for $M$.

Note that $M^\perp = \mathrm{N}(U^*)$. Hence, if $(c_n)_n$ is any sequence of scalars, we have

$$U^*((c_n)_n) = U^*(P(c_n)_n + (I-P)(c_n)_n) = U^*(P(c_n)_n).$$

In particular, we have

$$x_n = U^*e_n = U^*Pe_n, \ \forall n \in \mathbb{N}.$$

It remains to observe that $U^*$ is an invertible operator when regarded as an operator from $M$ to $H$. □

We are now ready for the first key theorem of this section.

**Theorem 2.4.5.** *Let $(x_n)_n$ be a Besselian frame for a Hilbert space $H$ with the analysis operator $U$. Then $\dim \mathrm{N}(U^*) < \infty$.*

**Proof.** Consider the closed subspace $M = \mathrm{R}(U)$ of $\ell^2$ and the orthogonal projection $P$ to $M$. We know from Lemma 2.4.4 that, if $(e_n)_n$ denotes the canonical basis for $\ell^2$, the sequence $(Pe_n)_n$ is a Parseval frame for $M$ that is similar to $(x_n)_n$. Moreover, the analysis operator of $(Pe_n)_n$ is precisely the inclusion $M \hookrightarrow H$. Notice that $(Pe_n)_n$ is also Besselian, being similar to $(x_n)_n$.

We must show that $\dim M^\perp < \infty$.

We prove by contradiction: suppose that $\dim M^\perp = \infty$. We shall now construct a sequence of scalars $(c_n)_n$ that is not in $\ell^2$, but for which the series $\sum_{n=1}^\infty c_n Pe_n$ converges.

Let $(\varphi_n)_n$ ba an ONB for $M^\perp$. Since

$$\varphi_1 = \sum_{n=1}^\infty \langle \varphi_1, e_n\rangle e_n \quad \text{and} \quad 0 = P\varphi_1 = \sum_{n=1}^\infty \langle \varphi_1, e_n\rangle Pe_n,$$

we have

$$\lim_{N\to\infty} \left\| \sum_{n=1}^N \langle \varphi_1, e_n\rangle e_n \right\| = 1 \quad \text{and} \quad \lim_{N\to\infty} \left\| \sum_{n=1}^N \langle \varphi_1, e_n\rangle Pe_n \right\| = 0. \tag{32}$$

Put $N_0 = 0$ and choose any $N_1$ such that $N_1 > N_0$ and

$$\left\| \sum_{n=1}^{N_1} \langle \varphi_1, e_n\rangle e_n \right\| > \frac{1}{\sqrt{2}} \quad \text{and} \quad \left\| \sum_{n=1}^{N_1} \langle \varphi_1, e_n\rangle Pe_n \right\| < \frac{1}{4}. \tag{33}$$

Observe that we have for any $m$

$$\left\| \varphi_m - \sum_{n=N_1+1}^{\infty} \langle \varphi_m, e_n\rangle e_n \right\|^2 = \left\| \sum_{n=1}^{N_1} \langle \varphi_m, e_n\rangle e_n \right\|^2 = \sum_{n=1}^{N_1} |\langle \varphi_m, e_n\rangle|^2.$$

Since $(\varphi_m)_m$ converges weakly to 0, we also have

$$\lim_{m\to\infty} \sum_{n=1}^{N_1} |\langle \varphi_m, e_n\rangle|^2 = 0. \tag{34}$$

Put $m_1 = 1$. It follows from (34) that there exists $m_2 > m_1$ such that

$$\left(\sum_{n=1}^{N_1} |\langle \varphi_{m_2}, e_n\rangle|^2\right)^{\frac{1}{2}} = \left\|\varphi_{m_2} - \sum_{n=N_1+1}^{\infty} \langle \varphi_{m_2}, e_n\rangle e_n\right\| < \frac{1}{16}. \tag{35}$$

Now we claim that

$$\left\|\sum_{n=N_1+1}^{\infty} \langle \varphi_{m_2}, e_n\rangle e_n\right\| > \frac{1}{\sqrt{2}}. \tag{36}$$

To see this, suppose the opposite, i.e. $\left\|\sum_{n=N_1+1}^{\infty} \langle \varphi_{m_2}, e_n\rangle e_n\right\| \leq \frac{1}{\sqrt{2}}$. This implies

$$1 = \|\varphi_{m_2}\| \leq \left\|\varphi_{m_2} - \sum_{n=N_1+1}^{\infty} \langle \varphi_{m_2}, e_n\rangle e_n\right\| + \left\|\sum_{n=N_1+1}^{\infty} \langle \varphi_{m_2}, e_n\rangle e_n\right\| \overset{(35)}{<} \frac{1}{16} + \frac{1}{\sqrt{2}}$$

which is a contradiction.

Thus, we have (36). From (36) we conclude that there exists $N_2' > N_1$ such that

$$N \geq N_2' \Longrightarrow \left\|\sum_{n=N_1+1}^{N} \langle \varphi_{m_2}, e_n\rangle e_n\right\| > \frac{1}{\sqrt{2}}. \tag{37}$$

Now $0 = P\varphi_{m_2} = \sum_{n=1}^{\infty} \langle \varphi_{m_2}, e_n\rangle Pe_n$ implies

$$\begin{aligned}
\left\|\sum_{n=N_1+1}^{\infty} \langle \varphi_{m_2}, e_n\rangle Pe_n\right\| &= \left\|\sum_{n=1}^{\infty} \langle \varphi_{m_2}, e_n\rangle Pe_n - \sum_{n=1}^{N_1} \langle \varphi_{m_2}, e_n\rangle Pe_n\right\| \\
&= \left\|\sum_{n=1}^{N_1} \langle \varphi_{m_2}, e_n\rangle Pe_n\right\| \\
&\leq \left\|\sum_{n=1}^{N_1} \langle \varphi_{m_2}, e_n\rangle e_n\right\| \\
&= \left(\sum_{n=1}^{N_1} |\langle \varphi_{m_2}, e_n\rangle|^2\right)^{\frac{1}{2}} \overset{(35)}{<} \frac{1}{16}.
\end{aligned}$$

Hence, there exists $N_2'' > N_1$ such that

$$N \geq N_2'' \Longrightarrow \left\|\sum_{n=N_1+1}^{N} \langle \varphi_{m_2}, e_n\rangle Pe_n\right\| < \frac{1}{8}. \tag{38}$$

From (37) and (38) we conclude that there exists $N_2$ (in fact, each $N_2 \geq N_2', N_2''$ will be good) for which we have

$$\left\| \sum_{n=N_1+1}^{N_2} \langle \varphi_{m_2}, e_n \rangle e_n \right\| > \frac{1}{\sqrt{2}} \quad \text{and} \quad \left\| \sum_{n=N_1+1}^{N_2} \langle \varphi_{m_2}, e_n \rangle P e_n \right\| < \frac{1}{8}. \tag{39}$$

So, by now we have $m_1 = 1, m_2$, and also $N_0 = 0, N_1, N_2$ such that

$$\left\| \sum_{n=1}^{N_1} \langle \varphi_{m_1}, e_n \rangle e_n \right\| > \frac{1}{\sqrt{2}}, \quad \left\| \sum_{n=1}^{N_1} \langle \varphi_{m_1}, e_n \rangle P e_n \right\| < \frac{1}{4},$$

$$\left\| \sum_{n=N_1+1}^{N_2} \langle \varphi_{m_2}, e_n \rangle e_n \right\| > \frac{1}{\sqrt{2}}, \quad \left\| \sum_{n=N_1+1}^{N_2} \langle \varphi_{m_2}, e_n \rangle P e_n \right\| < \frac{1}{8}.$$

Again, since $(\varphi_m)_m$ converges weakly to 0, we have

$$\lim_{m \to \infty} \sum_{n=1}^{N_2} |\langle \varphi_m, e_n \rangle|^2 = 0.$$

Hence, there exists $m_3 > m_2$ such that

$$\left( \sum_{n=1}^{N_2} |\langle \varphi_{m_3}, e_n \rangle|^2 \right)^{\frac{1}{2}} = \left\| \varphi_{m_3} - \sum_{n=N_2+1}^{\infty} |\langle \varphi_{m_3}, e_n \rangle \right\| < \frac{1}{32}. \tag{40}$$

Now we claim that

$$\left\| \sum_{n=N_2+1}^{\infty} \langle \varphi_{m_3}, e_n \rangle e_n \right\| > \frac{1}{\sqrt{2}}. \tag{41}$$

To see this, suppose the opposite, i.e. $\left\| \sum_{n=N_2+1}^{\infty} \langle \varphi_{m_3}, e_n \rangle e_n \right\| \leq \frac{1}{\sqrt{2}}$. This implies

$$1 = \|\varphi_{m_3}\| \leq \left\| \varphi_{m_3} - \sum_{n=N_2+1}^{\infty} \langle \varphi_{m_3}, e_n \rangle e_n \right\| + \left\| \sum_{n=N_2+1}^{\infty} \langle \varphi_{m_2}, e_n \rangle e_n \right\| \overset{(40)}{<} \frac{1}{32} + \frac{1}{\sqrt{2}}$$

which is a contradiction. So, we do have (41). It follows from (41) that there exists $N_3' > N_2$ such that

$$N \geq N_3' \Longrightarrow \left\| \sum_{n=N_2+1}^{N} \langle \varphi_{m_3}, e_n \rangle e_n \right\| > \frac{1}{\sqrt{2}}. \tag{42}$$

Now $0 = P\varphi_{m_3} = \sum_{n=1}^{\infty}\langle\varphi_{m_3}, e_n\rangle Pe_n$ implies

$$\begin{aligned}
\left\|\sum_{n=N_2+1}^{\infty}\langle\varphi_{m_3}, e_n\rangle Pe_n\right\| &= \left\|\sum_{n=1}^{\infty}\langle\varphi_{m_3}, e_n\rangle Pe_n - \sum_{n=1}^{N_2}\langle\varphi_{m_3}, e_n\rangle Pe_n\right\| \\
&= \left\|\sum_{n=1}^{N_2}\langle\varphi_{m_3}, e_n\rangle Pe_n\right\| \\
&\leq \left\|\sum_{n=1}^{N_2}\langle\varphi_{m_3}, e_n\rangle e_n\right\| \\
&= \left(\sum_{n=1}^{N_2}|\langle\varphi_{m_3}, e_n\rangle|^2\right)^{\frac{1}{2}} \overset{(40)}{<} \frac{1}{32}.
\end{aligned}$$

Hence, there exists $N_3'' > N_2$ such that

$$N \geq N_3'' \Longrightarrow \left\|\sum_{n=N_2+1}^{N}\langle\varphi_{m_3}, e_n\rangle Pe_n\right\| < \frac{1}{16}. \tag{43}$$

From (42) and (43) we conclude that there exists $N_3$ (in fact, we can choose any $N_3 \geq N_3', N_3''$) for which we have

$$\left\|\sum_{n=N_2+1}^{N_3}\langle\varphi_{m_3}, e_n\rangle e_n\right\| > \frac{1}{\sqrt{2}} \quad \text{and} \quad \left\|\sum_{n=N_2+1}^{N_3}\langle\varphi_{m_3}, e_n\rangle Pe_n\right\| < \frac{1}{16}. \tag{44}$$

Continuing by induction, we obtain sequences $(N_K)_{K=0}^{\infty}$ and $(m_K)_{K=1}^{\infty}$ for which we have, for all $K \geq 0$,

$$\left\|\sum_{n=N_K+1}^{N_{K+1}}\langle\varphi_{m_{K+1}}, e_n\rangle e_n\right\| > \frac{1}{\sqrt{2}}, \tag{45}$$

$$\left\|\sum_{n=N_K+1}^{N_{K+1}}\langle\varphi_{m_{K+1}}, e_n\rangle Pe_n\right\| < \frac{1}{2^{K+2}}. \tag{46}$$

In addition, we also have

$$\left\|\sum_{n=N_K+1}^{R}\langle\varphi_{m_{K+1}}, e_n\rangle Pe_n\right\| \leq \left\|\sum_{n=N_K+1}^{R}\langle\varphi_{m_{K+1}}, e_n\rangle e_n\right\| \leq 1, \quad \forall R \geq N_K + 1. \tag{47}$$

Consider now the sequence $(c_n)_n$ where

$$c_1, c_2, \ldots, c_{N_1} \quad \text{are defined as} \quad \frac{1}{\sqrt{1}}\langle\varphi_{m_1}, e_n\rangle,\ n = N_0 + 1 = 1, N_0 + 2 = 2, \ldots, N_1,$$

$$c_{N_1+1}, c_{N_1+2} \ldots, c_{N_2} \quad \text{are defined as} \quad \frac{1}{\sqrt{2}}\langle\varphi_{m_2}, e_n\rangle,\ n = N_1 + 1, N_1 + 2, \ldots, N_2,$$

...

$c_{N_K+1}, c_{N_K+2} \ldots, c_{N_{K+1}}$ are defined as $\dfrac{1}{\sqrt{K+1}}\langle \varphi_{m_{K+1}}, e_n\rangle,\ n = N_K+1, N_K+2, \ldots, N_{K+1}.$

Then we have for every $K$

$$\sum_{n=N_K+1}^{N_{K+1}} |c_n|^2 = \sum_{n=N_K+1}^{N_{K+1}} \frac{1}{K+1}|\langle \varphi_{m_{K+1}}, e_n\rangle|^2 \overset{(45)}{>} \frac{1}{2}\frac{1}{K+1}$$

so, $\sum_{n=1}^{\infty}|c_n|^2$ clearly diverges.

On the other hand, we claim that $\sum_{n=1}^{\infty} c_n Pe_n$ converges. We shall show that the associated sequence of partial sums is a Cauchy sequence.

So, take any $\epsilon > 0$ and choose $K_0$ such that $\frac{3}{\sqrt{K_0+1}} < \epsilon$. Let $R > N \geq N_{K_0}$. We first find $K$ and $L$ such that

$$N_K \leq N < N_{K+1} \quad \text{and} \quad N_{K+L} + 1 \leq R < N_{K+L+1}.$$

Then we have

$$\left\|\sum_{n=1}^{R} c_n Pe_n - \sum_{n=1}^{N} c_n Pe_n\right\| = \left\|\sum_{n=N+1}^{R} c_n Pe_n\right\| =$$

$$\left\|\sum_{n=N+1}^{N_{K+1}} c_n Pe_n\right\| + \left\|\sum_{n=N_{K+1}+1}^{N_{K+2}} c_n Pe_n\right\| + \ldots + \left\|\sum_{n=N_{K+L-1}+1}^{N_{K+L}} c_n Pe_n\right\| + \left\|\sum_{n=N_{K+L}+1}^{R} c_n Pe_n\right\|. \quad (48)$$

The first term in (48) is estimated in the following way:

$$\begin{aligned}
\left\|\sum_{n=N+1}^{N_{K+1}} c_n Pe_n\right\| &= \left\|\sum_{n=N_K+1}^{N_{K+1}} c_n Pe_n - \sum_{n=N_K+1}^{N} c_n Pe_n\right\| \\
&= \left\|\sum_{n=N_K+1}^{N_{K+1}} \frac{1}{\sqrt{K+1}}\langle \varphi_{m_{K+1}}, e_n\rangle Pe_n - \sum_{n=N_K+1}^{N} \frac{1}{\sqrt{K+1}}\langle \varphi_{m_{K+1}}, e_n\rangle Pe_n\right\| \\
&\overset{(46),(47)}{\leq} \frac{1}{\sqrt{K+1}}\frac{1}{2^{K+2}} + \frac{1}{\sqrt{K+1}}. \qquad (49)
\end{aligned}$$

Similarly, considering the last term in (48), we have

$$\left\|\sum_{n=N_{K+L}+1}^{R} c_n Pe_n\right\| = \left\|\sum_{n=N_{K+L}+1}^{R} \frac{1}{\sqrt{K+L+1}}\langle \varphi_{m_{K+L+1}}, e_n\rangle Pe_n\right\| \overset{(47)}{\leq} \frac{1}{\sqrt{K+L+1}}. \quad (50)$$

Finally, all the terms in (48) between the first and the last one are estimated using (46). Taking this into account together with (49) and (50), we now continue our computation from (48):

$$\left\|\sum_{n=1}^{R} c_n Pe_n - \sum_{n=1}^{N} c_n Pe_n\right\| = \left\|\sum_{n=N+1}^{R} c_n Pe_n\right\| =$$

$$\left\|\sum_{n=N+1}^{N_{K+1}} c_n P e_n\right\| + \left\|\sum_{n=N_{K+1}+1}^{N_{K+2}} c_n P e_n\right\| + \ldots + \left\|\sum_{n=N_{K+L-1}+1}^{N_{K+L}} c_n P e_n\right\| + \left\|\sum_{n=N_{K+L}+1}^{R} c_n P e_n\right\| \leq$$

$$\frac{1}{\sqrt{K+1}}\frac{1}{2^{K+2}} + \frac{1}{\sqrt{K+1}} + \frac{1}{\sqrt{K+2}}\frac{1}{2^{K+3}} + \ldots + \frac{1}{\sqrt{K+L}}\frac{1}{2^{K+L+1}} + \frac{1}{\sqrt{K+L+1}} \leq$$

$$\frac{1}{\sqrt{K+1}} + \frac{1}{\sqrt{K+L+1}} + \frac{1}{\sqrt{K+1}}\left(\frac{1}{2^{K+2}} + \ldots + \frac{1}{2^{K+L+1}}\right) \leq \frac{3}{\sqrt{K+1}} \leq \frac{3}{\sqrt{K_0+1}} < \epsilon.$$

□

To proceed with our analysis of near-Riesz bases and Bessel frames we need a simple auxiliary result on perturbations of ONB's.

**Lemma 2.4.6.** *Let $(e_n)_n$ be an ONB for a Hilbert space $H$. Suppose that a sequence $(z_n)_n$ in $H$ is such that $\sum_{n=1}^{\infty}\|e_n - z_n\|^2 < 1$. Then $(z_n)_n$ is a Riesz basis for $H$.*

**Proof.** Put $\sum_{n=1}^{\infty}\|e_n - z_n\|^2 = m^2$; by the assumption we have $m < 1$. We need to show that the operator $V$ on $H$ defined by $Ve_n = z_n$, $n \in \mathbb{N}$, is a well defined bounded invertible operator on $H$. To do that, it suffices to prove that $I - V$ is bounded operator which satisfies $\|I - V\| \leq m$.

Now we have, for any finite sum $y = \sum_{n=1}^{N} c_n e_n$,

$$\begin{aligned} \|(I-V)y\| &= \left\|\sum_{n=1}^{N} c_n(e_n - z_n)\right\| \\ &\leq \sum_{n=1}^{N} |c_n| \cdot \|e_n - z_n\| \\ &\leq \left(\sum_{n=1}^{N}|c_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N}\|e_n - z_n\|^2\right)^{\frac{1}{2}} \\ &\leq m\left(\sum_{n=1}^{N}|c_n|^2\right)^{\frac{1}{2}} = m\|y\|. \end{aligned}$$

This is enough to conclude that $I - V \in \mathbb{B}(\ell^2)$ and $\|I - V\| \leq m < 1$. □

We are now ready for the characterization of near-Riesz bases.

**Theorem 2.4.7.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. Then the following conditions are equivalent:*

*(a)* *$(x_n)_n$ is a near-Riesz basis;*

*(b)* *$(x_n)_n$ is Besselian;*

*(c)* *$\dim N(U^*) < \infty$.*

**Proof.** $(a) \Rightarrow (b)$ is already observed in Remark 2.4.2 and $(b) \Rightarrow (c)$ is Theorem 2.4.5. So, it only remains to prove $(c) \Rightarrow (a)$.

Suppose that $\dim \mathrm{N}(U^*) < \infty$. Denote again by $M$ the range $\mathrm{R}(U)$ of $U$. Note that $\mathrm{N}(U^*) = \mathrm{R}(U)^\perp = M^\perp$. Let $P \in \mathbb{B}(\ell^2)$ be the orthogonal projection to $M$. Let $(e_n)_n$ be the canonical basis for $\ell^2$. Recall that $(Pe_n)_n$ is a frame for $M$ and that the analysis operator of $(Pe_n)_n$ is the inclusion $M \hookrightarrow \ell^2$. Since $(Pe_n)_n$ is similar to $(x_n)_n$, it suffices to show that $(Pe_n)_n$ is a near-Riesz basis for $M$.

By the assumption we have $\dim M^\perp < \infty$ so, the rank of $I-P$ is finite; in particular, $I-P$ is a Hilbert-Schmidt operator. Thus,

$$\sum_{n=1}^{\infty} \|(I-P)e_n\|^2 < \infty.$$

From this we conclude that there exists $N$ such that

$$\sum_{n=N+1}^{\infty} \|e_n - Pe_n\|^2 < 1.$$

Define the sequence $(z_n)_n$ in $\ell^2$ by

$$z_n = \left\{ \begin{array}{cl} e_n, & n = 1, 2, \ldots, N \\ Pe_n, & n = N+1, N+2, \ldots \end{array} \right. .$$

Clearly, we now have

$$\sum_{n=1}^{\infty} \|e_n - z_n\|^2 < 1. \tag{51}$$

By Lemma 2.4.6, $(z_n)_n$ is a Riesz basis for $\ell^2$. Clearly, $(z_n)_{n\geq N+1} = (Pe_n)_{n\geq N+1}$ is then a Riesz basis for its closed linear span. Observe that, since $(Pe_n)_n$ is a frame for $M$, the co-dimension of $\overline{\mathrm{span}}\,\{Pe_n : n \geq N+1\}$ in $M$ is finite. It is now easy to conclude that there is a finite set $S$ contained in the set $\{1, 2, \ldots N\}$ such that the sequence $(Pe_n)_{n\in S} \cup (Pe_n)_{n\geq N+1}$ is a Riesz basis for $M$.

□

**Remark 2.4.8.** The difficult part of the proceeding proof - the implication $(b) \Rightarrow (c)$ - is in fact the content of Theorem 2.4.5.

Here we note that the equivalence of (a) and (c) in Theorem 2.4.5 can be obtained much easier if one avoids involving the Besselian property.

Indeed, $(c) \Rightarrow (a)$ is already demonstrated in the preceding proof. Let us prove directly that $(a) \Rightarrow (c)$.

So, suppose that $(x_n)_n$ is a near-Riesz basis for $H$ and denote again by $U$ its analysis operator. We may assume without loss of generality that $(x_n)_{n\geq k+1}$ is a Riesz basis for $H$. Denote by $(e_n)_n$ the canonical basis for $\ell^2$. Denote by $T \in \mathbb{B}(\ell^2, H)$ the invertible operator which satisfies $Te_n = x_{k+n}$ for all $n$. Let $S \in \mathbb{B}(\ell^2)$ be the unilateral shift. Since we have $U^*e_n = x_n$ for every $n$, it follows that $T = U^*S^k$. Let $M_k = \mathrm{span}\,\{e_1, e_2, \ldots e_k\}$. Clearly, $M_k = \mathrm{N}((S^k)^*)$ and $\mathrm{R}(S^k) = M_k^\perp$.

We now claim that $M_k^\perp \cap \mathrm{N}(U^*) = \{0\}$. To see this, take any $x \in M_k^\perp \cap \mathrm{N}(U^*)$. Since $\mathrm{R}(S^k) = M_k^\perp$, there exists $v$ such that $x = S^k v$. But then we have $0 = U^*x = U^*S^k v = Tv$ which implies $v = 0$ by invertibility of $T$.

Denote by $P_k$ the orthogonal projection to $M_k$. The preceding conclusion now implies that $P_k : \mathrm{N}(U^*) \to M_k$ is an injection. Indeed, if $P_k x = 0$ for some $x \in \mathrm{N}(U^*)$ then, clearly, $x \in M_k^\perp \cap \mathrm{N}(U^*) = \{0\}$; hence, $x = 0$.

Thus, $\mathrm{N}(U^*)$ is embedded into $M_k$ and hence it must be finite-dimensional.

The reason for choosing the extended version of Theorem 2.4.7 is the following theorem which tells us that all properties of frames introduced in Definition 2.4.1 are in fact equivalent.

**Theorem 2.4.9.** *Let $(x_n)_n$ ba a frame for a Hilbert space $H$ with the analysis operator $U$. The following statements are all equivalent:*

- *(a) $(x_n)_n$ is a near-Riesz basis;*
- *(b) $(x_n)_n$ is Besselian;*
- *(c) dim $\mathrm{N}(U^*) < \infty$;*
- *(d) $(x_n)_n$ is unconditional.*

**Proof.** We only need to prove that (d) is equivalent with any (and hence all) of the other three properties. Recall that $(b) \Rightarrow (d)$ is observed in Remark 2.4.2; so it suffices to prove $(d) \Rightarrow (b)$.

Here we provide the proof under the additional assumption that $(x_n)_n$ is bounded from below. Choose $m > 0$ such that $\|x_n\| \geq m$ for all $n$. Suppose we have a sequence of scalars $(c_n)_n$ such that $\sum_{n=1}^\infty c_n x_n$ converges. By the hypothesis, this series converges unconditionally. By Orlicz'c theorem we now have $\sum_{n=1}^\infty \|c_n x_n\|^2 < \infty$. This implies that $m^2 \sum_{n=1}^\infty |c_n|^2 \leq \sum_{n=1}^\infty |c_n|^2 \|x_n\|^2 = \sum_{n=1}^\infty \|c_n x_n\|^2 < \infty$; thus, $(c_n)_n$ is an $\ell^2$-sequence. $\square$

*Concluding remarks.* Almost all results of this section are obtained in [87]. The only two exemptions are Remark 2.4.8 for which we do not have a reference (but is certainly known to the experts) and implication $(d) \Rightarrow (b)$ from Theorem 2.4.9 which is in general case, without assuming that the frame under consideration is bounded below, proved in [36].

**Exercise 2.4.10.** Provide the details in the final argument of the proof of Theorem 2.4.7.

## 2.5 Excesses of frames

The excess of a frame $(x_n)_n$ is defined as the greatest number (possibly $\infty$) of elements that can be removed from $(x_n)_n$ yet leave the fundamental sequence in $H$.

**Definition 2.5.1.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$. The excess of $(x_n)_n$ is defined as*

$$e((x_n)_n) = \sup\{card(S) : \overline{span}\{x_n : n \notin S\} = H\}.$$

If $\mathrm{e}((x_n)_n) = m \in \mathbb{N}$, then it is evident from the definition that for any $k$, $1 \le k \le m$, one can find a set of indices $T$ with $\mathrm{card}(T) = k$ such that $\overline{\mathrm{span}}\{x_n : n \notin T\} = H$. Similarly, if $\mathrm{e}((x_n)_n) = \infty$ one can find such a set $T$ with $\mathrm{card}(T) = k$ for any $k \in \mathbb{N}$.

Theorem 2.5.5 below is the fundamental result on excesses of frames. We first need two auxiliary results. The first one is a lemma which should be compared to Remark 2.4.8.

**Lemma 2.5.2.** *Let $(x_n)_n$ be a near-Riesz basis for a Hilbert space $H$ with the analysis operator $U$. Suppose that $S$ is a finite set of indices for which $(x_n)_{n\in\mathbb{N}\setminus S}$ is a Riesz basis for $H$. Then $card(S) = dim\,N(U^*)$.*

**Proof.** First, by Theorem 2.4.7 we know that $\dim \mathrm{N}(U^*) < \infty$. Denote again by $M$ the range of $U$ and observe that $\mathrm{N}(U^*) = M^\perp$. Let $P \in \mathbb{B}(\ell^2)$ be the orthogonal projection to $M$ and let $(e_n)_n$ denote the canonical basis for $\ell^2$. Since by Lemma 2.4.4 the frame $(Pe_n)_n$ for $M$ is similar to $(x_n)_n$, it follows that $(Pe_n)_{n\in\mathbb{N}\setminus S}$ is a Riesz basis for $M$. In particular, $P|_{\overline{\mathrm{span}}\{e_n : n\in\mathbb{N}\setminus S\}} : \overline{\mathrm{span}}\{e_n : n \in \mathbb{N}\setminus S\} \to M$ is invertible (because each operator of Hilbert spaces that maps an ONB to a Riesz basis must be invertible).

For each $x \in \ell^2$ we have $Px \in M$, so there exists a unique $y \in \overline{\mathrm{span}}\{e_n : n \in \mathbb{N}\setminus S\}$ such that $Py = Px$, that is $x - y \in \mathrm{N}(P) = M^\perp$. This shows that

$$\ell^2 = \mathrm{N}(P) \dot{+} \overline{\mathrm{span}}\{e_n : n \in \mathbb{N}\setminus S\}.$$

(The sum is direct since $P|_{\overline{\mathrm{span}}\{e_n : n\in\mathbb{N}\setminus S\}}$ is an injection.) Finally, since dimensions of all direct complements of a closed subspace are equal, we conclude that

$$\dim(\mathrm{N}(U^*)) = \dim(\mathrm{N}(P)) = \dim((\overline{\mathrm{span}}\{e_n : n \in \mathbb{N}\setminus S\})^\perp) = \dim(\mathrm{span}\{e_n : n \in S\}) = \mathrm{card}(S).$$

□

**Lemma 2.5.3.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$. Suppose that $S$ is a finite set of indices such that $\overline{span}\{x_n : n \in \mathbb{N}\setminus S\} = H$. Then $(x_n)_{n\in\mathbb{N}\setminus S}$ is a frame for $H$.*

**Proof.** We may assume without loss of generality that $S = \{1, 2, \dots, k\}$, $k \in \mathbb{N}$. Obviously, $(x_n)_{n>k}$ is a Bessel sequence. Its analysis operator $U_1$ is given by $U_1 = (S^*)^k U$, where $S \in \mathbb{B}(\ell^2)$ is the unilateral shift, and $U$ is the analysis operator of $(x_n)_n$. Let $V$ be the analysis operator of any dual frame for $(x_n)_n$. Then we have $V^*U = I$. Put $V_1 = (S^*)^k V$. Denote by $(e_n)_n$ the canonical basis for $\ell^2$ and by $P_k$ the orthogonal projection to $\mathrm{span}\{e_1, e_2, \dots, e_k\}$. Then $V_1^*U_1 = V^*S^k(S^*)^kU = V^*(I - P_k)U = I - V^*P_kU$; thus, $I - V_1^*U_1$ is a compact operator. By Problem 181 in [76], $U_1$ has closed range. Hence, by Proposition 2.1.7, $U_1^*$ has closed range. By assumption the range of $U_1^*$ is dense in $H$, so $U_1^*$ is a surjection. This implies that $(x_n)_{n>k}$ is a frame. □

**Remark 2.5.4.** (a) Observe that Lemma 2.5.3 can alternatively be deduced from Proposition 2.3.5 (c),(d) (by applying these assertions finitely many times).

(b) Here we provide yet another proof of Lemma 2.5.3.

Suppose we have a frame $(x_n)_n$ for a Hilbert space $H$ such that $\overline{\text{span}}\,\{x_n : n > k\} = H$.

Assume first that $(x_n)_n$ is a Parseval frame. Suppose that $(x_n)_{n>k}$ is not a frame for $H$. We now claim that the operator $G \in \mathbb{B}(H)$ defined by $Gx = \sum_{n=k+1}^{\infty} \langle x, x_n \rangle x_n$ is not invertible. To see this, denote by $U_1$ the analysis operator of the sequence $(x_n)_{n>k}$ and observe that $G = U_1^* U_1$. Now, if $G$ is invertible, then $U_1^*$ is a surjection and hence $(x_n)_{n>k}$ is a frame for $H$ which is a contradiction.

Consider now the operator $E$ defined by $Ex = \sum_{n=1}^{k} \langle x, x_n \rangle x_n$. Since $(x_n)_n$ is a Parseval frame, we have $x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$ for all $x$ in $H$; hence, we have $E + G = I$. This shows us that $I - E$ is not invertible; thus, $1 \in \sigma(E)$. Since $E$ has finite rank, we conclude that 1 is an eigenvalue of $E$. Therefore there exists $x_0 \in H$ such that $\|x_0\| = 1$ and $Ex_0 = x_0$ (and also $Gx_0 = 0$).

Denote by $P$ the orthogonal projection to span$\,\{x_0\}$. Clearly, we have $PGP = 0$. On the other hand, we have $PGPx = \sum_{n=k+1}^{\infty} \langle x, Px_n \rangle Px_n$ for all $x$ in $H$. Here we have an increasing sequence of positive operators (the corresponding partial sums) that converges in the strong operator topology. Since all partial sums are dominated by the strong limit (which is in this situation the zero operator), we conclude that $\langle x, Px_n \rangle Px_n = 0$ for all $x$ and all $n > k$. This implies $Px_n = 0$ for all $n > k$. Thus, $x_0 \perp x_n$ for all $n > k$ which is a contradiction with our assumption that $\overline{\text{span}}\,\{x_n : n > k\} = H$.

In the general case when our frame $(x_n)_n$ is not Parseval we can work with the associated Parseval frame $((U^*U)^{-\frac{1}{2}} x_n)_n$ and apply the conclusion of the preceding discussion.

(c) However, the conclusion Lemma 2.5.3 is not correct if a "redundant set of indices" is infinite. More precisely: if $(x_n)_n$ is a frame for which there exists an infinite set $S \subset \mathbb{N}$ with the property $\overline{\text{span}}\,\{x_n : n \in \mathbb{N} \setminus S\} = H$, the the sequence $(x_n)_{n \in \mathbb{N} \setminus S}$ need not be a frame.

As an example, consider an orthonormal basis $(e_n)_n$ in a Hilbert space $H$ and the frame $e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots$. It is evident that its subsequence $(\frac{1}{\sqrt{n}}e_n)_n$ is fundamental in $H$, but not a frame for $H$.

**Theorem 2.5.5.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. Then $e((x_n)_n) = \dim \mathrm{N}(U^*)$.*

**Proof.** Suppose that $\{y_1, y_2, \ldots, y_m\}$ is linearly independent set in $\mathrm{N}(U^*) \leq \ell^2$. If $(e_n)_n$ denotes the canonical basis for $\ell^2$, we may write

$$y_j = (y_{ji})_{i=1}^{\infty} = \sum_{i=1}^{\infty} y_{ji} e_i, \;\; j = 1, 2, \ldots, m.$$

Note that the equality $U^* y_j = 0$ for $j = 1, 2, \ldots, m$, can be written as

$$\sum_{i=1}^{\infty} y_{ji} x_i = 0, \;\; j = 1, 2, \ldots, m,$$

or, in terms of infinite matrices, as

$$\begin{bmatrix} y_{11} & y_{12} & \dots \\ \vdots & \vdots & \\ y_{m1} & y_{m2} & \dots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The $m \times \infty$ matrix in the above equation has $m$ linearly independent rows and hence has $m$ linearly independent columns (this is seen by the argument used for finite matrices). Let us denote by $k_1, k_2, \dots, k_m$ the indices of $m$ linearly independent columns. We claim that the sequence $(x_n)_{n \neq k_1,\dots,k_m}$ is fundamental in $\overline{\operatorname{span}}\,\{x_n : n \in \mathbb{N}\} = H$. Note that this will imply $\mathrm{e}((x_n)_n) \geq m$ and hence $\mathrm{e}((x_n)_n) \geq \dim \mathrm{N}(U^*)$.

Suppose that $h \in \overline{\operatorname{span}}\,\{x_n : n \in \mathbb{N}\} = H$ satisfies $\langle h, x_n \rangle = 0$ for all $n \neq k_1, \dots, k_m$. We want to conclude that $h = 0$. Observe that

$$0 = \langle U^* y_j, h \rangle = \sum_{i=1}^{\infty} y_{ji} \langle x_i, h \rangle = \sum_{i=1}^{m} y_{jk_i} \langle x_{k_i}, h \rangle, \ \ \forall j = 1, \dots, m,$$

that is,

$$\begin{bmatrix} y_{1k_1} & \dots & y_{1k_m} \\ \vdots & & \vdots \\ y_{mk_1} & \dots & y_{mk_m} \end{bmatrix} \begin{bmatrix} \langle x_{k_1}, h \rangle \\ \vdots \\ \langle x_{k_m}, h \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

However, the matrix of the system is invertible, so this implies $\langle x_{k_1}, h \rangle = \ldots = \langle x_{k_m}, h \rangle = 0$. This, together with the assumption on $h$ gives us $h = 0$. So, we have proved that $\mathrm{e}((x_n)_n) \geq \dim \mathrm{N}(U^*)$.

It is now clear that $\dim \mathrm{N}(U^*) = \infty$ implies $\mathrm{e}((x_n)_n) = \infty$.

Suppose now that $\dim \mathrm{N}(U^*) = k \in \mathbb{N}$. By Theorem 2.4.7, $(x_n)_n$ is then a near-Riesz basis for $H$. Suppose that $S$ is a finite set of indices such that $(x_n)_{n \in \mathbb{N} \setminus S}$ is a Riesz basis for $H$. Now Lemma 2.5.2 gives us $\mathrm{card}(S) = \dim \mathrm{N}(U^*) = k$. We must show that $\mathrm{e}((x_n)_n) = k$ and by the first part of the proof it suffices to show that $\mathrm{e}((x_n)_n) \leq k$.

To see this, suppose the opposite: $\mathrm{e}((x_n)_n) > k$. By the observation following Definition 2.5.1, then there exists a set of indices $T \subset \mathbb{N}$ such that $\mathrm{card}(T) = k + 1$ such that the frame members $x_n$, $n \in T$, can be removed from $(x_n)_n$ and yet leave the fundamental sequence.

We can assume without loss of generality that $S = \{1, 2, \dots, k\}$. Since $(x_n)_{n \in \mathbb{N} \setminus S}$ is a Riesz basis for $H$, there exists an invertible operator $W \in \mathbb{B}(H)$ and an ONB $(e_n)_n$ for $H$ such that $W x_{k+n} = e_n$ for every $n \in \mathbb{N}$. The image of our frame $(x_n)_n$ under the action of $W$ is the sequence $(h_1, h_2, \dots, h_k, e_1, e_2, \dots)$, where $h_1, \dots, h_k$ are some elements of $H$. The assumption on the set $T$ now implies that we can find $k+1$ elements of this sequence which can be removed without destroying the spanning property. This is, obviously, impossible.

□

**Remark 2.5.6.** If $(x_n)_n$ is only a Bessel sequence the excess $\mathrm{e}((x_n)_n)$ is defined as

$$\mathrm{e}((x_n)_n) = \sup\,\{\mathrm{card}(S) : \overline{\operatorname{span}}\,\{x_n : n \in \mathbb{N} \setminus S\} = \overline{\operatorname{span}}\,\{x_n : n \in \mathbb{N}\}\}.$$

We note that the first part of the proof of the preceding theorem shows that we have $\mathrm{e}((x_n)_n) \geq \dim \mathrm{N}(U^*)$, where $U$ denotes the corresponding analysis operator. There are Bessel sequences for which this inequality is strict (see Exercise 2.5.13).

We proceed with some useful results on excesses of frames.

**Proposition 2.5.7.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. Then $e((x_n)_n) = \sum_{n=1}^{\infty} \left(1 - \langle (U^*U)^{-1}x_n, x_n\rangle\right)$.*

**Proof.** Recall from Corollary 2.2.8 that $U(U^*U)^{-1}U^*$ is the orthogonal projection to $\mathrm{R}(U)$; thus, $I - U(U^*U)^{-1}U^*$ is the orthogonal projection to $\mathrm{N}(U^*)$. Denote again by $(e_n)_n$ the canonical basis for $\ell^2$. Using Theorem 2.5.5 we now have

$$\begin{aligned}
\mathrm{e}((x_n)_n) &= \dim \mathrm{N}(U^*) \\
&= \mathrm{tr}\left(I - U(U^*U)^{-1}U^*\right) \\
&= \sum_{n=1}^{\infty} \langle (I - U(U^*U)^{-1}U^*)e_n, e_n\rangle \\
&= \sum_{n=1}^{\infty} \left(\langle e_n, e_n\rangle - \langle (U^*U)^{-1}U^*e_n, U^*e_n,\rangle\right) \\
&= \sum_{n=1}^{\infty} \left(1 - \langle (U^*U)^{-1}x_n, x_n\rangle\right).
\end{aligned}$$

$\square$

**Corollary 2.5.8.** *Let $(x_n)_n$ be a Parseval frame for a Hilbert space $H$. Then $e((x_n)_n) = \sum_{n=1}^{\infty} \left(1 - \|x_n\|^2\right)$.*

**Proposition 2.5.9.** *Let $(x_n)_n$ and $(v_n)_n$ be frames for a Hilbert space $H$ that are dual to each other. Then $e((x_n)_n) = e((v_n)_n)$.*

**Proof.** Denote by $U$ and $V$ the corresponding analysis operators. We must prove that $\dim \mathrm{N}(U^*) = \dim \mathrm{N}(V^*)$. Since we have $V^*U = I$, Lemma 2.2.6 (a) (with $S = V^*$ and $T = U$) gives us

$$\mathrm{N}(V^*) = (I - UV^*)(\mathrm{N}(U^*)).$$

From this we conclude

$$\dim \mathrm{N}(V^*) = \dim\left((I - UV^*)(\mathrm{N}(U^*))\right) \leq \dim \mathrm{N}(U^*).$$

The opposite inequality follows by symmetry since $V^*U = I$ is equivalent to $U^*V = I$. $\square$

There are many properties of frames depending on or described in terms of their excesses. Here we include just one of such results, namely a characterization of frames that possess Parseval duals.

**Theorem 2.5.10.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the optimal frame bounds $A_{opt}$ and $B$ and the analysis operator $U$. Then $(x_n)_n$ possesses a Parseval dual if and only if the following two conditions are satisfied:*

*(a)* $A_{opt} \geq 1$,

*(b)* $dim\,(R(U^*U - I)) \leq e((x_n)_n)$.

**Proof.** Suppose first that $(v_n)_n$ is a Parseval dual for $(x_n)$ and denote its analysis operator by $V$. By Corollary 2.2.12 (c), $V$ is of the form $V = U(U^*U)^{-1} + QW$, where $Q \in \mathbb{B}(\ell^2)$ is the orthogonal projection to $\mathrm{R}(U)^\perp$ and $W \in \mathbb{B}(H, \ell^2)$ is arbitrary.

Since $(v_n)_n$ is a Parseval frame, we have $V^*V = I$ i.e.,

$$((U^*U)^{-1}U^* + W^*Q)(U(U^*U)^{-1} + QW) = I.$$

Since $QU = 0$ and $U^*Q = 0$, this gives us

$$(U^*U)^{-1} + W^*QW = I. \tag{52}$$

In particular, this implies $(U^*U)^{-1} \leq I$ and hence $U^*U \geq I$. This proves $A_{\mathrm{opt}} \geq 1$.

Furthermore, by multiplying (52) from both sides by $(U^*U)^{\frac{1}{2}}$ we obtain

$$U^*U - I = (U^*U)^{\frac{1}{2}}(W^*QW)(U^*U)^{\frac{1}{2}}.$$

Since $(U^*U)^{\frac{1}{2}}$ is an invertible operator, from this we conclude

$$\begin{aligned}
\dim\,(\mathrm{R}(U^*U - I)) &= \dim\,(\mathrm{R}(U^*U)^{\frac{1}{2}}(W^*QW)(U^*U)^{\frac{1}{2}})) \\
&= \dim\,(\mathrm{R}(W^*QW)) \\
&\leq \dim\,(\mathrm{R}(Q)) \\
&= \dim\,((\mathrm{R}(U)^\perp) \\
&= e((x_n)_n).
\end{aligned}$$

To prove the converse, assume (a) and (b). We can write $U^*U = I \oplus T$ according to the decomposition $H = \mathrm{N}\,(U^*U - I) \oplus \overline{\mathrm{R}(U^*U - I)}$. Observe that here we have $T \geq 0$ and, since $A_{\mathrm{opt}} \geq 1$, $\sigma(T) \subseteq [1, B_{\mathrm{opt}}]$.

Consider a continuous function $g : [1, \infty) \to [0, 1)$ defined by $g(t) = \sqrt{1 - \frac{1}{t}}$. Put $G = g(T)$. We now use assumption (b) to find a partial isometry $L \in \mathbb{B}(H, \ell^2)$ whose initial space is $\overline{\mathrm{R}(U^*U - I)}$ with final space contained in $\mathrm{N}(U^*) = \mathrm{R}(U)^\perp$. Finally, denote by $P \in \mathbb{B}(H)$ the orthogonal projection onto $\overline{\mathrm{R}(U^*U - I)}$.

Let $V = U(U^*U)^{-1} + L(0 \oplus G)P$. Then

$$\begin{aligned}
V^*V &= \left((U^*U)^{-1}U^* + P(0 \oplus G)L^*\right)\left(U(U^*U)^{-1} + L(0 \oplus G)P\right) \\
&= (U^*U)^{-1} + (0 \oplus G^2) = (I \oplus T^{-1}) + (0 \oplus (I - T^{-1})) = I.
\end{aligned}$$

Let us now put $v_n = V^*e_n$, $n \in \mathbb{N}$, where $(e_n)_n$ is the canonical basis for $\ell^2$. Since $V^*V = I$, the sequence $(v_n)_n$ is a Parseval frame in $H$. Obviously, we also have $V^*U = I$ which means that $(v_n)_n$ is a dual of $(x_n)_n$. $\square$

The above condition (a) is not crucial, since it can be ensured by rescaling the original frame (although, the construction then yields only a tight dual frame with the frame bound different from 1). Condition (b) is essential; it tells us that the excess should be at least large as $d = \dim(\mathrm{R}(U^*U - I)) = \dim(\mathrm{N}(U^*U - I)^\perp)$. Note that the number $d$ can be interpreted as a kind of a measure of deviation of the original frame from being Parseval. Namely, the characterizing Parseval property $U^*U = I$ is trivially fulfilled on the subspace $\mathrm{N}(U^*U - I)$. So, any deviation from the Parseval property has its origin in the orthogonal complement $\mathrm{N}(U^*U - I)^\perp = \overline{\mathrm{R}(U^*U - I)}$.

In the following section we will discuss some properties of frames which are related to the condition $d < \infty$.

We end this section with a comment on frames with infinite excess. As observed in Remark 2.5.4 (c), if $(x_n)_n$ is a frame for $H$ for which there exists an infinite set $S \subset \mathbb{N}$ with the property $\overline{\mathrm{span}}\{x_n : n \in \mathbb{N} \setminus S\} = H$, then the sequence $(x_n)_{n\in\mathbb{N}\setminus S}$ need not be a frame. However, we have the following result from [19] (see also Theorem 8.44 in [81]) which we include without proof.

**Theorem 2.5.11.** *Let $(x_n)_n$ be a Parseval frame for a Hilbert space $H$, and let $(x_{p(n)})_n$ be a subsequence of $(x_n)_n$. Then the following conditions are equivalent:*

- *(a) For each $k \in \mathbb{N}$ the sequence $(x_n)_{n\neq p(k)}$ is fundamental (and hence a frame) and there exists a constant $C$ that is a lower bound for each frame $(x_n)_{n\neq p(k)}$.*
- *(b) $\sup\{\|x_{p(n)}\| : n \in \mathbb{N}\} < 1$.*

*In case (a) and (b) hold, for each $0 < \varepsilon < C$ there exists an infinite subsequence $(x_{r(p(n))})$ of $(x_{p(n)})$ such that $(x_n)_{n\in\mathbb{N}\setminus\{r(p(n)):n\in\mathbb{N}\}}$ is a frame for $H$ with frame bounds $C - \varepsilon$ and 1.*

**Example 2.5.12.** Recall from Example 2.1.18 that the sequence $(\sqrt{b}e^{2\pi inbt})_{n\in\mathbb{Z}}$ is a Parseval frame for $L^2([0,1])$ for any $0 < b < 1$. An application of the preceding theorem yields an infinite set $S \subset \mathbb{Z}$ such that the sequence $(\sqrt{b}e^{2\pi inbt})_{n\in\mathbb{Z}\setminus S}$ is a frame.

It is interesting to note in this context that the original sequence is finitely linearly independent (see [81], Example 8.7 and Exercise 8.42).

*Concluding remarks.* The notion of the excess of a frame was introduced in [19]. Theorem 2.5.5 and Proposition 2.5.7 are from the same paper. The proof from Remark 2.5.4 (b) is from [93]. Proposition 2.5.9 first appeared in [13]. The existence of Parseval duals was first discussed in [77]. It is proved there that a frame $(x_n)_n$ for a Hilbert space $H$ possesses a Parseval dual if and only if $(x_n)_n$ can be obtained by applying an oblique projection to an ONB for a larger Hilbert space $K$ which contains $H$ as a closed subspace. This property of frames is discussed in [4]. So, Theorem 2.5.10 is obtained as a combination of the results from [77] and [4]. The proof presented here is taken from [13].

**Exercise 2.5.13.** Let $(e_n)_n$ be an ONB for a Hilbert space $H$. Put $f = \sum_{n=1}^\infty \frac{1}{n}e_n$. Show that $(f, e_1, e_2, e_3, \ldots)$ is a Bessel sequence whose excess is equal to 1 and whose synthesis operator is an injection (cf. Remark 2.5.6).

**Exercise 2.5.14.** Let $(x_n)_n$ be a frame for a Hilbert space $H$ and let $T \in \mathbb{B}(H)$ be a surjection. Show that the excess of the frame $(Tx_n)_n$ is greater than or equal to the excess of $(x_n)_n$. In particular, show that similar frames have the same excess.

**Exercise 2.5.15.** Let $(x_n)_n$ and $(z_n)_n$ be frames for a Hilbert space $H$ with the analysis operators $U$ and $V$, respectively. We say that these two frames are pseudo-dual to each other if the operator $V^*U$ is invertible.

(a) If $(x_n)_n$ and $(z_n)_n$ are pseudo-dual to each other show that $((U^*V)^{-1}x_n)_n$ is dual to $(z_n)_n$.

(b) Show that pseudo-dual frames have the same excess.

**Exercise 2.5.16.** Let $H$ and $K$ be Hilbert spaces and $U \in \mathbb{B}(H,K)$. Suppose that there exists $V \in \mathbb{B}(H,K)$ such that the operator $I - V^*U$ is compact. Prove that $\mathrm{R}(U)$ is a closed subspace of $K$ and $\dim(\mathrm{N}(U)) < \infty$ ([76], Problem 181). Observe that $I - U^*V$ is also compact, so the same conclusions apply to $V$.

**Exercise 2.5.17.** Let $(x_n)_n$ and $(v_n)_n$ be frames for a Hilbert space $H$ with the analysis operators $U$ and $V$ respectively such that $I - VU^*$ is a compact operator. Show that $(x_n)_n$ and $(v_n)_n$ are then near-Riesz bases.

Note that if $(x_n)_n$ is a near-Riesz basis and if $(y_n)_n$ is its canonical dual (whose analysis operator we denote by $V$), then $I - VU^* = I - U(U^*U)^{-1}U^*$ is a finite rank operator, and hence compact. Thus, we have the following characterization of near-Riesz bases: a frame $(x_n)_n$ with the analysis operator $U$ is a near-Riesz basis if and only if there exists a frame $(v_n)_n$ with the analysis operator $V$ such that $I - VU^*$ is a compact operator.

**Exercise 2.5.18.** Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the optimal upper frame bound $B_{\text{opt}}$. Suppose that the series $\sum_{n=1}^{\infty}\left(B_{\text{opt}} - \|x_n\|^2\right)$ converges. Prove that $(x_n)_n$ is then a near-Riesz basis. Show that the converse is not true; construct an example of a frame $(x_n)_n$ with finite excess such that $\sum_{n=1}^{\infty}\left(B_{\text{opt}} - \|x_n\|^2\right)$ diverges.

**Exercise 2.5.19.** Let $(e_n)_n$ be an ONB for a Hilbert space $H$. Show that the sequence $e_1, e_1, e_2, e_3, e_4, \ldots$ is a frame for $H$ that possesses a Parseval dual and use the proof of Theorem 2.5.10 to find such a dual frame.

## 2.6 Finite extensions of Bessel sequences

In this section we discuss finite extensions of Bessel sequences to frames. Here we work exclusively in infinite-dimensional spaces since the problem is trivial if the underlying space is finite-dimensional. Observe that each finite sequence of vectors is obviously Bessel. In general, when we work with finite sequences (consisting of, say, $k$ elements) it is natural to assume that the corresponding analysis operator takes values in $\mathbb{F}^k$. However, since here the underlying space is infinite-dimensional, it is convenient to adopt the following convention: if $(x_n)_{n=1}^k$ is a finite sequence[1] in a Hilbert space $H$ we will understand that its analysis operator takes values in $\ell^2$; in other words, we will tacitly assume that $x_1, x_2, \ldots, x_k$ are followed by infinitely many null-vectors.

Suppose we have a Bessel sequence $(x_n)_n$ in a Hilbert space $H$. Consider the following question: does there exist a sequence $(f_n)_{n=1}^k$ in $H$ such that the extended sequence $(f_n)_{n=1}^k \cup (x_n)_{n=1}^\infty$ is a frame for $H$?

If $(x_n)_n$ is a Bessel sequence in $H$ one defines its *deficit* ([19]) as the least cardinal $d$ such that there exists a subset $G$ of $H$ of cardinality $d$ so that $\overline{\text{span}}\,((x_n)_n \cup G) = H$. If $(x_n)_n$ is already fundamental in $H$ (as it is the case when $(x_n)_n$ is a frame for $H$), we understand that its deficit is equal to 0. If $U$ denotes the analysis operator of $(x_n)_n$, one easily concludes that the deficit of $(x_n)_n$ is equal to $\dim(\mathrm{N}(U))$. This is simply because we have

$$\dim\left(\overline{\text{span}}\,\{x_n : n \in \mathbb{N}\}\right) = \dim\left(\overline{\mathrm{R}(U^*}\right) = \dim\, H - \dim\left(\mathrm{N}(U)\right).$$

So, if we want to obtain a frame from a Bessel sequence by adding only finitely many vectors, then necessarily its deficit should be finite. However, this is not enough.

**Example 2.6.1.** Consider the canonical orthonormal basis $(e_n)_n$ for $\ell^2$ and the sequence $(x_n)_n$ defined by $x_1 = e_1$, $x_n = e_{n-1} + e_n$, $n \geq 2$. Clearly, $(x_n)_n$ is a Bessel sequence in $\ell^2$ with the analysis operator $U = S + I$, where $S$ is the unilateral shift on $\ell^2$. Since $S$ has no eigenvalues, we have $\dim\left(\mathrm{N}(U)\right) = 0$; hence, the deficit of this sequence is equal to 0.

However, one can not extend $(x_n)_n$ to a frame by adding finitely many vectors. This can be seen directly (we omit the details), but also, since $\mathrm{R}\,(S + I)$ is not a closed subspace of $\ell^2$, by applying Proposition 2.6.2 below and Exercise 2.5.16.

In order to characterize all Bessel sequences which admit finite extensions to frames we now provide another necessary condition.

**Proposition 2.6.2.** *Let $(x_n)_n$ be a Bessel sequence in $H$ for which there exists a finite sequence $(f_n)_{n=1}^k$ in $H$ such that the extended sequence $(f_n)_{n=1}^k \cup (x_n)_n$ is a frame for $H$. Then there exists a Bessel sequence $(v_n)_n$ in $H$ such that the operator $I - V^*U$ has finite rank, where $U$ and $V$ denote the analysis operators of $(x_n)_n$ and $(v_n)_n$, respectively.*

**Proof.** Denote by $U_1$ the analysis operator of the frame $(f_n)_{n=1}^k \cup (x_n)_n$. Let us take any dual frame of $(f_n)_{n=1}^k \cup (x_n)_n$ and denote it, for convenience, by $(g_n)_{n=1}^k \cup (v_n)_n$ (in other words,

[1] In general, when we work with an infinite sequence, we write, as before, $(x_n)_n$ assuming tacitly that the index set is $\mathbb{N}$; if, on the other hand, a sequence under consideration consists of $k$ vectors, $k \in \mathbb{N}$, we will write $(x_n)_{n=1}^k$ to avoid any possibility of confusion.

the first $k$ elements $g_1, \ldots, g_k$ are followed by $v_1, v_2, \ldots$). Let $V_1$ be its analysis operator. We now have for all $x \in H$

$$x = V_1^* U_1 x = \sum_{n=1}^{k} \langle x, f_n \rangle g_n + \sum_{n=1}^{\infty} \langle x, x_n \rangle v_n = \sum_{n=1}^{k} \langle x, f_n \rangle g_n + V^* U x;$$

thus

$$(I - V^* U)x = \sum_{n=1}^{k} \langle x, f_n \rangle g_n, \ \forall x \in H.$$

This shows that $I - V^* U$ is a finite rank operator. $\square$

It turns out that the converse of the preceding proposition is also true. In fact, we will prove the converse in a stronger form.

**Theorem 2.6.3.** *Let $(x_n)_n$ and $(v_n)_n$ be Bessel sequences in $H$ with Bessel bounds $B$ and $D$ and the analysis operators $U$ and $V$, respectively. Suppose that $I - V^* U$ is a compact operator. Then there exist finite sequences $(f_n)_{n=1}^k$ and $(h_n)_{n=1}^l$ such that $(f_n)_{n=1}^k \cup (x_n)_n$ and $(h_n)_{n=1}^l \cup (v_n)_n$ are frames for $H$ with upper frame bounds $B$ resp. $D$.*

**Proof.** By Exercise 2.5.16 we have $\dim(\mathrm{N}(U)) < \infty$, so one can find a finite frame $(f_n)_{n=1}^k$ for $\mathrm{N}(U)$ with upper frame bound $B$. Let us denote by $F$ the corresponding analysis operator. Take any dual frame $(g_n)_{n=1}^k$ for $(f_n)_{n=1}^k$ with the analysis operator $G$. We assume that all $f_j$'s and $g_j$'s belong to $\mathrm{N}(U)$. We regard $F$ and $G$ as operators from $H$ to $\ell^2$ assuming that both $F$ and $G$ act trivially on $\mathrm{N}(U)^\perp$. Then we have $G^* F = P$, where $P$ denotes the orthogonal projection to $\mathrm{N}(U)$. Note also that both $\mathrm{R}(F)$ and $\mathrm{R}(G)$ are contained in $M_k = \mathrm{span}\,\{e_1, \ldots, e_k\}$, where $(e_n)_n$ denotes the canonical basis for $\ell^2$.

Consider now the extended sequence $f_1, \ldots, f_k, x_1, x_2, \ldots$. Observe that its analysis operator $U_1$ is given by $U_1 = F + S^k U$, where $S$ denotes the unilateral shift on $\ell^2$. Let $W = G + S^k (U^\dagger)^*$ (since $\mathrm{R}(U)$ is by Exercise 2.5.16 closed, $U^\dagger$ does exist). Then, using the equalities $F^* S^k = G^* S^k = 0$ and $U^\dagger U = I - P$, we obtain

$$W^* U_1 = (G^* + U^\dagger (S^*)^k)(F + S^k U) = G^* F + U^\dagger U = P + (I - P) = I.$$

This implies that $U_1$ is bounded from below; thus, $(f_n)_{n=1}^k \cup (x_n)_{n=1}^\infty$ is a frame.

For $x \in \mathrm{N}(U)$ we have $0 = \|Ux\|^2 = \sum_{n=1}^\infty |\langle x, x_n \rangle|^2$ and $\sum_{n=1}^k |\langle x, f_n \rangle|^2 \le B\|x\|^2$. On the other hand, if $x \in \mathrm{N}(U)^\perp$ then $\sum_{n=1}^k |\langle x, f_n \rangle|^2 = 0$.

Let us now take an arbitrary $x \in H$ and write $x = a + b$ with $a \in \mathrm{N}(U)$ and $b \in \mathrm{N}(U)^\perp$. Then

$$\begin{aligned} \sum_{n=1}^{k} |\langle x, f_n \rangle|^2 + \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 &= \sum_{n=1}^{k} |\langle a, x_n \rangle|^2 + \|U(a+b)\|^2 \\ &\le B\|a\|^2 + \|Ub\|^2 \\ &\le B(\|a\|^2 + \|b\|^2) \\ &= B\|x\|^2. \end{aligned}$$

The assertions concerning $(v_n)_{n=1}^{\infty}$ follow by the same arguments using compactness of the operator $I - U^*V$. $\square$

**Remark 2.6.4.** As the preceding proof shows, the extension of a Bessel sequence to a frame is not unique, even if we insist (as we did) on the same upper frame (Bessel) bound. It is also clear that the minimal number of elements that should be added to a given Bessel sequence $(x_n)_n$ in order to obtain a frame is the deficit of $(x_n)_n$, i.e. $\dim(\mathrm{N}(U))$. In that sense a minimal choice is $(\sqrt{B}w_1, \ldots, \sqrt{B}w_d)$, where $(w_1, \ldots, w_d)$ is an ONB for $\mathrm{N}(U)$.

Proposition 2.6.2 and Theorem 2.6.3 motivate the following definition:

**Definition 2.6.5.** *We say that Bessel sequences $(x_n)_n$ and $(v_n)_n$ with the analysis operators $U$ and $V$ are essentially dual to each other if $I - V^*U$ is a compact operator.*

If $I - V^*U$ is compact then, obviously, $I - U^*V$ is compact as well; hence essential duality of Bessel sequences is a symmetric relation.

Now we can summarize the statements of Proposition 2.6.2 and Theorem 2.6.3 in the following simple way:

**Theorem 2.6.6.** *A Bessel sequence $(x_n)_n$ in a Hilbert space $H$ has a finite extension to a frame for $H$ if and only if there exists a Bessel sequence essentially dual to $(x_n)_n$.*

Next we discuss finite extensions of Bessel sequences to Parseval frames. Again, we will first obtain necessary conditions.

Suppose we have a Bessel sequence $(x_n)_n$ in $H$ for which there exists a finite sequence $(f_n)_{n=1}^k$ such that $(f_n)_{n=1}^k \cup (x_n)_n$ is a Parseval frame for $H$. Denote by $B_{\mathrm{opt}}$ the optimal Bessel bound of $(x_n)_n$. Since

$$\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq \sum_{n=1}^{k} |\langle x, f_n \rangle|^2 + \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 = \|x\|^2, \ \ \forall x \in H,$$

we conclude that $B_{\mathrm{opt}} \leq 1$.

Let $U$ be the analysis operator of $(x_n)_n$. Denote by $F$ the analysis operator of $(f_n)_{n=1}^k$; since this sequence is finite, $F$ is a finite rank operator. Now observe that the analysis operator $U_1$ of the sequence $(f_n)_{n=1}^k \cup (x_n)_n$ is given by $U_1 = F + S^kU$, where, as before, $S$ denotes the unilateral shift on $\ell^2$.

Since by our assumption $(f_n)_{n=1}^k \cup (x_n)_n$ is a Parseval frame for $H$, we have $U_1^*U_1 = I$. From this we obtain

$$I = (F + S^kU)^*(F + S^kU) = F^*F + F^*S^kU + U^*(S^k)^*F + U^*U.$$

Let $K = F^*F + F^*S^kU + U^*(S^k)^*F$. Then $K$ is a finite rank operator and $I - U^*U = K$. In particular, the operator $I - U^*U$ is not invertible because it has finite-dimensional range. This in turn implies that 1 belongs to the spectrum of $U^*U$ and this, together with our previous conclusion $B_{\mathrm{opt}} \leq 1$, implies $B_{\mathrm{opt}} = 1$.

The statement of the following theorem appears in a similar form in [96]. Although the proof in [96] uses g-frames, the key argument is essentially the same as in our proof below.

**Theorem 2.6.7.** *Let $(x_n)_n$ be a Bessel sequence in a Hilbert space $H$ with the optimal Bessel bound $B_{opt}$ and the analysis operator $U$. The following conditions are equivalent:*

*(a) There exists a finite sequence $(f_n)_{n=1}^k$ in $H$ such that $(f_n)_{n=1}^k \cup (x_n)_n$ is a Parseval frame for $H$,*

*(b) $B_{opt} = 1$ and dim $(R(I - U^*U)) < \infty$.*

**Proof.** (a) $\Rightarrow$ (b) is already proved in the preceding discussion. Let us prove (b) $\Rightarrow$ (a).

Since $B_{\text{opt}} = 1$, the square root $(I-U^*U)^{\frac{1}{2}}$ is a well defined positive operator. Observe that $\mathrm{N}((I - U^*U)^{\frac{1}{2}}) = \mathrm{N}(I - U^*U)$. By taking orthogonal complements we get $\mathrm{R}((I - U^*U)^{\frac{1}{2}}) = \mathrm{R}(I - U^*U)$ (since, by assumption, this subspace is finite-dimensional, the closure signs are superfluous).

Let $k = \dim(\mathrm{R}(I - U^*U)) < \infty$ and $M_k = \operatorname{span}\{e_1, \ldots, e_k\} \leq \ell^2$; here again $(e_n)_n$ denotes the canonical basis for $\ell^2$. Take any partial isometry $F \in \mathbb{B}(H, \ell^2)$ with the initial subspace $\mathrm{R}(I - U^*U)$ and the final subspace $M_k$. Notice that $\mathrm{R}(F) = M_k \perp \mathrm{R}(S^kU)$.

Let $U_1 = F(I - U^*U)^{\frac{1}{2}} + S^kU$. We claim that $U_1$ is an isometry. Indeed, we have for any $x \in H$

$$\begin{aligned}
\|U_1x\|^2 &= \|F(I - U^*U)^{\frac{1}{2}}x + S^kUx\|^2 \\
&= \|F(I - U^*U)^{\frac{1}{2}}x\|^2 + \|S^kUx\|^2 \\
&= \|(I - U^*U)^{\frac{1}{2}}x\|^2 + \|Ux\|^2 \\
&= \langle (I - U^*U)x, x\rangle + \langle U^*Ux, x\rangle \\
&= \|x\|^2.
\end{aligned}$$

Since $U_1$ is an isometry, $(U_1^*e_n)_n$ is a Parseval frame for $H$. Observe that we have $U_1^* = (I-U^*U)^{\frac{1}{2}}F^* + U^*(S^*)^k$ which implies $U_1^*e_{k+j} = U^*e_j = x_j$, $\forall j \in \mathbb{N}$. Thus, our original Bessel sequence $(x_n)_n$ is extended to a Parseval frame by the elements $f_j = (I - U^*U)^{\frac{1}{2}}F^*e_j$, $j = 1, 2, \ldots, k$. ☐

**Remark 2.6.8.** Suppose that $l \geq k = \dim(\mathrm{R}(I - U^*U))$ and take a partial isometry $F'$ with the initial subspace $\mathrm{R}(I-U^*U)$ and the final subspace *contained* in $M_l = \operatorname{span}\{e_1, \ldots, e_l\} \leq \ell^2$. Then the same argument as above applies if we replace $U_1$ by $U_1' = F'(I - U^*U)^{\frac{1}{2}} + S^lU$. This would result with an extension of the original Bessel sequence to a Parseval frame by adding $l$ elements.

The minimal number of elements that one must add to a given Bessel sequence in order to obtain a Parseval frame is $k = \dim(\mathrm{R}(I - U^*U))$. Such minimal extensions are described in Corollary 2.6.10.

**Remark 2.6.9.** We also note that for the proof of (b) $\Rightarrow$ (a) in the preceding theorem it suffices to assume $B_{\text{opt}} \leq 1$ and $\dim(\mathrm{R}(I - U^*U)) < \infty$.

**Corollary 2.6.10.** *Let $(x_n)_n$ be a Bessel sequence in a Hilbert space $H$ with a Bessel bound less than or equal to 1 and such that $\dim(R(I-U^*U)) = k < \infty$. Let $x_j = (I-U^*U)^{\frac{1}{2}} w_j$, $j = 1, \ldots, k$, where $(w_1, \ldots, w_k)$ is an orthonormal basis for $R(I-U^*U)$. Then $(f_n)_{n=1}^k \cup (x_n)_n$ is a Parseval frame for $H$.*

When one compares the statement of Theorem 2.6.7 with those of Theorem 2.6.3 and Corollary 2.6.6 it is natural to ask the following question: is it enough, in order to ensure a finite extension of a given Bessel sequence to a Parseval frame, to assume that $I-U^*U$ is only a compact operator (together with $B \leq 1$)?

The answer is negative. Namely, if $I-U^*U$ is a compact operator, then by Atkinson's theorem ([76], Problem 181), there exists a bounded operator $V$ such that $I-V^*U$ has finite rank, but one can not conclude that the rank of $I-U^*U$ is finite.

Here is an example. Consider the canonical basis $(e_n)_n$ for $\ell^2$ and the sequence $(x_n)_n$ defined by $x_n = \sqrt{\frac{n}{n+1}} e_n$, $n \in \mathbb{N}$. Clearly, $(x_n)_{n=1}^\infty$ is a frame for $\ell^2$; in fact, a Riesz basis with the upper frame bound $B_{\text{opt}} = 1$. If we denote by $U$ its analysis operator, then $U^*Ux = \sum_{n=1}^\infty \frac{n}{n+1} \langle x, e_n \rangle e_n$, $\forall x \in \ell^2$. This implies $(I-U^*U)x = \sum_{n=1}^\infty \frac{1}{n+1} \langle x, e_n \rangle e_n$, $\forall x \in \ell^2$; thus, $I-U^*U$ is a compact operator. However; the sequence $(x_n)_n$ can not be extended to a Parseval frame by adding a finite number of elements. This can be seen directly, but it is easier to apply Theorem 2.6.7: namely, it is evident that the operator $I-U^*U$ has infinite rank.

Theorem 2.6.7 shows that Bessel sequences and, in particular, frames for which $I-U^*U$ is a finite rank operator are, in a sense, almost Parseval. Our next theorem provides two more characterizing properties of such frames which also show, in a different way, a close relation with the class of Parseval frames.

**Theorem 2.6.11.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$. The following conditions are equivalent:*

*(a)* $\dim(R(I-U^*U)) < \infty$,

*(b) There exists a sequence $(h_n)_n$ in a finite-dimensional subspace $L$ of $H$ such that $(x_n+h_n)_n$ is a Parseval frame for $H$,*

*(c) $x = \sum_n \langle x, x_n \rangle x_n$, $\forall x \in M$, where $M$ is a closed subspace of $H$ of finite co-dimension.*

**Proof.** (a) $\Rightarrow$ (b). Since $I-U^*U$ is a self-adjoint operator and $\dim(\mathrm{R}(I-U^*U)) < \infty$, we have $H = \mathrm{N}(I-U^*U) \oplus \mathrm{R}(I-U^*U)$. This implies

$$\mathrm{R}(U) = U(\mathrm{N}(I-U^*U)) + U(\mathrm{R}(I-U^*U)).$$

Moreover, we claim that this sum is direct. Indeed, if $Ux = Uy$ for some $x \in \mathrm{N}(I-U^*U)$ and $y \in \mathrm{R}(I-U^*U)$ then, by injectivity of $U$, we conclude $x = y$ and hence $x = y = 0$.

Now observe that all direct complements of a closed subspace in a given space are of the same dimension. Thus,

$$\dim(\mathrm{R}(U) \ominus U(\mathrm{N}(I-U^*U))) = \dim(U(\mathrm{R}(I-U^*U))) = \dim(\mathrm{R}(I-U^*U))$$

where the last inequality follows from injectivity of $U$.

This allows us to find an isometry $F_0 : \mathrm{R}(I - U^*U) \to \mathrm{R}(U)$ such that $\mathrm{R}(F_0) \perp U(\mathrm{N}(I - U^*U))$. Put $U_0 = U|_{\mathrm{N}(I-U^*U)}$. Since we have $U^*Ux = x$ for $x \in \mathrm{N}(I - U^*U)$, $U_0$ is also an isometry. Finally, let $V = U_0 \oplus F_0$; since the images of $U_0$ and $F_0$ are mutually orthogonal, $V$ is an isometry. So, if we define $g_n = V^*e_n$, $n \in \mathbb{N}$, where $(e_n)_n$ is the canonical basis for $\ell^2$, the sequence $(g_n)_n$ is a Parseval frame for $H$. Obviously, $F = V - U$ is a finite rank operator. Namely, $F$ acts trivially on $\mathrm{N}(I - U^*U)$ and $\dim(\mathrm{R}(I - U^*U)) < \infty$. Put $h_n = F^*e_n$, $n \in \mathbb{N}$. Then $(h_n)_n$ is a finite-dimensional perturbation of $(x_n)_n$ such that $(g_n = x_n + h_n)_n$ is a Parseval frame for $H$.

(b) $\Rightarrow$ (a). If we assume (b) then $(h_n)_n$ is a Bessel sequence, as the difference of two Bessel sequences. Let $T$ denotes its analysis operator. By our assumption, $T^*$ has finite rank. Hence, $T$ is also a finite rank operator. Since $(U^* + T^*)e_n = x_n + h_n$, $\forall n \in \mathbb{N}$, and $(x_n + h_n)_n$ is a Parseval frame for $H$, $U + T$ is an isometry. Thus,

$$I = (U + T)^*(U + T) = U^*U + U^*T + T^*U + T^*T.$$

Let $F = U^*T + T^*U + T^*T$. Then $F$ is a finite rank operator and we have $I - U^*U = F$.

(a) $\Rightarrow$ (c). Let $M = \mathrm{N}(I - U^*U)$. By (a), $M^\perp = \mathrm{R}(I - U^*U)$ is finite-dimensional. For $x \in M$ we have $x = U^*Ux$ i.e. $x = \sum_{n=1}^\infty \langle x, x_n\rangle x_n$.

(c) $\Rightarrow$ (a). Suppose we have $x = \sum_{n=1}^\infty \langle x, x_n\rangle x_n$ for all $x \in M$ with $\dim(M^\perp) < \infty$. This implies $U^*Ux = x$, $\forall x \in M$. Hence, $M \subseteq \mathrm{N}(I - U^*U)$, and this implies $\mathrm{R}(I - U^*U) \subseteq M^\perp$. $\square$

**Remark 2.6.12.** (a) In contrast to Theorem 2.6.7, a general assumption in the preceding theorem is that $(x_n)_n$ is a frame, not merely a Bessel sequence. The reason for that is the proof of the above implication (a) $\Rightarrow$ (b) where we have used injectivity of the analysis operator $U$.

(b) Note that in condition (c) in the above theorem we do not claim that the frame elements $x_n$ belong to $M$. In this light, we may say that $(x_n)_n$ is, in a sense, an outer Parseval frame for the subspace $M$.

**Remark 2.6.13.** Here we demonstrate an alternative proof of (a) $\Rightarrow$ (b) from the preceding theorem.

Consider the decomposition $H = \mathrm{N}(I - U^*U) \oplus \mathrm{R}(I - U^*U)$. Since $\mathrm{N}(I - U^*U)$ is an eigenspace for the operator $U^*U$, its orthogonal complement is invariant for $U^*U$. So, both subspaces in the above decomposition are invariant for $U^*U$ and we can write $U^*U$ in the form $U^*U = I \oplus \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} = I \oplus \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with respect to an ONB $(w_1, \dots, w_k)$ for $\mathrm{R}(I - U^*U)$ consisting of eigenvectors of $U^*U$. Denote by $A$ and $B$ frame bounds of $(x_n)_n$ and note that $A \le \lambda_1, \dots, \lambda_k \le B$. Let us write, with respect to the same decomposition of $H$, $x_n = v_n + z_n$, $n \in \mathbb{N}$.

Now observe that both subspaces are also invariant for $(U^*U)^{-\frac{1}{2}}$ and that $(U^*U)^{-\frac{1}{2}}$ acts as the identity operator on $\mathrm{N}(I - U^*U)$.

Put $\overline{x_n} = (U^*U)^{-\frac{1}{2}}x_n$, $n \in \mathbb{N}$. Recall that $(\overline{x_n})_n$ is the Parseval frame for $H$ canonically associated with $(x_n)_n$. Obviously, for all $n \in \mathbb{N}$ we have $\overline{x_n} = v_n + w_n$, where $w_n = \begin{bmatrix} \lambda_1^{-\frac{1}{2}} & & \\ & \ddots & \\ & & \lambda_k^{-\frac{1}{2}} \end{bmatrix} z_n$, $n = 1, 2, \ldots, k$. The sequences $(z_n)_n$ and $(w_n)_n$ belong to a finite-dimensional subspace $\mathrm{R}(I - U^*U)$, so does their difference and we have $x_n + (w_n - z_n) = v_n + z_n + w_n - z_n = \overline{x_n}$, $\forall n \in \mathbb{N}$.

*Concluding remarks.* The results of this section are originally published in [14] However, Theorem 2.6.7 is first proved in [96].

**Exercise 2.6.14.** Let $(f_n)_n$ and $(g_n)_n$ be Bessel sequences in a Hilbert space $H$. Prove that there exist Bessel sequences $(h_n)_n$ and $(k_n)_n$ in $H$ such that $(f_n)_n \cup (h_n)_n$ and $(g_n)_n \cup (k_n)_n$ form a pair of dual frames for $H$. *Remark.* This is the statement of Proposition 2.1 from [54]. Observe that here, in contrast to our considerations in this section, the extensions $(h_n)_n$ and $(k_n)_n$ are, in general, infinite sequences.

## 2.7 Perturbations of frames

The classical Paley-Wiener theorem states the following: Let $(x_n)_n$ be a basis for a Banach space $B$, and let $(v_n)_n$ be a sequence of vectors in $B$. If there exists a constant $\lambda \in [0,1)$ such that

$$\left\|\sum_{n=1}^{N} c_n(x_n - v_n)\right\| \le \lambda \left\|\sum_{n=1}^{N} c_n x_n\right\|$$

for all scalars $c_1, c_2, \dots, c_N$ and every $N \in \mathbb{N}$, then $(v_n)_n$ is a basis for $B$.

In this section we discuss similar results for frames. There are several papers in the literature concerned with the perturbation theorems for frames. Here we begin our considerations with the most general of these theorems from which the (chronologically) preceeding results follow as corollaries. First we need a lemma that is interesting in its own. A classical result states that a bounded operator $T$ on a Banach space is invertible if $\|I - T\| < 1$. The following lemma shows that $T$ is invertible under a much weaker condition; observe that even boundedness of $T$ is not a priori assumed.

**Lemma 2.7.1.** *Let $H$ be a Hilbert space, and let $T : H \to H$ be a linear operator for which there exist constants $\lambda_1, \lambda_2 \in [0,1)$ such that*

$$\|Tx - x\| \le \lambda_1\|x\| + \lambda_2\|Tx\|, \ \ \forall x \in H. \tag{53}$$

*Then $T$ is an invertible bounded operator and*

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \le \|Tx\| \le \frac{1+\lambda_1}{1-\lambda_2}\|x\|, \ \ \frac{1-\lambda_2}{1+\lambda_1}\|x\| \le \|T^{-1}x\| \le \frac{1+\lambda_2}{1-\lambda_1}\|x\|, \ \ \forall x \in H. \tag{54}$$

**Proof.** First observe that

$$\|Tx\| \le \|Tx - x\| + \|x\| \overset{(53)}{\le} \lambda_1\|x\| + \lambda_2\|Tx\| + \|x\|, \ \ \forall x \in H \tag{55}$$

and

$$\|Tx\| \ge \|x\| - \|Tx - x\| \overset{(53)}{\ge} \|x\| - \lambda_1\|x\| - \lambda_2\|Tx\|, \ \ \forall x \in H. \tag{56}$$

Clearly, (55) and (56) give us the first two inequalities in (54). In particular, $T$ is bounded and bounded from below. Also observe that, once we prove that $T$ is invertible, the remaining two inequalities in (54) follow from the first two by replacing $x$ by $T^{-1}x$.

The rest of the proof consists of proving that $T$ is invertible. Since $T$ is bounded from below, we only need to show that $T$ is surjective.

Let $\lambda = \max\{\lambda_1, \lambda_2\}$ and $\mu = \frac{1-\lambda}{1+\lambda}$.

For any $\alpha \le 0$ and all $x$ in $H$ we have

$$\begin{aligned}
\|x - Tx\| &\overset{(53)}{\le} \lambda\|x\| + \lambda\|Tx\| \\
&\le \lambda\|x\| + \lambda\|Tx - \alpha x\| + \lambda\|\alpha x\| \\
&= \lambda\|x\| + \lambda\|Tx - \alpha x\| - \lambda\alpha\|x\| \\
&= \lambda\|\alpha x - Tx\| + \lambda(1-\alpha)\|x\|.
\end{aligned}$$

From this we conclude that

$$\lambda\|\alpha x - Tx\| \geq \|x - Tx\| - \lambda(1-\alpha)\|x\|. \tag{57}$$

Similarly, for any $\alpha \leq 0$ and all $x$ in $H$ we have

$$\|x - Tx\| = \|(1-\alpha)x + (\alpha x - Tx)\| \geq (1-\alpha)\|x\| - \|\alpha x - Tx\|.$$

This gives us

$$\|\alpha x - Tx\| \geq -\|x - Tx\| + (1-\alpha)\|x\|. \tag{58}$$

By combining (57) and (58) we obtain

$$\|\alpha x - Tx\| \geq \frac{(1-\lambda)(1-\alpha)}{1+\lambda}\|x\| \geq \mu\|x\|, \ \ \forall \alpha \leq 0, \forall x \in H. \tag{59}$$

In particular, this shows that for each $\alpha \leq 0$ the operator $\alpha I - T$ is bounded from below and hence injective.

Let us now define

$$E = \{\alpha \leq 0 : \|\alpha x - T^*x\| \geq \frac{\mu}{2}\|x\|, \ \forall x \in H\}.$$

The set $E$ is closed and non-empty because $\alpha = -(1 + \|T\|) \in E$.

By definition, for each $\alpha \in E$, the operator $\alpha I - T^*$ is bounded from below. Proposition 2.1.7 (b) now implies that $\alpha I - T$ is surjective. Thus, $\alpha I - T$ is an invertible operator for each $\alpha$ in $E$. To end the proof, we only need to show that $0 \in E$.

Recall now that each bounded invertible operator $L$ satisfies $(L^{-1})^* = (L^*)^{-1}$. Thus,

$$\left\|(\alpha I - T^*)^{-1}\right\| = \left\|\left((\alpha I - T)^{-1}\right)^*\right\| = \left\|(\alpha I - T)^{-1}\right\| \overset{(59)}{\leq} \frac{1}{\mu}, \ \ \forall \alpha \in E.$$

From this we obtain

$$\|(\alpha I - T^*)x\| \geq \mu\|x\|, \ \ \forall \alpha \in E, \forall x \in H. \tag{60}$$

Let us now take any $\alpha \in E$ and $\delta$ such that $0 < \delta \leq \frac{\mu}{2}$. Then we have

$$\|(\alpha+\delta)x - T^*x\| \geq \|\alpha x - T^*x\| - \delta\|x\| \overset{(60)}{\geq} \mu\|x\| - \delta\|x\| \geq \frac{\mu}{2}\|x\|, \ \ \forall \alpha \in E, \forall x \in H.$$

This shows that $\alpha \in E \Rightarrow [\alpha, \alpha + \frac{\mu}{2}] \cap \{\alpha : \alpha \leq 0\} \subseteq E$. If $0 \leq \alpha + \frac{\mu}{2}$, this gives us that $0 \in E$. The other possibility is that $\alpha + \frac{\mu}{2} < 0$. If this is the case, we repeat the argument replacing $\alpha$ by $\alpha + \frac{\mu}{2}$. Clearly, after finitely many steps, we obtain that $0 \in E$. $\square$

**Example 2.7.2.** Let $(e_n)_n$ be an ONB of a Hilbert space $H$. Consider the operator $T$ on $H$ defined by $Te_n = e_n + \frac{1}{n}e_{n+1}$, $n \in \mathbb{N}$, and then extended by linearity. Observe that $\|Tx - x\| = \left\|\sum_{n=1}^{\infty}\langle x, e_n\rangle \frac{1}{n}e_{n+1}\right\|$, for every $x$ in $H$. Thus, $T$ is bounded and $\|T - I\| \leq 1$. Moreover, since $Te_1 - e_1 = e_2$, we conclude that $\|T - I\| = 1$. So we can not use the classical result to show that $T$ is in fact invertible. However, in this situation Lemma 2.7.1 applies.

To see this, first observe that $\|T\| \leq 2$. We also have for each $x$ in $H$

$$\|Tx\| = \left\| \langle x, e_1 \rangle e_1 + \sum_{n=2}^{\infty} \left( \langle x, e_n \rangle + \frac{1}{n-1} \langle x, e_{n-1} \rangle \right) e_n \right\| \geq |\langle x, e_1 \rangle|. \tag{61}$$

Using this, we obtain

$$\|Tx - x\|^2 = \sum_{n=1}^{\infty} \left| \frac{1}{n} \langle x, e_n \rangle \right|^2 \leq |\langle x, e_1 \rangle|^2 + \frac{1}{4}\|x\|^2 \overset{(61)}{\leq} \|Tx\|^2 + \frac{1}{4}\|x\|^2 \leq \left( \|Tx\| + \frac{1}{2}\|x\| \right)^2,$$

and finally

$$\|Tx - x\| \leq \|Tx\| + \frac{1}{2}\|x\| \leq \frac{7}{8}\|Tx\| + \frac{1}{8}\|T\| \cdot \|x\| + \frac{1}{2}\|x\| = \frac{7}{8}\|Tx\| + \frac{3}{4}\|x\|, \ \ \forall x \in H.$$

So, by Lemma 2.7.1, we conclude that $T$ is invertible. Note in passing that this shows that the sequence $(e_n + \frac{1}{n} e_{n+1})_n$ is a Riesz basis for $H$.

**Theorem 2.7.3.** *Let $H$ and $K$ be Hilbert spaces, let $U \in \mathbb{B}(H, K)$ be an invertible operator, and let $V : H \to K$ ba a linear operator. Suppose that there exist constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|Ux - Vx\| \leq \lambda_1 \|Ux\| + \lambda_2 \|Vx\|, \ \ \forall x \in H. \tag{62}$$

*Then $V$ is also an invertible bounded operator and*

$$\frac{1-\lambda_1}{1+\lambda_2}\|Ux\| \leq \|Vx\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|Ux\|, \ \ \forall x \in H, \tag{63}$$

$$\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{\|U\|}\|y\| \leq \|V^{-1}x\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|U^{-1}\| \cdot \|y\|, \ \ \forall y \in K. \tag{64}$$

**Proof.** Define a linear operator $T : K \to K$ by $Ty = VU^{-1}y$, $y \in K$. Using (62) with $x = U^{-1}y$, we obtain that

$$\|y - Ty\| \leq \lambda_1 \|y\| + \lambda_2 \|Ty\|, \ \ \forall y \in K. \tag{65}$$

Lemma 2.7.1 now implies that $T$ is bounded and invertible. Thus, $V$ is also a bounded and invertible operator. Inequalities (63) and (64) follow easily from (54). □

If we a priori know that $V$ is bounded, then we are even allowed to take $\lambda = 1$.

**Corollary 2.7.4.** *Let $H$ and $K$ be Hilbert spaces, let $U \in \mathbb{B}(H, K)$ be an invertible operator. Suppose that $V \in (H, K)$ is an operator for which there exists a constant $\lambda_1 \in [0, 1)$ such that*

$$\|Ux - Vx\| \leq \lambda_1 \|Ux\| + \|Vx\|, \ \ \forall x \in H. \tag{66}$$

*Then $V$ is invertible and $\|V^{-1}\| \leq \frac{2}{1-\lambda_1}\|U^{-1}\|$.*

**Proof.** Let $\epsilon > 0$. Consider again $VU^{-1} : K \to K$; note that here $VU^{-1}$ is bounded. Using (66) with $x = U^{-1}y$ we obtain that

$$\|y - VU^{-1}y\| \leq \lambda_1\|y\| + \|VU^{-1}y\| \leq (\lambda_1 + \epsilon\|VU^{-1}\|)\|y\| + (1-\epsilon)\|VU^{-1}y\|, \ \ \forall y \in K. \tag{67}$$

We can now choose $\epsilon$ small enough to have $\epsilon_1 := \lambda_1 + \epsilon\|VU^{-1}\| < 1$. Then (67) becomes

$$\|y - VU^{-1}y\| \leq \epsilon_1\|y\| + (1-\epsilon)\|VU^{-1}y\|, \ \ \forall y \in K. \tag{68}$$

This enables us to apply Lemma 2.7.1 and conclude that $VU^{-1}$ is invertible. Hence, $V$ is invertible. Moreover, Lemma 2.7.1 gives us the estimate

$$\left\|(VU^{-1})^{-1}\right\| \leq \frac{1+1-\epsilon}{1-\epsilon_1} = \frac{1+1-\epsilon}{1-(\lambda_1+\epsilon\|VU^{-1}\|)}.$$

From this we obtain by letting $\epsilon \to 0$

$$\left\|UV^{-1}\right\| \leq \frac{2}{1-\lambda_1}.$$

Finally, this gives us

$$\|V^{-1}\| = \|U^{-1}UV^{-1}\| \leq \|U^{-1}\| \cdot \|UV^{-1}| \leq \frac{2}{1-\lambda_1}\|U^{-1}\|.$$

□

We are now ready for our first result on perturbations of frames.

**Theorem 2.7.5.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with frame bounds $A$ and $B$. Suppose that $(v_n)_n$ is a sequence in $H$ for which there exist constants $\lambda_1, \lambda_2, \mu \geq 0$ such that* $\max\left\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\right\} < 1$ *and*

$$\left\|\sum_{n=1}^{N} c_n(x_n - v_n)\right\| \leq \lambda_1\left\|\sum_{n=1}^{N} c_n x_n\right\| + \lambda_2\left\|\sum_{n=1}^{N} c_n v_n\right\| + \mu\left(\sum_{n=1}^{N}|c_n|^2\right)^{\frac{1}{2}} \tag{69}$$

*for all scalars $c_1, c_2, \ldots, c_N$ and every $N \in \mathbb{N}$. Then $(v_n)_n$ is a frame for $H$ with frame bounds* $A\left(1 - \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{A}}}{1+\lambda_2}\right)^2$ *and* $B\left(1 + \frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right)^2$.

**Proof.** Denote by $U$ the analysis operator of $(x_n)_n$. Recall that $\|U\| \leq \sqrt{B}$.

Let $T$ be the operator defined on $c_{00}$ by $Tc = \sum_{n=1}^{N} c_n v_n$, $c = (c_n)_{n=1}^{N} \in c_{00}$. Then (69) can be rewritten as

$$\|U^*c - Tc\| \leq \lambda_1\|U^*c\| + \lambda_2\|Tc\| + \mu\|c\|, \ \ \forall c = (c_n)_{n=1}^{N} \in c_{00}. \tag{70}$$

This gives us

$$\|Tc\| \leq \|U^*c - Tc\| + \|U^*c\| \leq (1+\lambda_1)\|U^*c\| + \lambda_2\|Tc\| + \mu\|c\|$$

wherefrom we obtain

$$\|Tc\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|U^*c\| + \frac{\mu}{1-\lambda_2}\|c\|, \ \ \forall c = (c_n)_{n=1}^N \in c_{00}. \tag{71}$$

Inequality (71) shows that $T$ is a bounded operator on $c_{00}$. Thus, it can be extended to a bounded operator from $\ell^2$ to $H$ which we denote by $V^*$; moreover we now conclude from (70) and (71) that

$$\|U^*c - V^*c\| \leq \lambda_1\|U^*c\| + \lambda_2\|V^*c\| + \mu\|c\|, \ \ \forall c = (c_n)_n \in \ell^2 \tag{72}$$

and

$$\|V^*c\| \leq \frac{\sqrt{B}(1+\lambda_1)+\mu}{1-\lambda_2}\|c\|, \ \ \forall c = (c_n)_n \in \ell^2. \tag{73}$$

Denote by $(e_n)_n$ the canonical basis for $\ell^2$ and observe that we have $V^*e_n = Te_n = v_n$ for all $n$. Thus, $(v_n)_n$ is a Bessel sequence in $H$ whose Bessel bound is

$$B\left(\frac{1+\lambda_1+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right)^2 = B\left(1+\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right)^2.$$

Consider now the canonical dual $(y_n)_n$ of $(x_n)_n$. Recall that $y_n = (U^*U)^{-1}x_n$ for each $n$ in $\mathbb{N}$. Its frame bounds are $\frac{1}{B}$ and $\frac{1}{A}$ and the analysis operator is equal to $U(U^*U)^{-1}$

$$U(U^*U)^{-1}x = \left(\langle x, (U^*U)^{-1}x_n\rangle\right)_n = (\langle x, xy_nU(U^*U)^{-1})_n, \ \ \forall x \in H.$$

From this we have

$$\left\|U(U^*U)^{-1}x\right\|^2 = \sum_{n=1}^{\infty}\left|\langle x, (U^*U)^{-1}x_n\rangle\right|^2 \leq \frac{1}{A}\|x\|^2, \ \ \forall x \in H. \tag{74}$$

We now apply (72) and (74) to $c = U(U^*U)^{-1}x \in \ell^2$, for each $x \in H$. In this way we obtain

$$\begin{aligned}\left\|x - V^*U(U^*U)^{-1}x\right\| &\overset{(72)}{\leq} \lambda_1\|x\| + \lambda_2\|V^*U(U^*U)^{-1}x\| + \mu\|U(U^*U)^{-1}x\| \\ &\overset{(74)}{\leq} \left(\lambda_1 + \frac{\mu}{\sqrt{A}}\right)\|x\| + \lambda_2\|V^*U(U^*U)^{-1}x\|, \ \ \forall x \in H.\end{aligned} \tag{75}$$

Lemma 2.7.1 now implies that $V^*U(U^*U)^{-1}$ is an invertible operator. In particular, $V^*$ is then surjective. This is enough to conclude that $(v_n)_n$ is a frame for $H$. The first inequality in (54) gives us the estimate for its lower frame bound. □

**Corollary 2.7.6.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with frame bounds $A$ and $B$. Suppose that $(v_n)_n$ is a sequence in $H$ for which there exist constants $\lambda, \mu \geq 0$ such that $\lambda + \frac{\mu}{\sqrt{A}} < 1$ and*

$$\left\|\sum_{n=1}^{N} c_n(x_n - v_n)\right\| \leq \lambda\left\|\sum_{n=1}^{N} c_n x_n\right\| + \mu\left(\sum_{n=1}^{N}|c_n|^2\right)^{\frac{1}{2}} \tag{76}$$

*for all scalars $c_1, c_2, \ldots, c_N$ and every $N \in \mathbb{N}$. Then $(v_n)_n$ is a frame for $H$ with frame bounds $A\left(1-(\lambda+\frac{\mu}{\sqrt{A}})\right)^2$ and $B\left(1+\lambda+\frac{\mu}{\sqrt{B}}\right)^2$.*

**Proof.** This is the special case of Theorem 2.7.5 with $\lambda_1 = \lambda$ and $\lambda_2 = 0$. $\square$

**Corollary 2.7.7.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with frame bounds $A$ and $B$. Suppose that $(v_n)_n$ is a sequence in $H$ for which there exists a constant $\mu \geq 0$ such that $\frac{\mu}{\sqrt{A}} < 1$ and*

$$\sum_{n=1}^{\infty} \|x_n - v_n\|^2 \leq \mu^2. \tag{77}$$

*Then $(v_n)_n$ is a frame for $H$ with frame bounds $A\left(1 - \frac{\mu}{\sqrt{A}}\right)^2$ and $B\left(1 + \frac{\mu}{\sqrt{B}}\right)^2$.*

**Proof.** Take any $N \in \mathbb{N}$ and arbitrary scalars $c_1, c_2, \ldots, c_N$. Then we have

$$\begin{aligned}
\left\|\sum_{n=1}^{N} c_n(x_n - v_n)\right\| &\leq \sum_{n=1}^{N} |c_n| \cdot \|x_n - v_n\| \\
&\leq \left(\sum_{n=1}^{N} \|x_n - v_n\|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} |c_n|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{n=1}^{\infty} \|x_n - v_n\|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} |c_n|^2\right)^{\frac{1}{2}} \\
&\overset{(77)}{\leq} \mu \left(\sum_{n=1}^{N} |c_n|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

We are now in the position to apply Theorem 2.7.5 with $\lambda_1 = \lambda_2 = 0$. $\square$

**Example 2.7.8.** Consider, for $0 < b \leq 1$ the frame $(e^{2\pi inbt})_{n\in\mathbb{Z}}$ for $L^2([0,1])$. Recall that $(e^{2\pi inbt})_{n\in\mathbb{Z}}$ is tight with the frame bound $\frac{1}{b}$.

Take any sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of real numbers and consider the sequence $(e^{2\pi i\lambda_n t})_{n\in\mathbb{Z}}$ in $L^2([0,1])$. Using the estimate $\left|e^t - 1\right| \leq t$ one easily obtains that

$$\left\|e^{2\pi inbt} - e^{2\pi i\lambda_n t}\right\|_2^2 \leq \frac{4\pi^2}{3}|nb - \lambda_n|^2, \ \ \forall n \in \mathbb{Z}.$$

Hence, if the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ has the property

$$\sum_{n\in\mathbb{Z}} |nb - \lambda_n|^2 < \frac{3}{4\pi^2}\frac{1}{b} \tag{78}$$

we have

$$\sum_{n\in\mathbb{Z}} \left\|e^{2\pi inbt} - e^{2\pi i\lambda_n t}\right\|_2^2 \leq \frac{4\pi^2}{3}\sum_{n\in\mathbb{Z}} |nb - \lambda_n|^2 < \frac{1}{b}$$

and the preceding corollary applies. This allows us to conclude that under the assumption (78) the sequence $(e^{2\pi i\lambda_n t})_{n\in\mathbb{Z}}$ is a frame for $L^2([0,1])$. (Cf. Exercise 8.10 in [81].)

**Remark 2.7.9.** Suppose that $(x_n)_n$ and $(v_n)_n$ are as in Theorem 2.7.5. Then $(x_n)_n$ is a Riesz basis for $H$ if and only if $(v_n)_n$ is a Riesz basis for $H$. To see this, recall that we have concluded in the last paragraph of the proof of Theorem 2.7.5 that $V^*U(U^*U)^{-1}$ is an invertible operator, where $U$ and $V$ are the analysis operators of $(x_n)_n$ and $(v_n)_n$, respectively. So, obviously, $U$ is invertible if and only if $V$ is invertible.

Note that the same conclusion applies to the sequences $(x_n)_n$ and $(v_n)_n$ from Corollary 2.7.6 and Corollary 2.7.7. This last fact should be compared with Lemma 2.4.6.

**Remark 2.7.10.** Suppose again that $(x_n)_n$ and $(v_n)_n$ are as in Theorem 2.7.5. Since $V^*U(U^*U)^{-1}$ is an invertible operator, $(x_n)_n$ and $(v_n)_n$ are pseudo-dual frames (see Exercise 2.5.15). Therefore, by Exercise 2.5.15, $(x_n)_n$ and $(v_n)_n$ have the same excess.

In particular, $(x_n)_n$ is a near-Riesz basis if and only $(v_n)_n$ is a near-Riesz basis.

Suppose that $(x_n)_n$ is a frame for a Hilbert space $H$ with the analysis operator $U$ and the lower frame bound $A$. Then we know that $\|Ux\| \geq \sqrt{A}\|x\|$, $\forall x \in H$. Consider the open ball $K(U, \sqrt{A})$ in the norm-topology of $\mathbb{B}(H, \ell^2)$. Each operator $V \in K(U, \sqrt{A})$ is by Exercise 2.1.19 bounded from below which implies that $V^*$ is a surjection. Hence, if denote by $(e_n)_n$ the canonical basis for $\ell^2$, the sequence $(v_n)_n$ defined by $v_n = V^*e_n$, $n \in \mathbb{N}$, is a frame for $H$. This tells us that each operator in the ball $K(U, \sqrt{A})$ is the analysis operator of some frame for $H$.

This conclusion can be also deduced from Theorem 2.7.5. To show this, take any $V \in K(U, \sqrt{A})$ and consider the sequence $(v_n)_n$ defined by $v_n = V^*e_n$, $n \in \mathbb{N}$. Put $\|U - V\| = \mu < \sqrt{A}$. Then we have, for every $N$ in $\mathbb{N}$ and any choice of scalars $c_1, c_2, \ldots, c_N$,

$$\left\| \sum_{n=1}^{N} c_n(x_n - v_n) \right\| = \left\|(U^* - V^*)((c_n)_{n=1}^N)\right\| \leq \|U^* - V^*\| \cdot \|(c_n)_{n=1}^N\| = \mu \left( \sum_{n=1}^{N} |c_n|^2 \right)^{\frac{1}{2}}.$$

So, the sequence $(v_n)_n$ satisfies condition (69) with $\lambda_1 = \lambda_2 = 0$. Therefore, by Theorem 2.7.5, $(v_n)_n$ is a frame for $H$.

One can now raise the following question: does there exist a Parseval frame in this "operator-neighborhood" of $(x_n)_n$? In other words: can we find an isometry in the ball $K(U, \sqrt{A})$?

If $U \in \mathbb{B}(H, K)$ is an operator of Hilbert spaces, we define *minimum modulus* $\gamma(U)$ of $U$ by

$$\gamma(U) = \inf\{\|Ux\| : x \in \mathrm{N}(U)^{\perp},\ \|x\| = 1\}. \tag{79}$$

(see Exercise 2.7.14). Observe that, if $U$ is bounded from below, then $\gamma(U)$ is the optimal lower bound for $U$, i.e. the greatest number that satisfies $\|Ux\| \geq \gamma(U)\|x\|$, for all $x \in H$.

**Proposition 2.7.11.** *Let $H$ and $K$ be Hilbert spaces, and let $U \in \mathbb{B}(H, K)$ be bounded from below. Put $\|U\| = \sqrt{B}$ and $\gamma(U) = \sqrt{A}$. Then there exists an isometry in the ball $K(U, \sqrt{A})$ if and only if the following two conditions are satisfied:*

$$A > \frac{1}{4}, \tag{80}$$

$$\sqrt{B} < \sqrt{A} + 1. \tag{81}$$

**Proof.** Suppose first that there exists an isometry $V \in K(U, \sqrt{A})$. Let $\|U - V\| = \mu < \sqrt{A}$. Then we have

$$\|Ux\| \geq \|Vx\| - \|Vx - Ux\| = \|x\| - \|Vx - Ux\| \geq (1 - \mu)\|x\|, \ \ \forall x \in H.$$

From this we conclude that $\sqrt{A} = \gamma(U) \geq 1 - \mu$; thus, $1 \leq \mu + \sqrt{A} < 2\sqrt{A}$. Therefore, $\frac{1}{4} < A$.

Similarly, we have

$$\|Ux\| \leq \|Ux - Vx\| + \|Vx\| \leq (\mu + 1)\|x\|, \ \ \forall x \in H,$$

which gives us $\|U\| \leq \mu + 1$. Therefore, $\sqrt{B} \leq \mu + 1 < \sqrt{A} + 1$.

To prove the converse, assume (80) and (81). First observe that $\sigma(U^*U) \subseteq [A, B]$. Let $U = VP$ be the polar decomposition of $U$. Notice that $P = \sqrt{U^*U}$ and hence $\sigma(P) \subseteq [\sqrt{A}, \sqrt{B}]$. Secondly, since $P$ is invertible and $\|Px\| = \|Ux\|$ for all $x$, $V$ is an isometry. We also have

$$\|V - U\| = \|V - VP\| \leq \|V\| \cdot \|I - P\| = \|I - P\|.$$

So, in order to finish the proof, it is enough to conclude that $\|I - P\| < \sqrt{A}$.

Since $I - P$ is self-adjoint and $\sigma(I - P) = 1 - \sigma(P)$, we have

$$\|I - P\| = \max\{|\lambda| : \lambda \in (1 - \sigma(P))\} = \max\{|1 - \lambda| : \lambda \in \sigma(P)\}.$$

Observe now that $\sqrt{A}, \sqrt{B} \in \sigma(P)$. Since $\lambda \mapsto 1 - \lambda$ is a monotone function, we conclude that

$$\max\{|1 - \lambda| : \lambda \in \sigma(P)\} = \max\left\{|1 - \sqrt{A}|, |1 - \sqrt{B}|\right\}.$$

We can now split the argument in three cases: (i) $B \leq 1$, (ii) $A \geq 1$, and (iii) $A < 1 < B$. It is now easy to conclude by inspection: if $B \leq 1$ then $\max\left\{|1 - \sqrt{A}|, |1 - \sqrt{B}|\right\} = 1 - \sqrt{A}$, if $A \geq 1$ then $\max\left\{|1 - \sqrt{A}|, |1 - \sqrt{B}|\right\} = \sqrt{B} - 1$, and if $A < 1 < B$ then $\max\left\{|1 - \sqrt{A}|, |1 - \sqrt{B}|\right\}$ is equal either to $1 - \sqrt{A}$ or to $\sqrt{B} - 1$. Since inequalities (80) and (81) imply $1 - \sqrt{A} < \sqrt{A}$ and $\sqrt{B} - 1 < \sqrt{A}$, in each of the above three cases we have $\max\left\{|1 - \sqrt{A}|, |1 - \sqrt{B}|\right\} < \sqrt{A}$. $\square$

**Corollary 2.7.12.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the optimal frame bounds $A_{opt}$ and $B_{opt}$ that satisfy inequalities (80) and (81). Then there exist a Parseval frame $(v_n)_n$ for $H$ and a positive number $\mu < \sqrt{A_{opt}}$ such that $\|x_n - v_n\| \leq \mu$, for all $n$ in $\mathbb{N}$.*

*Concluding remarks.* The formulation of the Paley-Wiener theorem from the beginning of the section is due to Boas (cf. [119], see also [63]). Lemma 2.7.1 is proved in [38] in the setting of Banach spaces. The original Hilbert space result goes back to [85]. Example 2.7.2 is also borrowed from [38] The main result in [38] is Theorem 2.7.5. Corollary 2.7.6 is proved in [52] and Corollary 2.7.7 first appeared in [53]. Proposition 2.7.11 and Corollary 2.7.12 are proved in [23], where one can find some other related results. The argument in Remark 2.7.10 is borrowed from [13], but the fact that frames from Theorem 2.7.5 have the same excess is first proved in [38].

**Exercise 2.7.13.** Let $\{e_1, e_2, \ldots, e_n\}$, $n \in \mathbb{N}$, be an orthonormal set in a Hilbert space $H$. Suppose that $f_1, f_2, \ldots, f_n$ are vectors in $H$ such that $\|e_i - f_i\| < \frac{1}{\sqrt{n}}$ for all $i = 1, 2, \ldots, n$. Show that the set $\{f_1, f_2, \ldots, f_n\}$ is linearly independent ([75]).

**Exercise 2.7.14.** Let $H$ and $K$ be Hilbert spaces and $T \in \mathbb{B}(H, K)$. Prove that

(a) $\gamma(T) > 0$ if and only if $T$ has closed range. When this is the case, we have $\gamma(T) = \frac{1}{\|T^\dagger\|}$.

(b) $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{\frac{1}{2}} = \gamma(TT^*)^{\frac{1}{2}}$.

## 2.8 Reconstruction from frame coefficients with erasures

Frames are often used in process of encoding and decoding signals. It is the redundancy property of frames that makes them robust to erasures and corrupted data.

In applications, we first use a given frame $(x_n)_n$ to compute the frame coefficients $\langle x, x_n\rangle$ of a vector (signal) $x$ (analyzing or encoding $x$) and then apply the reconstruction formula to reconstruct (synthesizing or decoding) $x$ using a suitable dual frame. During the processing the frame coefficients or data transmission some of the coefficients could get lost. Thus, a natural question arises: how to reconstruct the original signal in a best possible way with erasure-corrupted frame coefficients? One possible approach to this problem is to choose the original frame (or an appropriate dual of the original frame) in order to minimize the error. Another approach is oriented towards the perfect reconstruction of the original signal.

However, it is intuitively clear that the perfect reconstruction is impossible in full generality. For example, if we work with a Riesz basis or with a frame with a finite excess, and if lose $k$ frame coefficients of some vector $x$, where $k$ is greater than the excess of the frame, than it is impossible to reconstruct $x$, unless $x$ belongs to the subspace spanned by the remaining frame members.

It turns out that the perfect reconstruction is possible as long as the erased coefficients are indexed by a set that satisfies the minimal redundancy condition - the condition we have already seen in Lemma 2.5.3.

**Definition 2.8.1.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$. We say that a finite set of indices $E$ satisfies the minimal redundancy condition for $(x_n)_n$ if $\overline{span}\{x_n : n \in \mathbb{N} \setminus E\} = H$.*

If the set $E$ satisfies the minimal redundancy condition for a frame $(x_n)_n$, then we will see that the perfect reconstruction of each signal $x$ is always possible even if the coefficients $\langle x, x_n\rangle$, $n \in E$, are lost. Here again, there are two possibilities. First, one can try to reconstruct the lost coefficients using the non-erased ones, and then use the reconstruction formula with any frame dual to the original one. An alternative approach consists of finding a dual frame, depending on the index set of erased coefficients, in order to compensate for errors. More precisely, in the second approach one wants to find a frame $(v_n)_n$ dual to the original frame $(x_n)_n$ such that

$$v_n = 0, \quad \forall n \in E. \tag{82}$$

Obviously, such an "$E^c$-supported" frame $(v_n)_n$ (with $E^c$ denoting the complement of $E$ in the index set), enables the perfect reconstruction using the reconstruction formula $x = \sum_{n=1}^{\infty}\langle x, x_n\rangle v_n$ without knowing or recovering the lost coefficients $\langle x, x_n\rangle$, $n \in E$.

In this section we will discuss both approaches to the perfect reconstruction described above.

We begin by describing "the bridging" - a technique for reconstructing the erased coefficients which is introduced in [93].

Let $(x_n)_n$ be a frame for $H$. Denote by $(y_n)_n$ its canonical dual. Suppose that a set $E$ consisting of $k$ elements, $k \in \mathbb{N}$, satisfies the minimal redundancy condition for $(x_n)_n$. Further, suppose that for some $x \in H$ the coefficients $\langle x, x_n\rangle$, $n \in E$, are lost.

We may assume without loss of generality that $E = \{1, 2, \ldots, k\}$. Let

$$L_E x = \sum_{n=1}^{k}\langle x, x_n\rangle y_n, \quad \text{and} \quad R_E x = \sum_{n=k+1}^{\infty}\langle x, x_n\rangle y_n.$$

Observe that

$$x = L_E x + R_E x. \tag{83}$$

By our assumption, we have $R_E x$, but $L_E x$ is not known.

Consider now a finite set of indices $S$ disjoint from $E$; let $S = \{j_1, j_2, \ldots, j_q\}$, $q \in \mathbb{N}$. The idea is to replace each of the lost coefficients $\langle x, x_n \rangle$ by $\langle x, x'_n \rangle$ such that

$$x'_n \in \text{span}\,\{x_{j_1}, x_{j_2}, \ldots, x_{j_q}\}, \ \ \forall n = 1, 2, \ldots, k. \tag{84}$$

Let

$$B_E x = \sum_{n=1}^{k} \langle x, x'_n \rangle y_n, \quad \text{and} \quad \tilde{x} = B_E x + R_E x. \tag{85}$$

Consider also

$$\tilde{N}_E = I - R_E - B_E = L_E - B_E, \ \ \tilde{N}_E x = \sum_{n=1}^{k} \langle x, x_n - x'_n \rangle y_n \tag{86}$$

We can regard $L_E, R_E, B_E, \tilde{N}_E$ as operators on $H$. Suppose for a moment that $\tilde{N}_E$ is nilpotent of index 2. Since $R_E + B_E = I - \tilde{N}_E$, this implies that $R_E + B_E$ is invertible and $(R_E + B_E)^{-1} = I + \tilde{N}_E$. From this we obtain

$$x \overset{(85)}{=} (R_E + B_E)^{-1} \tilde{x} = (I + \tilde{N}_E)\tilde{x} = \tilde{x} + \tilde{N}_E \tilde{x}. \tag{87}$$

From this we conclude: if we can choose $x'_n$'s as in (84) such that $\tilde{N}_E$ that is given by (86) is nilpotent of index 2, then we can obtain $\tilde{x}$ from (85), and the original vector $x$ can be perfectly reconstructed using (87).

Observe also that the operator $\tilde{N}_E$ will be nilpotent of index 2 if we can choose $x'_1, x'_2, \ldots, x'_n$ in such a way that

$$y_m \perp (x_n - x'_n), \ \ \forall m, n = 1, 2, \ldots, k. \tag{88}$$

**Theorem 2.8.2.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ and let $(y_n)_n$ be its canonical dual. Suppose that the set $E = \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, satisfies the minimal redundancy condition for $(x_n)_n$. Then there exist a set $S = \{j_1, j_2, \ldots, j_q\} \subseteq \mathbb{N} \setminus E$ with $q = dim(span\,\{y_1, y_2, \ldots y_k\})$ and vectors $x'_1, x'_2, \ldots, x'_n$ in $span\,\{x_{j_1}, x_{j_2}, \ldots, x_{j_q}\}$ such that (88) is satisfied. In particular, the operator $\tilde{N}_E$ defined by (86) is nilpotent of index 2.*

**Proof.** Let $H_k = \text{span}\,\{y_1, y_2, \ldots y_k\}$. Then we have $q = \dim H_k$. Let $(b_n)_n$ be an ONB for $H_k^{\perp}$. Since the co-dimension of $H_k^{\perp}$ is equal to $q$, and since (because of the minimal redundancy condition) $\overline{\text{span}}\,\{x_n : n \geq k+1\} = H$, we can find $q$ vectors $x_{j_1}, x_{j_2}, \ldots, x_{j_q}$ in the set $\{x_n : n \geq k+1\}$ such that $(b_n)_n \cup (x_{j_i})_{i=1}^q$ is a Riesz basis for $H$. Then we have

$$x_m = \sum_{i=1}^{q} c_{j_i}^{(m)} x_{j_i} + \sum_{n=1}^{\infty} c_n^{(m)} b_n, \ \ \forall m = 1, 2, \ldots, k. \tag{89}$$

Put

$$x'_m = \sum_{i=1}^{q} c_{j_i}^{(m)} x_{j_i}, \ \ \forall l = 1, 2, \ldots, k. \tag{90}$$

Then (89) implies that $x_m - x'_m \in H_k^{\perp}$; thus $(x_m - x'_m) \perp y_n$ for all $n, m = 1, 2, \ldots, k$. $\square$

**Remark 2.8.3.** It is clear from the proof that the above theorem (together with the preceding considerations) remains true if we replace the canonical dual $(y_n)_n$ by any other dual frame $(v_n)_n$ of $(x_n)_n$.

The set $S$ from Proposition 2.8.2 is called the bridging set.

**Remark 2.8.4.** One should note that there is a small problem in the proof of Theorem 2.8.2 when the ambient space $H$ is finite-dimensional.

To see this, suppose that $\dim H = N$ and that we are given a frame $(x_n)_{n=1}^M$ such that the set of indices $E = \{1, 2, \dots, k\}$ that satisfies the minimal redundancy condition consists of $k \geq N$ elements. Denote again $H_k = \text{span}\,\{y_1, y_2, \dots y_k\}$ and $\dim H_k = q$.

If $H_k = H$, i.e. if $q = N$, then the desired condition (88) forces $x_n' = x_n$ for all $n = 1, 2, \dots, k$. If so, then actually the idea of the bridging method collapses. However one can always recalculate the lost coefficients $\langle x, x_1\rangle, \dots, \langle x, x_k\rangle$ by expressing each $x_n$, $n = 1, \dots k$ as a linear combination of $x_{k+1}, \dots x_M$. Of course, such method of reconstruction of the lost coefficients is at our disposal in every case.

We now turn to the second approach to recovering erasures in which one tries to find a suitable dual frame that enables the perfect reconstruction without recovering the lost coefficients. It is actually easy to see that such dual frames do exist.

**Remark 2.8.5.** Suppose that $(x_n)_n$ is a frame for $H$ for which a finite set $E$ satisfies the minimal redundancy condition. Then there exists a frame $(v_n)_n$ for $H$ dual to $(x_n)_n$ such that $v_n = 0$ for all $n \in E$. Indeed, since, by Lemma 2.5.3, $(x_n)_{n\in E^c}$ is a frame for $H$, by taking an arbitrary dual frame $(v_n)_{n\in E^c}$ of $(x_n)_{n\in E^c}$ and putting $v_n = 0$ for $n \in E$, we get a dual frame $(v_n)_n$ of $(x_n)_n$ with the desired property.

However, from the application point of view this is not enough; what we really need is a concrete construction of such a dual $(v_n)_n$.

In the proposition that follows we give alternative descriptions of the minimal redundancy condition. We first need some additional notation.

Consider an arbitrary frame $(x_n)_n$ for $H$ with the analysis operator $U$ and a finite set of indices $E = \{n_1, n_2, \dots, n_k\}$. Obviously, sequences $(x_n)_{n\in E^c}$ and $(x_n)_{n\in E}$ are Bessel. Denote the corresponding analysis operators by $U_{E^c}$ and $U_E$, respectively. Notice that $(x_n)_{n\in E}$ is finite, so $U_E$ takes values in $\mathbb{F}^k$. It is evident that the corresponding frame operators are given by $U_{E^c}^* U_{E^c} x = \sum_{n\in E^c}\langle x, x_n\rangle x_n$, $U_E^* U_E x = \sum_{n\in E}\langle x, x_n\rangle x_n$, $x \in H$, and hence

$$U_{E^c}^* U_{E^c} = U^*U - U_E^* U_E. \tag{91}$$

Further, if $(y_n)_n$ is the canonical dual of $(x_n)_n$, its analysis operator is $V = U(U^*U)^{-1}$. The analysis operators of Bessel sequences $(y_n)_{n\in E^c}$ and $(y_n)_{n\in E}$ will be denoted by $V_{E^c}$ and $V_E$, respectively. Observe that $V_{E^c} = U_{E^c}(U^*U)^{-1}$ and $V_E = U_E(U^*U)^{-1}$. Since $V^*U = I$, we obtain (in the same way as (91))

$$V_{E^c}^* U_{E^c} = I - V_E^* U_E. \tag{92}$$

**Proposition 2.8.6.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$ and the canonical dual $(y_n)_n$. Let $E = \{n_1, n_2, \dots, n_k\}$ be a finite set of indices. The following statements are equivalent:*

*(a) $E$ satisfies the minimal redundancy condition for $(x_n)_n$,*

*(b) $R(U) \cap \mathrm{span}\,\{e_n : n \in E\} = \{0\}$ where $(e_n)_n$ is the canonical basis for $\ell^2$,*

*(c) $I - V_E^* U_E \in \mathbb{B}(H)$ is invertible,*

*(d) The matrix*

$$\begin{bmatrix} \langle y_{n_1}, x_{n_1}\rangle & \langle y_{n_2}, x_{n_1}\rangle & \dots & \langle y_{n_k}, x_{n_1}\rangle \\ \langle y_{n_1}, x_{n_2}\rangle & \langle y_{n_2}, x_{n_2}\rangle & \dots & \langle y_{n_k}, x_{n_2}\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_{n_1}, x_{n_k}\rangle & \langle y_{n_2}, x_{n_k}\rangle & \dots & \langle y_{n_k}, x_{n_k}\rangle \end{bmatrix} - I$$

*is invertible.*

**Proof.** We can assume without loss of generality that $E = \{1, 2, \dots, k\}$.

(a)$\Leftrightarrow$(b) Suppose that we have $s \in \mathrm{R}(U) \cap \mathrm{span}\,\{e_n : n \in E\}$, $s \neq 0$. Equivalently, there exists $x \in H$, $x \neq 0$, such that $x \perp x_n$ for all $n \in E^c$. By continuity of the inner product, this is equivalent to $x \perp \overline{\mathrm{span}}\,\{x_n : n \in E^c\}$. So, the intersection $\mathrm{R}(U) \cap \mathrm{span}\,\{e_n : n \in E\}$ is non-trivial if and only if the sequence $(x_n)_{n\in E^c}$ is not fundamental in $H$.

(a)$\Leftrightarrow$(c) By Lemma 2.5.3, $E$ satisfies the minimal redundancy condition for $(x_n)_n$ if and only if $(x_n)_{n\in E^c}$ is a frame for $H$, which is the case if and only if the operator $U_{E^c}^* U_{E^c}$ is invertible. Since

$$I - V_E^* U_E \overset{(92)}{=} V_{E^c}^* U_{E^c} = (U^*U)^{-1} U_{E^c}^* U_{E^c},$$

this is further equivalent to invertibility of $I - V_E^* U_E \in \mathbb{B}(H)$.

(c)$\Leftrightarrow$(d) By a well known result, $I - V_E^* U_E \in \mathbb{B}(H)$ is invertible if and only if $I - U_E V_E^* \in \mathbb{B}(\mathbb{F}^k)$ is invertible. But the matrix of $U_E V_E^* - I$ in the canonical basis of $\mathbb{F}^k$ is precisely

$$\begin{bmatrix} \langle y_1, x_1\rangle & \langle y_2, x_1\rangle & \dots & \langle y_k, x_1\rangle \\ \langle y_1, x_2\rangle & \langle y_2, x_2\rangle & \dots & \langle y_k, x_2\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_1, x_k\rangle & \langle y_2, x_k\rangle & \dots & \langle y_k, x_k\rangle \end{bmatrix} - I.$$

$\square$

The following theorem provides a concrete description of a dual frame with the desired property (as in Remark 2.8.5) in terms of the canonical dual.

**Theorem 2.8.7.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the canonical dual $(y_n)_n$. Suppose that a finite set of indices $E = \{n_1, n_2, \dots, n_k\}, k \in \mathbb{N}$, satisfies the minimal redundancy condition for $(x_n)_n$. For each $n \in E^c$ let $(\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nk})$ be a (unique) solution of* *the system*

$$\left(\begin{bmatrix} \langle y_{n_1}, x_{n_1}\rangle & \langle y_{n_2}, x_{n_1}\rangle & \dots & \langle y_{n_k}, x_{n_1}\rangle \\ \langle y_{n_1}, x_{n_2}\rangle & \langle y_{n_2}, x_{n_2}\rangle & \dots & \langle y_{n_k}, x_{n_2}\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_{n_1}, x_{n_k}\rangle & \langle y_{n_2}, x_{n_k}\rangle & \dots & \langle y_{n_k}, x_{n_k}\rangle \end{bmatrix} - I\right) \begin{bmatrix} \alpha_{n1} \\ \alpha_{n2} \\ \vdots \\ \alpha_{nk} \end{bmatrix} = \begin{bmatrix} \langle y_n, x_{n_1}\rangle \\ \langle y_n, x_{n_2}\rangle \\ \vdots \\ \langle y_n, x_{n_k}\rangle \end{bmatrix}. \tag{93}$$

*Put*

$$v_{n_1} = v_{n_2} = \ldots = v_{n_k} = 0, \quad v_n = y_n - \sum_{i=1}^{k} \alpha_{ni} y_{n_i}, \quad n \neq n_1, n_2, \ldots, n_k. \tag{94}$$

*Then $(v_n)_n$ is a frame for $H$ dual to $(x_n)_n$.*

**Proof.** Let $U$ be the analysis operator of $(x_n)_n$. Recall from Corollary 2.2.12 that a frame $(v_n)_n$ with the analysis operator $V$ is dual to $(x_n)_n$ if and only if $V^*$ is of the form $V^* = (U^*U)^{-1}U^*F$ where $F \in \mathbb{B}(\ell^2)$ is the oblique projection to $\mathrm{R}(U)$ parallel to some closed subspace $Y$ of $\ell^2$ such that $\ell^2 = \mathrm{R}(U) \dot{+} Y$.

Hence, to obtain a dual frame $(v_n)_n$ with the property $v_n = 0$ for all $n \in E$, we only need to find a closed direct complement $Y$ of $\mathrm{R}(U)$ in $\ell^2$ such that $e_n \in Y$ for all $n \in E$. Then we will have

$$Fe_n = 0, \quad \forall n \in E,$$

and, consequently,

$$v_n = V^*e_n = (U^*U)^{-1}U^*Fe_n = 0, \quad \forall n \in E.$$

Since $E$ satisfies the minimal redundancy condition for $(x_n)_n$, Proposition 2.8.6 tells us that $\mathrm{R}(U) \cap \mathrm{span}\,\{e_n : n \in E\} = \{0\}$. Denote by $Z$ the orthogonal complement of $\mathrm{R}(U) \dot{+} \mathrm{span}\,\{e_n : n \in E\}$. (Indeed, this is a closed subspace, being a sum of two closed subspaces, one of which is finite-dimensional.) In other words, let

$$\ell^2 = \Big(\mathrm{R}(U) \dot{+} \mathrm{span}\,\{e_n : n \in E\}\Big) \oplus Z. \tag{95}$$

This may be rewritten in the form

$$\ell^2 = \mathrm{R}(U) \dot{+} \left(\mathrm{span}\,\{e_n : n \in E\} \oplus Z\right). \tag{96}$$

Put

$$Y = \mathrm{span}\,\{e_n : n \in E\} \oplus Z. \tag{97}$$

Clearly, $Y$ is a closed direct complement of $\mathrm{R}(U)$ in $\ell^2$ with the desired property.

Assume, without loss of generality, that $E = \{1, 2, \ldots, k\}$. Recall that the synthesis operator of our desired dual $(v_n)_n$ is $V^* = (U^*U)^{-1}U^*F$, so $v_n$'s are given by

$$v_n = (U^*U)^{-1}U^*Fe_n, \quad \forall n \in \mathbb{N}. \tag{98}$$

We want to express $(v_n)_n$ in terms of the canonical dual frame $(y_n)_n$. Recall that

$$y_n = (U^*U)^{-1}U^*e_n, \quad \forall n \in \mathbb{N}. \tag{99}$$

Let $p_n \in \mathrm{R}(U)$ and $a_n \in \mathrm{R}(U)^\perp$ be such that

$$e_n = p_n + a_n, \quad \forall n \in \mathbb{N}. \tag{100}$$

Since $a_n \in \mathrm{R}(U)^\perp = \mathrm{N}(U^*)$, we can rewrite (99) in the form

$$y_n = (U^*U)^{-1}U^*p_n, \quad \forall n \in \mathbb{N}. \tag{101}$$

Recall now that $U(U^*U)^{-1}U^*$ is the orthogonal projection onto $\mathrm{R}(U)$. Hence, by applying $U$ to (101) we get

$$Uy_n = p_n, \quad \forall n \in \mathbb{N}. \tag{102}$$

On the other hand, using (96), we can find $r_n \in \mathrm{R}(U), b_n \in \mathrm{span}\{e_1, e_2, \ldots, e_k\}$ and $c_n \in Z$ such that

$$e_n = r_n + b_n + c_n, \quad \forall n \in \mathbb{N}. \tag{103}$$

Since $F$ is the oblique projection to $\mathrm{R}(U)$ along $\mathrm{span}\{e_1, e_2, \ldots, e_k\} \oplus Z$, we have

$$Fe_n = r_n, \quad \forall n \in \mathbb{N}. \tag{104}$$

Observe that

$$b_n = e_n,\ r_n = 0,\ c_n = 0, \quad \forall n = 1, 2, \ldots, k. \tag{105}$$

Since each $b_n$ belongs to $\mathrm{span}\{e_1, e_2, \ldots, e_k\}$, there exist coefficients $\alpha_{ni}$ such that

$$b_n = \sum_{i=1}^{k} \alpha_{ni} e_i, \quad \forall n \in \mathbb{N}. \tag{106}$$

Note that (105) implies

$$\alpha_{ni} = \delta_{ni}, \quad \forall n, i = 1, 2, \ldots, k. \tag{107}$$

We now have for all $n \in \mathbb{N}$

$$\begin{aligned} e_n &\overset{(103)}{=} r_n + b_n + c_n \\ &\overset{(106)}{=} r_n + \sum_{i=1}^{k} \alpha_{ni} e_i + c_n \\ &\overset{(100)}{=} r_n + \sum_{i=1}^{k} \alpha_{ni}(p_i + a_i) + c_n \\ &= \left( r_n + \sum_{i=1}^{k} \alpha_{ni} p_i \right) + \left( \sum_{i=1}^{k} \alpha_{ni} a_i + c_n \right). \end{aligned}$$

Observe that $\left( r_n + \sum_{i=1}^{k} \alpha_{ni} p_i \right) \in \mathrm{R}(U)$, while $\left( \sum_{i=1}^{k} \alpha_{ni} a_i + c_n \right) \in \mathrm{R}(U)^\perp$. Thus, comparing this last equality with (100) we obtain

$$r_n = p_n - \sum_{i=1}^{k} \alpha_{ni} p_i, \quad a_n = \sum_{i=1}^{k} \alpha_{ni} a_i + c_n, \quad \forall n \in \mathbb{N}. \tag{108}$$

Finally, we conclude that for all $n \in \mathbb{N}$

$$\begin{aligned} v_n &\overset{(98)}{=} (U^*U)^{-1}U^*Fe_n \\ &\overset{(104)}{=} (U^*U)^{-1}U^*r_n \\ &\overset{(108)}{=} (U^*U)^{-1}U^*\left(p_n - \sum_{i=1}^{k}\alpha_{ni}p_i\right) \\ &\overset{(101)}{=} y_n - \sum_{i=1}^{k}\alpha_{ni}y_i. \end{aligned} \tag{109}$$

Note that (109) and (107) show that $v_1 = v_2 = \ldots = v_k = 0$, as required.

So far we have described our desired dual frame $(v_n)_n$ in terms of the canonical dual $(y_n)_n$. Obviously, to obtain $v_n$'s one has to compute all the coefficients $\alpha_{ni}$, $i = 1, 2, \ldots, k$, $n \geq k+1$. To do that, let us first note the following useful consequence of the preceding computation. We claim that

$$\langle v_n, x_i\rangle = -\alpha_{ni}, \quad \forall i = 1, 2, \ldots, k, \quad \forall n \geq k+1. \tag{110}$$

Indeed, for $i = 1, 2, \ldots, k$ and $n \geq k+1$ we have

$$\begin{aligned} \langle v_n, x_i\rangle &= \langle v_n, U^*e_i\rangle \\ &\overset{(98)}{=} \langle U(U^*U)^{-1}U^*Fe_n, e_i\rangle \\ &= \langle Fe_n, e_i\rangle \\ &\overset{(104)}{=} \langle r_n, e_i\rangle \\ &\overset{(103)}{=} \langle e_n - b_n - c_n, e_i\rangle \quad (\text{since } i < n \text{ and } c_n \perp e_i) \\ &= -\langle b_n, e_i\rangle \\ &\overset{(106)}{=} -\alpha_{ni}. \end{aligned}$$

For each $n \geq k+1$ we can rewrite (110), using (109), as

$$\left\langle y_n - \sum_{j=1}^{k}\alpha_{nj}y_j, x_i\right\rangle = -\alpha_{ni}, \quad \forall i = 1, 2, \ldots, k$$

or, equivalently,

$$\sum_{j=1}^{k}\langle y_j, x_i\rangle\alpha_{nj} - \alpha_{ni} = \langle y_n, x_i\rangle, \quad \forall i = 1, 2, \ldots, k.$$

The above equalities can be regarded as a system of $k$ equations in unknowns $\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nk}$ that can be written in the matrix form as

$$\left(\begin{bmatrix} \langle y_1, x_1\rangle & \langle y_2, x_1\rangle & \ldots & \langle y_k, x_1\rangle \\ \langle y_1, x_2\rangle & \langle y_2, x_2\rangle & \ldots & \langle y_k, x_2\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_1, x_k\rangle & \langle y_2, x_k\rangle & \ldots & \langle y_k, x_k\rangle \end{bmatrix} - I\right)\begin{bmatrix} \alpha_{n1} \\ \alpha_{n2} \\ \vdots \\ \alpha_{nk} \end{bmatrix} = \begin{bmatrix} \langle y_n, x_1\rangle \\ \langle y_n, x_2\rangle \\ \vdots \\ \langle y_n, x_k\rangle \end{bmatrix}, \tag{111}$$

where $I$ denotes the unit $k \times k$ matrix. By Proposition 2.8.6 the matrix of the above system is invertible; hence, the system has a unique solution $(\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nk})$ for each $n \geq k+1$.

**Remark 2.8.8.** (a) Clearly, if $(x_n)_n$ is a Parseval frame, our constructed dual frame $(v_n)_n$ is expressed in terms of the original frame members $x_n$'s.

(b) Note that the matrix of the system (93) is independent not only of $n$, but also of all $x \in H$. Thus, the inverse matrix can be computed in advance, without knowing for which $x$ the coefficients $\langle x, x_{n_1}\rangle$, $\langle x, x_{n_2}\rangle, \ldots, \langle x, x_{n_k}\rangle$ will be lost.

(c) The frame $(v_n)_n$ from Theorem 2.8.7 coincides with the canonical dual if and only if $x_{n_1} = x_{n_2} = \ldots = x_{n_k} = 0$.

Finally we note the following obvious corollary to Theorem 2.8.7 in the case $m = 1$:

**Corollary 2.8.9.** *Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the analysis operator $U$ and the canonical dual $(y_n)_n$. Suppose that a set $E = \{m\}$ satisfies the minimal redundancy condition for $(x_n)_n$. Let $v_m = 0$ and*

$$v_n = y_n + \frac{\langle y_n, x_m\rangle}{1 - \langle y_m, x_m\rangle} y_m, \quad \forall n \neq m. \tag{112}$$

*Then $(v_n)_n$ is a frame for $H$ dual to $(x_n)_n$.*

*Concluding remarks.* The discussion in this section is far from complete. For more results we refer the readers to [26, 24, 44, 70, 80, 86, 93, 95, 94] and references therein. Since most of these papers are concerned with finite frames, we will turn back to some of them in the next chapter that is devoted to finite frames. Theorem 2.8.2 is proved in [93]. We refer the reader to that paper for more details concerning various aspects of construction of bridging vectors. Proposition 2.8.6 first appeared in [6]. The equivalence (a)⇔(d) in Proposition 2.8.6 is also proved in Lemma 2.3 from [80] for finite frames using a different technique. More on Theorem 2.8.7 and several related results can be found in [6]. The existence of frames dual to $(x_n)_n$ with pre-determined elements indexed by indices from $E$ is also proved in Theorem 5.2 from [93], but only for finite frames in finite-dimensional spaces.

**Exercise 2.8.10.** Let $(e_n)_n$ be an ONB for a Hilbert space $H$. Consider $a \in H$ such that $\langle a, e_n\rangle \neq 0$ for each $n$ in $\mathbb{N}$. Let $M = (\text{span}\,\{a\})^{\perp}$ and let $P \in \mathbb{B}(H)$ be the orthogonal projection to $M$. Show that $(Pe_n)_n$ is a Parseval frame for $M$ for which the set $E = \{n_0\}$ has the minimal redundancy, for each $n_0 \in \mathbb{N}$. Find the excess of the frame $(Pe_n)_{n \neq n_0}$.

**Exercise 2.8.11.** Suppose that $(x_n)_n$ is a frame for a Hilbert space $H$ for which the set $E = \{1, 2, \ldots, k\}$ ,$k \in \mathbb{N}$, has the minimal redundancy property. Choose any $h_1, h_2, \ldots, h_k$ in $H$. Show that there exists a sequence $(h_n)_{n \geq k+1}$ such that $(h_n)_n$ is a frame for $H$ dual to $(x_n)_n$ ([93], Theorem 5.2).

**Exercise 2.8.12.** For two vectors $x$ and $y$ in a Hilbert space let $\theta_{x,y}$ denote the rank-one operator defined by $\theta_{x,y} v = \langle v, y\rangle x$, $v \in H$. Suppose that $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$,

$k \in \mathbb{N}$, are such that the operator $F = I - \sum_{n=1}^{k} \theta_{x_n,y_n}$ is invertible. Find $F^{-1}$. *Remark.* A closed form formula for $F^{-1}$ is provided in Theorem 6.2 in [93], but under the additional assumption that the set $\{x_1, x_2, \ldots, x_k\}$ is linearly independent. It is proved in [6] that this additional assumption is unnecessary.

# 3 Finite frames

## 3.1 Basics of finite frame theory

Here we consider frames for $N$-dimensional real or complex spaces with $N \in \mathbb{N}$. For the ambient space we may take, without loss of generality, the space of all one-column matrices $M_{N1}(\mathbb{F})$ which we denote by $H_N$.

Recall from Remark 2.1.5 that a finite sequence $(x_n)_{n=1}^M$ is a frame for $H_N$ if and only if $\operatorname{span}\{x_n : 1 \leq n \leq M\} = H_N$. Note also that each finite sequence is Bessel.

Suppose that $(x_n)_{n=1}^M$ is a frame for $H_N$ and write

$$x_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{N1} \end{bmatrix}, x_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{N2} \end{bmatrix}, \ldots, x_M = \begin{bmatrix} x_{1M} \\ x_{2M} \\ \vdots \\ x_{NM} \end{bmatrix}.$$

Observe that the corresponding analysis operator $U$ takes values in $\ell^2_M = \mathbb{F}^M$. It is now evident that the matrix $[U^*]$ of the synthesis operator $U^* \in \mathbb{B}(\ell^2_M, H_N)$ in the pair of canonical bases is given by

$$[U^*] = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1M} \\ x_{21} & x_{22} & \ldots & x_{2M} \\ \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & \ldots & x_{NM} \end{bmatrix} \tag{1}$$

so that we can write

$$[U^*] = \begin{bmatrix} x_1 & x_2 & \ldots & x_M \end{bmatrix}. \tag{2}$$

Thus, each frame (and each Bessel sequence) in $H_N$ can be identified with its synthesis operator; more precisely, with the matrix representation of its synthesis operator in the pair of canonical bases for $\ell^2_M$ and $H_N$.

**Definition 3.1.1.** *A frame $(x_n)_{n=1}^M$ for $H_N$ is said to be uniform if there exists a constant $c$ such that $\|x_n\|^2 = c$ for all $n = 1, 2, \ldots, M$.*

**Proposition 3.1.2.** *Let $(x_n)_{n=1}^M$ be a frame for $H_N$ with the analysis operator $U$. Then the optimal frame bounds coincide with the smallest and the largest eigenvalue of the frame operator $U^*U$. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ are all eigenvalues of $U^*U$, then*

$$\sum_{k=1}^{N} \lambda_k = \sum_{n=1}^{M} \|x_n\|^2.$$

*In particular, if $(x_n)_{n=1}^M$ is Parseval and uniform, then*

$$\|x_n\|^2 = \frac{N}{M}, \ \forall n = 1, 2, \ldots, M.$$

**Proof.** The first assertion is already proved in the preceding chapter; see formula (7) in Proposition 2.1.11.

To prove the second assertion, denote by $(f_k)_{k=1}^N$ the ONB for $H_N$ for which we have $U^*Uf_k = \lambda_k f_k$ for all $k = 1, 2, \ldots, N$. Now we compute:

$$\begin{aligned}
\sum_{k=1}^N \lambda_k &= \sum_{k=1}^N \lambda_k \|f_k\|^2 \\
&= \sum_{k=1}^N \langle \lambda_k f_k, f_k\rangle \\
&= \sum_{k=1}^N \langle U^*Uf_k, f_k\rangle \\
&= \sum_{k=1}^N \|Uf_k\|^2 \\
&= \sum_{k=1}^N \left(\sum_{n=1}^M |\langle f_k, x_n\rangle|^2\right) \\
&= \sum_{n=1}^M \left(\sum_{k=1}^N |\langle x_n, f_k\rangle|^2\right) \\
&= \sum_{n=1}^M \|x_n\|^2.
\end{aligned}$$

The last assertion now follows immediately. □

The following proposition provides more details on the correspondence of finite frames and their synthesis operators.

**Proposition 3.1.3.** *Let $T : \ell_M^2 \to H_N$ be a linear operator, let $(f_k)_{k=1}^N$ be an ONB for $H_N$ and let $(\lambda_k)_{k=1}^N$ be a sequence of positive numbers. Denote by $(e_n)_{n=1}^M$ the canonical basis for $\ell_M^2$ and by $[T]_e^f$ the matrix representation of $T$ with respect to $(e_n)_{n=1}^M$ and $(f_k)_{k=1}^N$. Then the following conditions are equivalent:*

*(a) $(Te_n)_{n=1}^M$ is a frame for $H_N$ and $TT^*$ has eigenvectors $(f_k)_{k=1}^N$ and associated eigenvalues $(\lambda_k)_{k=1}^N$;*

*(b) The rows of $[T]_e^f$ are orthogonal and the square of the norm of the $j$-th row is equal to $\lambda_j$ for each $j = 1, 2, \ldots, N$;*

*(c) The columns of $[T]_e^f$ make up a frame for $H_N$ and $[T]_e^f[T^*]_f^e = diag(\lambda_1, \lambda_2, \ldots, \lambda_N)$.*

**Proof.** Put $[T]_e^f = (t_{ij})$ and notice that then we have $[T^*]_f^e = (\overline{t_{ji}})$.

$(a) \Rightarrow (b)$. We have, for all $i, j = 1, 2, \ldots, N$,

$$\begin{aligned}
\langle (t_{i1}, \ldots, t_{iM}), (t_{j1}, \ldots, t_{jM}) \rangle &= \sum_{n=1}^{M} t_{in} \overline{t_{jn}} \\
&= \left\langle \sum_{n=1}^{M} \overline{t_{jn}} e_n, \sum_{n=1}^{M} \overline{t_{in}} e_n \right\rangle \\
&= \langle T^* f_j, T^* f_i \rangle \\
&= \langle TT^* f_j, f_i \rangle \\
&= \lambda_j \delta_{ij}.
\end{aligned}$$

$(b) \Rightarrow (c)$. From (b) we conclude that $[T]_e^f$ has $N$ linearly independent rows. Hence $\mathrm{r}(T) = N$, $T$ is a surjection, and this tells us that $(Te_n)_{n=1}^M$ is a frame for $H_N$. Consider now the linear map $\varphi : H_N \to H_N$ which every $x$ maps to the column of its coordinates with respect to $(f_k)_{k=1}^N$. Obviously, $\varphi$ is an isomorphism. Observe now that the $n$-th column of $[T]_e^f$ is in fact $\varphi(Te_n)$, for all $n = 1, 2, \ldots, M$. Since $(Te_n)_{n=1}^M$ is a frame for $H_N$ and $\varphi$ is an isomorphism, this implies that the columns of $[T]_e^f$ also form a frame for $H_N$. The second claim in (c) is evident from (b).

$(c) \Rightarrow (a)$. If we assume (c) the first assertion in (a) follows (precisely as above) from the fact that $n$-th column of $[T]_e^f$ is in fact $\varphi(Te_n)$, for all $n = 1, 2, \ldots, M$. Finally, notice that the assumed equality $[T]_e^f [T^*]_f^e = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_N)$ can be rewritten as $[TT^*]_f^f = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_N)$, where $[TT^*]_f^f$ denotes the matrix representation of the operator $TT^*$ with respect to the ONB $(f_k)_{k=1}^N$. $\square$

**Corollary 3.1.4.** *For any $N \times M$ matrix $X = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1M} \\ x_{21} & x_{22} & \ldots & x_{2M} \\ \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & \ldots & x_{NM} \end{bmatrix}$ the following conditions are equivalent:*

*(a) $XX^* = I$;*

*(b) The rows of $X$ form an ON system in $\ell_M^2$;*

*(c) The columns of $X$, $x_n = \begin{bmatrix} x_{1n} \\ \vdots \\ x_{Nn} \end{bmatrix}$, $n = 1, 2, \ldots, M$, make up a Parseval frame for $H_N$.*

**Proof.** $(a) \Leftrightarrow (b)$ is obvious.

$(b) \Rightarrow (c)$. Let $T : \ell_M^2 \to H_N$ be the linear operator whose matrix representation with respect to the pair of canonical bases is $X$. So, if $(e_n)_{n=1}^M$ and $(f_k)_{k=1}^N$ denote the canonical bases in $\ell_M^2$ and $H_N$, respectively, then $[T]_e^f = X$. Thus, if we assume (b), we see that the conditions in (b) from the preceding proposition are fulfilled with $\lambda_k = 1$ for all $k = 1, 2, \ldots, N$. Using the implication $(b) \Rightarrow (c)$ from the preceding proposition, we conclude that $(x_n)_{n=1}^M$ is a frame for $H_N$. Moreover, if we denote by $U$ its analysis operator, we know that $X$ is the matrix of the synthesis operator $U^*$ in the pair of canonical bases, so $XX^* = I$ is equivalent to $U^*U = I$. Hence, $(x_n)_{n=1}^M$ is Parseval.

$(c) \Rightarrow (b)$. This follows from the fact observed in the preceding paragraph that $XX^* = I$ is equivalent to $U^*U = I$. □

**Example 3.1.5.** Let $X = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1M} \\ x_{21} & x_{22} & \ldots & x_{2M} \\ \vdots & \vdots & & \vdots \\ x_{M1} & x_{M2} & \ldots & x_{MM} \end{bmatrix}$ be any unitary $M \times M$ matrix, $M \in \mathbb{N}$. Fix $N \leq M$ and any $N$-tuple $(i_1, i_2, \ldots, i_N)$ of indices such that $1 \leq i_1 < i_2 < \ldots < i_N \leq M$. Let

$$x_n = \begin{bmatrix} x_{i_1 n} \\ x_{i_2 n} \\ \vdots \\ x_{i_N n} \end{bmatrix}, \quad n = 1, 2, \ldots, M.$$

Then, by the preceding corollary, $(x_n)_{n=1}^M$ is a Parseval frame for $H_N$. So, if we take any $N$ rows of a unitary $M \times M$ matrix, we obtain a Parseval frame for $H_N$ consisting of $M$ vectors.

It should be noted that the same conclusion can also be obtained from the observation from the beginning of Section 2.1; see also Proposition 2.1.16 and Proposition 2.1.17.

**Example 3.1.6.** Given $M \in \mathbb{N}$, we let $\omega = e^{\frac{2\pi i}{M}}$. Then the discrete Fourier transform $M \times M$ matrix DFT(M) is defined as

$$DFT(M) = (\frac{1}{\sqrt{M}}\omega^{jk})_{j,k=0}^{M-1} = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{M-1} \\ 1 & \omega^2 & (\omega^2)^2 & \ldots & (\omega^{M-1})^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{M-1} & (\omega^2)^{M-1} & \ldots & (\omega^{M-1})^{M-1} \end{bmatrix}. \tag{3}$$

It is not difficult to show that $DFT(M)$ is a unitary matrix. Thus, for each $N < M$, any selection of $N$ rows yields a uniform Parseval frame for the complex space $H_N$. In particular, if we choose the first $N$ rows, we get the frame $(x_n)_{n=1}^M$,

$$x_1 = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, x_2 = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 \\ \omega \\ \vdots \\ \omega^{N-1} \end{bmatrix}, \ldots, x_M = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 \\ \omega^{M-1} \\ \vdots \\ (\omega^{M-1})^{N-1} \end{bmatrix}.$$

that is called the complex harmonic frame.

**Example 3.1.7.** Let $M \in \mathbb{N}$. If $M = 2k$ we define

$$D_{M=2k}(\mathbb{R}) = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ 1 & \cos\frac{2\pi}{M} & \cos\frac{2\cdot 2\pi}{M} & \cdots & \cos\frac{(M-1)2\pi}{M} \\ 0 & \sin\frac{2\pi}{M} & \sin\frac{2\cdot 2\pi}{M} & \cdots & \sin\frac{(M-1)2\pi}{M} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos\frac{2(k-1)\pi}{M} & \cos\frac{2\cdot 2(k-1)\pi}{M} & \cdots & \cos\frac{(M-1)2(k-1)\pi}{M} \\ 0 & \sin\frac{2(k-1)\pi}{M} & \sin\frac{2\cdot 2(k-1)\pi}{M} & \cdots & \sin\frac{(M-1)2(k-1)\pi}{M} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{4}$$

One can check that $D_{M=2k}(\mathbb{R})$ is a unitary matrix. For $N$ odd, $N = 2l+1 < M$, we take first $N$ rows of the above matrix to obtain a real harmonic frame $(x_n)_{n=0}^{M-1}$ for $H_N$;

$$x_n = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \cos\frac{n\cdot 2\pi}{M} \\ \sin\frac{n\cdot 2\pi}{M} \\ \vdots \\ \cos\frac{n\cdot 2\pi\frac{N-1}{2}}{M} \\ \sin\frac{n\cdot 2\pi\frac{N-1}{2}}{M} \end{bmatrix}, \quad n = 0, 1, \ldots, M-1.$$

If $M = 2k+1$ we define

$$D_{M=2k+1}(\mathbb{R}) = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ 1 & \cos\frac{2\pi}{M} & \cos\frac{2\cdot 2\pi}{M} & \cdots & \cos\frac{(M-1)2\pi}{M} \\ 0 & \sin\frac{2\pi}{M} & \sin\frac{2\cdot 2\pi}{M} & \cdots & \sin\frac{(M-1)2\pi}{M} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos\frac{2k\pi}{M} & \cos\frac{2\cdot 2k\pi}{M} & \cdots & \cos\frac{(M-1)2k\pi}{M} \\ 0 & \sin\frac{2k\pi}{M} & \sin\frac{2\cdot 2k\pi}{M} & \cdots & \sin\frac{(M-1)2k\pi}{M} \end{bmatrix} \tag{5}$$

Again, $D_{M=2k+1}(\mathbb{R})$ is a unitary matrix. For $N$ even, $N = 2l < M$, we omit the first row and take next $N$ rows to obtain a real harmonic frame $(x_n)_{n=0}^{M-1}$ for $H_N$;

$$x_n = \sqrt{\frac{2}{M}} \begin{bmatrix} \cos\frac{n\cdot 2\pi}{M} \\ \sin\frac{n\cdot 2\pi}{M} \\ \vdots \\ \cos\frac{n\cdot 2\pi\frac{N}{2}}{M} \\ \sin\frac{n\cdot 2\pi\frac{N}{2}}{M} \end{bmatrix}, \quad n = 0, 1, \ldots, M-1.$$

**Corollary 3.1.8.** *For all $M$ and $N$ such that $M > N$ there exists a uniform Parseval frame for $H_N$ consisting of $M$ elements.*

An important problem both in theory and applications is a question of construction methods of finite frames; in particular, of frames with some additional properties (uniform, equiangular, etc). This also includes methods for extending given finite sequence of vectors to a frame. It turns out that the following theorem, which we include without proof, serves as an important tool for extending finite sequences to frames.

**Theorem 3.1.9.** *([47]) Let $S$ be a positive operator on $H_N$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N > 0$ be the eigenvalues of $S$. Fix $M \geq N$ and real numbers $a_1 \geq a_2 \geq \ldots \geq a_M > 0$. Then the following conditions are equivalent:*

*(a) There exists a frame $(x_n)_{n=1}^M$ for $H_N$ with the analysis operator $U$ such that $U^*U = S$ and $\|x_n\| = a_n$ for all $n = 1, 2, \ldots, M$;*

*(b) For every $k$, $1 \leq k \leq N$,*

$$\sum_{i=1}^{k} a_i^2 \leq \sum_{i=1}^{k} \lambda_i \quad \textit{and} \quad \sum_{i=1}^{M} a_i^2 = \sum_{i=1}^{N} \lambda_i.$$

We now demonstrate several results which provide methods for extending finite sequences of vectors to frames. The first one is fairly simple.

**Proposition 3.1.10.** *Let $(x_n)_{n=1}^M$ be a sequence of unit vectors in $H_N$. Then there exists a uniform tight frame $(x_{nj})_{n,j=1}^{M,N}$ for $H_N$ such that $x_{n1} = x_n$ for every $n = 1, 2, \ldots, M$.*

**Proof.** For each $n$ we can find an ONB $(x_{nj})_{j=1}^N$ for $H_N$ such that $x_{n1} = x_n$. □

The following proposition provides an extension of a given sequence to a frame by adding much smaller number of vectors.

**Proposition 3.1.11.** *Let $(x_n)_{n=1}^M$ be a sequence of vectors in $H_N$ such that $x_n \neq 0$ for at least one $n$. Then there exists a sequence $(h_j)_{j=2}^N$ such that $(x_n)_{n=1}^M \cup (h_j)_{j=2}^N$ is a tight frame for $H_N$.*

**Proof.** Let $U$ be the analysis operator of $(x_n)_{n=1}^M$. Put $S = U^*U$ and observe that $Sx = \sum_{n=1}^M \langle x, x_n \rangle x_n$, $x \in H_N$. Let $(g_j)_{j=1}^N$ be the ON eigenbasis for $S$ with respective eigenvalues $(\lambda_j)_{j=1}^N$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Notice that $\lambda_1 > 0$ since $S \neq 0$.

For $2 \leq j \leq N$, let $h_j = \sqrt{\lambda_1 - \lambda_j} g_j$. Denote by $U_1$ the analysis operator of $(x_n)_{n=1}^M \cup (h_j)_{j=2}^N$. Let $S_1 = U_1^* U_1$. Then we have

$$\begin{aligned}
S_1 x &= \sum_{n=1}^{M} \langle x, x_n \rangle x_n + \sum_{j=2}^{N} \langle x, h_j \rangle h_j \\
&= \sum_{j=1}^{N} \lambda_j \langle x, g_j \rangle g_j + \sum_{j=2}^{N} (\lambda_1 - \lambda_j) \langle x, g_j \rangle g_j \\
&= \lambda_1 \sum_{j=1}^{N} \langle x, g_j \rangle g_j \\
&= \lambda_1 x.
\end{aligned}$$

$\square$

The frame that is obtained by the method from the preceding proof is tight, but not uniform (even if the original sequence consists of vectors of equal norms).

**Proposition 3.1.12.** *Let $(x_n)_{n=1}^M$ be a sequence of unit vectors in $H_N$ with the optimal Bessel bound $B$. Then there exists a sequence $(g_j)_{j=1}^K$, $K \in \mathbb{N}$, of unit vectors such that $(x_n)_{n=1}^M \cup (g_j)_{j=1}^K$ is a uniform tight frame with tight frame bound $A \leq B + 2$.*

**Proof.** Let $U$ denote the analysis operator of $(x_n)_{n=1}^M$. Write $S = U^*U$ and denote by $(e_i)_{i=1}^N$ the eigenbasis for $H_N$ with respective eigenvalues $(\lambda_i)_{i=1}^N$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$. Observe that $B = \lambda_1$. We also have

$$\begin{aligned}
\sum_{j=1}^N \lambda_j &= \sum_{j=1}^N \lambda_j \langle e_j, e_j \rangle \\
&= \sum_{j=1}^N \langle U^*Ue_j, e_j \rangle \\
&= \sum_{j=1}^N \|Ue_j\|^2 \\
&= \sum_{j=1}^N \left( \sum_{n=1}^M |\langle e_j, x_n \rangle|^2 \right) \\
&= \sum_{n=1}^M \left( \sum_{j=1}^N |\langle x_n, e_j \rangle|^2 \right) \\
&= \sum_{n=1}^M \|x_n\|^2 \\
&= M. \qquad (6)
\end{aligned}$$

For $\varepsilon \in [0, 1]$ consider $f(\varepsilon) = N(\lambda_1 + 1 + \varepsilon) - M$. Notice that $f(0) = N\lambda_1 + N - M$ and $f(1) = N\lambda_1 + 2N - M$. Thus, there exists $\varepsilon \in [0, 1]$ for which we have $f(\varepsilon) = N(\lambda_1 + 1 + \varepsilon) - M = K \in \mathbb{N}$. Observe that (6) implies $K \geq N$.

Let us now define the operator $S_0$ on $H_N$ by $S_0 e_j = ((\lambda_1 + 1 + \varepsilon) - \lambda_j)e_j$, $j = 1, 2, \ldots, N$. Note that $S_0 \geq 0$.

Since each of the eigenvalues of $S_0$ is greater than 1, letting $a_i = 1$ for $i = 1, 2, \ldots, K$, we immediately obtain the first condition (the inequality) in Theorem 3.1.9 (b). Also,

$$\sum_{j=1}^N ((\lambda_1 + 1 + \varepsilon) - \lambda_j) = N(\lambda_1 + 1 + \varepsilon) - \sum_{j=1}^N \lambda_j \overset{(6)}{=} N(\lambda_1 + 1 + \varepsilon) - M = K = \sum_{i=1}^K a_i^2.$$

We are now in position to apply Theorem 3.1.9: there exists a sequence $(g_j)_{j=1}^K$ of unit vectors which is a frame for $H_N$ having $S_0$ for its frame operator (that is, $S_0 x = \sum_{j=1}^K \langle x, g_j \rangle g_j$, $x \in H_N$).

Consider now $(x_n)_{n=1}^M \cup (g_j)_{j=1}^K$. We know that

$$(S+S_0)x = \sum_{n=1}^{M} \langle x, x_n \rangle x_n + \sum_{j=1}^{K} \langle x, g_j \rangle g_j, \ \ \forall x \in H_N,$$

and, on the other hand,

$$(S+S_0)e_j = \lambda_j e_j + ((\lambda_1 + 1 + \varepsilon) - \lambda_j)e_j = (\lambda_1 + 1 + \varepsilon)e_j, \ \ \forall j = 1, 2, \ldots, N. \tag{7}$$

Put $\lambda_1 + 1 + \varepsilon = A$. Then $A \leq \lambda_1 + 2 = B + 2$ and (7) tells us that $(x_n)_{n=1}^M \cup (g_j)_{j=1}^K$ is an $A$-tight frame for $H_N$. □

We end these considerations by showing that any finite sequence in $H_N$ can be extended to a tight frame for $H_N$.

**Proposition 3.1.13.** *Let $(x_n)_{n=1}^M$ be a finite sequence in $H_N$ with a Bessel bound $B$. Then there exists a sequence $(g_j)_{j=1}^N$ in $H_N$ such that $(x_n)_{n=1}^M \cup (g_j)_{j=1}^N$ is a $B$-tight frame for $H_N$.*

**Proof.** Denote by $U$ the analysis operator of $(x_n)_{n=1}^M$. Then we know that $U^*U \leq B \cdot I$; thus, $B \cdot I - U^*U \geq 0$. By Exercise 3.1.20, there exists a sequence $(g_j)_{j=1}^N$ in $H_N$ with the analysis operator $V$ such that $V^*V = B \cdot I - U^*U$. This implies that

$$\sum_{n=1}^{M} \langle x, x_n \rangle x_n + \sum_{j=1}^{N} \langle x, g_j \rangle g_j = U^*Ux + V^*Vx = U^*Ux + (B \cdot I - U^*U)x = B \cdot x, \ \ \forall x \in H_N;$$

thus, $(x_n)_{n=1}^M \cup (g_j)_{j=1}^N$ is a $B$-tight frame for $H_N$. □

Consider a frame $(x_n)_{n=1}^M$ for $H_N$ and assume we are given the image of a signal $x \in H_N$ under the analysis operator $U$:

$$Ux = \left(\langle x, x_n \rangle\right)_{n=1}^M.$$

Theoretically, one can reconstruct $x$ using the reconstruction formula

$$x = \sum_{n=1}^{M} \langle x, x_n \rangle y_n,$$

where $(y_n)_{n=1}^M$ is the canonical dual frame. Recall that $y_n = (U^*U)^{-1}x_n$, $n = 1, 2, \ldots, M$. In applications the reconstruction formula might not be utilizable because inversion is computationally expensive and numerically instable. Proposition that follows (Frame algorithm) provides us with an iterative method to derive a converging sequence of approximations of $x$ from the knowledge of frame coefficients $\langle x, x_n \rangle$, $n = 1, 2, \ldots, M$.

**Proposition 3.1.14.** *Let $(x_n)_{n=1}^M$ be a frame for $H_N$ with frame bounds $A, B$ and the analysis operator $U$. Given a vector $x \in H_N$, define a sequence $(v_k)_{k=0}^\infty$ in $H_N$ by*

$$v_0 = 0, \ \ v_k = v_{k-1} + \frac{2}{A+B} U^*U(x - v_{k-1}), \ \ \forall k \in \mathbb{N}.$$

*Then* $x = \lim_{k\to\infty} v_k$ *and the rate of convergence is*

$$\|x - v_k\| \le \left(\frac{B-A}{B+A}\right)^k \|x\|, \quad k \ge 0.$$

**Proof.** First observe that for all $x \in H_N$ we have

$$\left\langle \left(I - \frac{2}{A+B}U^*U\right)x, x\right\rangle = \|x\|^2 - \frac{2}{A+B}\sum_{n=1}^{M}|\langle x, x_n\rangle|^2 \le \|x\|^2 - \frac{2A}{A+B}\|x\|^2 = \frac{B-A}{B+A}\|x\|^2.$$

In a similar way one obtains

$$-\frac{B-A}{B+A}\|x\|^2 \le \left\langle \left(I - \frac{2}{A+B}U^*U\right)x, x\right\rangle.$$

Hence,

$$\left\| I - \frac{2}{A+B}U^*U \right\| \le \frac{B-A}{B+A}. \tag{8}$$

By definition of $v_k$ we have, for any $k \ge 0$,

$$x - v_k = x - v_{k-1} - \frac{2}{A+B}U^*U(x - v_{k-1}) = \left(I - \frac{2}{A+B}U^*U\right)(x - v_{k-1}).$$

Iterating this computation we obtain

$$x - v_k = \left(I - \frac{2}{A+B}U^*U\right)^k (x - v_0).$$

We now use (8) to get

$$\|x - v_k\| = \left\| \left(I - \frac{2}{A+B}U^*U\right)^k (x - v_0)\right\| \le \left\| I - \frac{2}{A+B}U^*U \right\|^k \|x\| \le \left(\frac{B-A}{B+A}\right)^k \|x\|.$$

□

**Remark 3.1.15.** (a) We assume in the preceding algorithm that $A \ne B$, i.e. that our frame is not tight. If, on the other hand, $A = B$, then $U^*U = A \cdot I$ and the reconstruction formula reads $x = \sum_{n=1}^{m} \langle x, x_n\rangle \frac{1}{A} x_n$.

(b) Observe that the iteration formula contains $x$, but the algorithm uses only the frame coefficients $\langle x, x_n\rangle$, $n = 1, 2, \dots, M$, since

$$U^*U(x - v_{k-1}) = \sum_{n=1}^{m}(\langle x, x_n\rangle - \langle v_{k-1}, x_n\rangle)x_n.$$

(c) The inspection of the proof shows that the proposition is also true for infinite frames.

(d) One drawback of the frame algorithm is the fact that the convergence rate depends on the ratio of the frame bounds in a sensitive way. A large ratio of the frame bounds leads to very slow convergence. To tackle this problem, in [72], the Chebyshev method and the conjugate gradient method were introduced.

*Concluding remarks.* The material in this Section is standard. We refer the reader to [45] for a comprehensive exposition of the finite frame theory.

Theorem 3.1.9 is proved in [47]. Proposition 3.1.11, Proposition 3.1.12, and Proposition 3.1.13 first appeared in [48]. Here, the reader is also referred to [31] and [65]. In general, frame constructions have a long history; in particular, there are various methods for construction of uniform Parseval frames since those frames are most advantageous for applications. A relevant concept in some constructions is the so-called frame potential; see [20] and [40]. More on harmonic frames can be found in [44], [70], and [116]. In [42] the so-called *Spectral Tetris* algorithm for constructing of uniform tight frames is introduced; see also [43], [45], and [50]

A frame $(x_n)_{n=1}^M$ is said to be equiangular if there exists a constant $c$ such that $|\langle x_n, x_m\rangle| = c$ for all $n \neq m$. For theoretical aspects of equiangular frames and their applications we refer the reader to [17], [18], [24], [66], [86], [112], and [113].

A frame $(x_n)_{n=1}^M$ is said to be scalable if there exist non-negative numbers $\alpha_1, \alpha_2, \dots \alpha_M$ such that $(\alpha_n x_n)_{n=1}^M$ is a Parseval frame. Scalable frames are introduced and studied in [92].

Here we also mention some of the papers devoted to various methods (and appropriately designed frames) compensating for erasures, noise reduction and similar disturbances in signal transmissions: [44], [24], [71], [86], [94], [95], [80]

For connection of finite frames to sparsity methodologies we refer to [43], [91], and [45].

**Exercise 3.1.16.** Let

$$x_1 = \sqrt{\frac{2}{3}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_2 = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}, x_3 = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{-\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

Show that $(x_n)_{n=1}^3$ is a uniform equiangular Parseval frame for $H_2$. For obvious reasons, this frame is called the Mercedes-Benz frame.

**Exercise 3.1.17.** Prove that every finite frame can be completed to an invertible matrix by adding a suitable set of rows. (Here we identify frames with the matrix representations of their synthesis operators in the pair of canonical bases.)

**Exercise 3.1.18.** Show that the matrices given by (3), (4), and (5) from Example 3.1.6 and Example 3.1.7 are unitary.

**Exercise 3.1.19.** Prove that DFT(M) diagonalizes the cyclic shift $Z_M = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & & & 0 \end{bmatrix}$.

**Exercise 3.1.20.** Let $S$ be a positive operator on a Hilbert space $H$. Prove that there is a Bessel sequence $(g_n)_n$ in $H$ with the analysis operator $V$ such that $V^*V = S$. If $\dim H = N < \infty$ show that one can find a finite sequence in $H$, consisting of precisely $N$ vectors, with the desired property.

**Exercise 3.1.21.** Let $(x_n)_{n=1}^M$ be a Parseval frame for $H_N$. If $T$ is any linear operator on $H_N$ show that $\operatorname{tr} T = \sum_{n=1}^M \langle T x_n, x_n\rangle$.

### 3.2 Full spark frames

**Definition 3.2.1.** *A frame $(x_n)_{n=1}^M$ for $H_N$ is said to be 1-robust if for every $j$, $1 \leq j \leq M$, the reduced sequence $(x_n)_{n=1}^{j-1} \cup (x_n)_{n=j+1}^M$ is a frame for $H_N$. (In other words, $(x_n)_{n=1}^M$ is 1-robust if each $\{j\}$, $j = 1, 2, \ldots, M$, satisfies the minimal redundancy condition.)*

*Similarly, we say that $(x_n)_{n=1}^M$ is $K$-robust, $K \in \mathbb{N}$, if any set of different indices $\{i_1, i_2, \ldots, i_K\} \subset \{1, 2, \ldots, M\}$ of cardinality $K$ satisfies the minimal redundancy condition for $(x_n)_{n=1}^M$.*

One can show (see Exercise 3.2.24) that each Parseval frame $(x_n)_n$ with the property that $\|x_n\| < 1$ for all $n$ is 1-robust.

Clearly, a $K$ robust frame is resistant to erasures of *any* $K$ frame coefficients. So, in applications it is most convenient to work with $K$-robust frames with maximal possible $K$. Obviously, a frame $(x_n)_{n=1}^M$ for $H_N$ is maximally robust if it is $(M-N)$-robust and when this is the case, for any set of indices $\{i_1, i_2, \ldots, i_{M-N}\} \subset \{1, 2, \ldots, M\}$ of cardinality $M - N$, the reduced sequence $(x_n)_{n \in \{1,2,\ldots,M\} \setminus \{i_1, i_2, \ldots, i_{M-N}\}}$ is a basis for $H_N$. Such frames are called *full spark frames.*

**Definition 3.2.2.** *([61]) Let $T \in M_{NM}$ be a matrix with columns $S_1, S_2, \ldots, S_M$. The spark of $T$ is the cardinality of the smallest linearly dependent subset of $\{S_1, S_2, \ldots, S_M\}$.*

Equivalently, spark$(T)$ can be formulated in terms of the Hamming weight. Recall that the Hamming weight is defined, for any $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix} \in H_M$, by

$$\|v\|_0 = \text{card}\,\{j \in \{1, 2, \ldots, M\} : v_j \neq 0\}.$$

Observe now that $Tv = v_1 S_1 + v_2 S_2 + \ldots + v_M S_M$. From this we conclude that

$$\text{spark}(T) = \min\,\{\|v\|_0 : Tv = 0, v \neq 0\}. \tag{9}$$

When $T$ has a zero-column, then, obviously, spark$(T) = 1$. Similarly, spark$(T) = 2$ means that all columns of $T$ are non-trivial and at least two columns are proportional.

If $T$ is a $N \times M$ matrix with $M \leq N$ it can happen that the set of all its columns $\{S_1, S_2, \ldots, S_M\}$ is linearly independent. When this is the case, we understand that spark$(T) = \infty$.

If $T$ is a $N \times M$ matrix with $M > N$ (which is the case of our interest since frames for $H_N$ are represented by such "long" matrices), then any set of $N + 1$ columns of $T$ is linearly dependent. Hence, the spark of each "long" matrix $(M > N)$ is at most $N + 1$.

**Definition 3.2.3.** *We say that $T \in M_{NM}$, $M > N$, is a full spark matrix if spark$(T) = N+1$.*

**Remark 3.2.4.** It is useful to note the following immediate observation: $T \in M_{NM}$, $N < M$, is a full spark matrix if and only if any set of its columns of cardinality $N$ is linearly independent.

**Remark 3.2.5.** Suppose that $(x_n)_{n=1}^M$ is a frame for $H_N$. Let

$$x_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{Nn} \end{bmatrix}, \quad n = 1, 2, \ldots, M.$$

As before, we identify $(x_n)_{n=1}^M$ with the matrix $X = [U^*]$ of its synthesis operator in the pair of canonical bases for $\ell_M^2$ and $H_N$:

$$X = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1M} \\ x_{11} & x_{12} & \ldots & x_{1M} \\ \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & \ldots & x_{NM} \end{bmatrix}.$$

It is now immediate from Definition 3.2.1 together with the subsequent comments and Remark 3.2.4 that $(x_n)_{n=1}^M$ is a full spark frame (that is, maximally robust) if and only if $X$ is a full spark matrix.

**Example 3.2.6.** For $N < M$ and any sequence $(\alpha_n)_{n=1}^M$ of distinct scalars let

$$V_{N,\alpha_1,\alpha_2,\ldots,\alpha_M} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_M \\ \vdots & \vdots & & \vdots \\ \alpha_1^{N-1} & \alpha_2^{N-1} & \ldots & \alpha_M^{N-1} \end{bmatrix}.$$

Then (Vandermonde) $V_{N,\alpha_1,\alpha_2,\ldots,\alpha_M}$ is a full spark matrix.

A subject of a prominent interest in frame theory is to develop methods for construction of full spark frames, possibly with some additional properties (e.g. uniform, equiangular etc). Note that immediately from the preceding example we get

**Example 3.2.7.** The complex harmonic frame $(x_n)_{n=1}^M$ from Example 3.1.6 is full spark.

**Remark 3.2.8.** Recall that the complex harmonic frame is obtained by taking the first $N$ rows and deleting the remaining $M-N$ rows from the descrete Fourier transform matrix DFT(M). We know that deleting *any* $M-N$ rows from DFT(M) yields a uniform Parseval frame for $H_N$. However, such frames are not necessarily full spark.

Consider

$$DFT(4) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

After deleting the second and the forth row we get the matrix

$$X = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

which represents a uniform Parseval frame for $H_2$ consisting of 4 vectors. Obviously, $\text{spark}(X) = 2$ and hence this frame is not full spark.

In this context, the following classical result (which we include without proof) is useful.

**Theorem 3.2.9.** *(Chebotarëv, [2], [109]) Let $M$ be prime. Then every square submatrix of DFT(M) is invertible.*

**Corollary 3.2.10.** *Let $M$ be prime and $N < M$. Then every choice of $N$ rows from DFT(M) produces a full spark uniform Parseval frame for $H_N$.*

The first class of real full spark uniform tight frames is constructed in [103]. The same paper also provides a technique for constructing full spark tight frames based on some properties of orthogonal polynomials ([111]).

Another construction of full spark tight frames appears in [2]. The proof of the following theorem, which we omit, is based on Chebotarëv's theorem.

**Theorem 3.2.11.** *([2], [114]) Let $M$ be prime and pick any $N \le M$ rows of DFT(M); denote by $F$ the resulting $N \times M$ matrix. Next, pick any $K \le N$ and take $D$ to be the $N \times N$ diagonal matrix whose first $K$ diagonal entries are $\sqrt{\frac{M+K-N}{MN}}$, and whose remaining $N-K$ diagonal entries are $\sqrt{\frac{M+K}{MN}}$. Then concatenating $DF$ with the first $K$ elements of the canonical basis for $H_N$ produces a full spark uniform tight frame for $H_N$ consisting of $M+K$ vectors.*

**Example 3.2.12.** Let $M = 5, N = 3, K = 1$, and $\omega = e^{\frac{2\pi i}{5}}$. Let $F$ be the matrix obtained by taking the first, the second, and the fifth row of $DFT(5)$:

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}.$$

Next we take

$$D = \begin{bmatrix} \sqrt{\frac{1}{5}} & 0 & 0 \\ 0 & \sqrt{\frac{2}{5}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{5}} \end{bmatrix}.$$

Then by Theorem 3.2.11 the columns of

$$X = \begin{bmatrix} \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & 1 \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{2}{5}}\omega & \sqrt{\frac{2}{5}}\omega^2 & \sqrt{\frac{2}{5}}\omega^3 & \sqrt{\frac{2}{5}}\omega^4 & 0 \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{2}{5}}\omega^4 & \sqrt{\frac{2}{5}}\omega^3 & \sqrt{\frac{2}{5}}\omega^2 & \sqrt{\frac{2}{5}}\omega & 0 \end{bmatrix}.$$

*make up a full spark uniform tight frame for $H_3$.*

In the remaining part of this section we will characterize all finite full spark frames and provide another technique for constructing such frames.

**Definition 3.2.13.** *An $N \times M$ matrix $T$ is said to be totally non-singular if all its square submatrices are invertible.*

**Theorem 3.2.14.** *Let $(x_n)_{n=1}^N$ be a basis for $H_N$ and let $T = (t_{ij}) \in M_{NK}$, $K \in \mathbb{N}$, be a totally non-singular matrix. Define $x_{N+1}, x_{N+2}, \dots, x_{N+K} \in H_N$ by*

$$x_{N+j} = \sum_{i=1}^{N} t_{ij} x_i, \quad \forall j = 1, 2, \dots, K. \tag{10}$$

*Then $(x_n)_{n=1}^{N+K}$ is a full spark frame for $H_N$.*

*Conversely, each full spark frame for $H_N$ is of this form. More precisely, if $(x_n)_{n=1}^{N+K}$ is a full spark frame for $H_N$, then there is a totally non-singular matrix $T = (t_{ij}) \in M_{NK}$ such that $x_{N+1}, x_{N+2}, \dots, x_{N+K}$ are of the form (10).*

**Proof.** Suppose that we are given a basis $(x_n)_{n=1}^N$ for $H_N$ and a totally non-singular matrix $T = (t_{ij}) \in M_{NK}$. Consider $(x_n)_{n=1}^{N+K}$ with $x_{N+1}, x_{N+2}, \dots, x_{N+K} \in H_N$ defined by (10). Let $k$ be a natural number such that $1 \leq k \leq N, K$. Consider two arbitrary sets of indices of cardinality $k$; $I = \{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, N\}$ and $J = \{j_1, j_2, \dots, j_k\} \subseteq \{1, 2, \dots, K\}$ and let $I^c = \{1, 2, \dots, N\} \setminus I$. We must prove that a reduced sequence

$$(x_n)_{n \in I^c} \cup (x_{N+j})_{j \in J} \tag{11}$$

is a basis for $H_N$.

Note that the case $k = 0$ is trivial. On the other hand, if $N \leq K$ and $k = N$ (so that all $x_n$'s are omitted), then our assumption on $T$ guarantees that the resulting sequence is a basis for $H_n$. Thus, we only need to consider the case $1 \leq k < N$ and the sequence of the form (11).

Denote by $C \in M_N$ the matrix that is obtained by representing our reduced sequence (11) in the basis $(x_n)_{n=1}^N$. It suffices to show that $C$ is an invertible matrix. We shall show, using an argument from the proof of Theorem 6 in [2], that $\det C \neq 0$. By suitable changes of rows and columns of $C$, where only the first $N-k$ columns of $C$ are involved, we get a block-matrix $C'$ of the form

$$C' = \begin{bmatrix} I_{N-k} & T' \\ 0 & T'' \end{bmatrix}$$

where $I_{N-k} \in M_{N-k}$ is a unit matrix while $T' \in M_{N-k,k}$ and $T'' \in M_k$ are some submatrices of $T$ (up to appropriate permutation of rows and columns). By the hypothesis, $T''$ is invertible. Hence, $\det C' = \det I_{N-k} \cdot \det T'' = \det T'' \neq 0$ and this obviously implies $\det C \neq 0$.

To prove the converse, suppose that $(x_n)_{n=1}^{N+K}$ is an arbitrary full spark frame for $H_N$. In particular, $(x_n)_{n=1}^N$ is a basis for $H_N$, so there exist numbers $t_{ij}$ such that $x_{N+1}, x_{N+2}, \dots, x_{N+K}$ are of the form (10). We must prove that each square submatrix of $T = (t_{ij}) \in M_{NK}$ is invertible.

Consider two sets of indices $I = \{i_1, i_2, \dots, i_k\}$ and $J = \{j_1, j_2, \dots, j_k\}$ with $1 \leq k \leq N, K$ and the corresponding $k \times k$ submatrix $T_{I,J} = (t_{ij})_{i \in I, j \in J}$ of $T$. Denote again $I^c = \{1, 2, \dots, N\} \setminus I$ and consider the corresponding reduced sequence

$$(x_n)_{n \in I^c} \cup (x_{N+j})_{j \in J}.$$

By the assumption, these $N$ vectors make up a basis for $H_N$. Denote by $C$ the matrix representation of this basis with respect to $(x_n)_{n=1}^N$ and notice that $C$ is an invertible matrix. In particular, the rows of $C$ are linearly independent. Observe now that the $k \times N$ submatrix $C_I$ of $C$ that corresponds to the rows indexed by $I$ is a block-matrix of the form

$$C_I = \left[ \begin{array}{c|c} 0 & T_{I,J} \end{array} \right].$$

In particular, the rows of $C_I$ are linearly independent. This immediately implies that the rows of $T_{I,J}$ are linearly independent. Thus, $T_{I,J}$ is invertible. $\square$

We proceed with some applications of Theorem 3.2.14. Let us begin by providing examples of totally non-singular matrices.

Recall from [67] that a square matrix $T$ is called *totally positive* if all its minors are positive real numbers. Clearly, each totally positive matrix is totally non-singular. We will construct a class of infinite totally positive symmetric matrices which can be used, via Theorem 3.2.14, for producing new examples of full spark frames. A construction that follows may be of its own interest.

For a matrix $T = (t_{ij}) \in M_n$ and two sets of indices $I, J \subseteq \{1, 2, \dots, n\}$ of the same cardinality we denote by $\Delta(T)_{I,J}$ the corresponding minor; i.e. the determinant of a submatrix $T_{I,J} = (t_{ij})_{i \in I, j \in J}$. A minor $\Delta(T)_{I,J}$ is called *solid* if both $I$ and $J$ consist of consecutive indices. More specifically, a minor $\Delta(T)_{I,J}$ is called *initial* if it is solid and $1 \in I \cup J$. Observe that each matrix entry is the lower-right corner of exactly one initial minor. In our construction we will make use of the following efficient criterion for total positivity which was proved by M. Gasca and J.M. Peña in [69] (see also Theorem 9 in [67]): a square matrix is totally positive if and only if all its initial minors are positive.

To describe our construction we need to introduce one more notational convention. Given an infinite matrix $T = (t_{ij})_{i,j=1}^{\infty}$ and $n \in \mathbb{N}$, we denote by $T^{(n)}$ a submatrix in the upper-left $n \times n$ corner of $T$, that is $T^{(n)} = (t_{ij})_{i,j=1}^{n}$. Its minors will be denoted by $\Delta(T^{(n)})_{I,J}$.

**Theorem 3.2.15.** *Let $(a_n)_n$ and $(b_n)_n$ be sequences of natural numbers such that $b_1 = a_2$ and $a_n b_{n+1} - b_n a_{n+1} = 1$ for all $n \in \mathbb{N}$. Then there exists an infinite matrix $T = (t_{ij})_{i,j=1}^{\infty}$ with the following properties:*

1. $t_{ij} \in \mathbb{N}$, $\forall i, j \in \mathbb{N}$;
2. $t_{ij} = t_{ji}$, $\forall i, j \in \mathbb{N}$;
3. $t_{1n} = t_{n1} = a_n$, $\forall n \in \mathbb{N}$, *and* $t_{2n} = t_{n2} = b_n$, $\forall n \in \mathbb{N}$.
4. *all minors of* $T^{(n)}$ *are positive (i.e.* $T^{(n)}$ *is totally positive), for each* $n \in \mathbb{N}$ *;*
5. *For each* $n \in \mathbb{N}$, *it holds*
   $\Delta(T^{(n)})_{\{n\},\{1\}} = t_{n1} = a_n,$
   $\Delta(T^{(n)})_{\{n,n-1\},\{1,2\}} = 1,$
   $\Delta(T^{(n)})_{\{n,n-1,n-2\},\{1,2,3\}} = 1,$

$\Delta(T^{(n)})_{\{n,n-1,n-2,n-3\},\{1,2,3,4\}} = 1,$

$\ldots$

$\Delta(T^{(n)})_{\{n,n-1,\ldots,1\},\{1,2,\ldots,n\}} = \det T^{(n)} = 1$

*(i.e. all solid minors of $(T^{(n)})$ with the lower-left corner coinciding with the lower-left corner of $(T^{(n)})$, except possibly $\Delta(T^{(n)})_{\{n\},\{1\}}$, are equal to 1).*

**Proof.** We shall construct $T$ by induction starting from $T^{(1)} = [a_1]$. Observe that $T^{(2)} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$; note that $T^{(2)}$ is symmetric since by assumption we have $b_1 = a_2$.

Suppose that we have a symmetric totally positive matrix with integer coefficients $T^{(n)} \in M_n$ which satisfies the above conditions (1)-(5),

$$T^{(n)} = \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_2 & b_2 & b_3 & \ldots & b_n \\ a_3 & b_3 & t_{33} & \ldots & t_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & b_{n-1} & t_{n-1,3} & \ldots & t_{n-1,n} \\ a_n & b_n & t_{n3} & \ldots & t_{nn} \end{bmatrix}.$$

Put

$$T^{(n+1)} = \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_n & a_{n+1} \\ a_2 & b_2 & b_3 & \ldots & b_n & b_{n+1} \\ a_3 & b_3 & t_{33} & \ldots & t_{3n} & x_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & b_{n-1} & t_{n-1,3} & \ldots & t_{n-1,n} & x_{n-1} \\ a_n & b_n & t_{n3} & \ldots & t_{nn} & x_n \\ a_{n+1} & b_{n+1} & x_3 & \ldots & x_n & x_{n+1} \end{bmatrix}. \qquad (12)$$

Note that, by the hypothesis on sequences $(a_n)_n$ and $(b_n)_n$, we have

$$\det \begin{bmatrix} a_n & b_n \\ a_{n+1} & b_{n+1} \end{bmatrix} = 1.$$

We must find numbers $x_3, x_4, \ldots, x_n, x_{n+1}$ such that $T^{(n+1)}$ satisfies (1)-(5). Consider a $3 \times 3$ minor in the lower-left corner of $T^{(n+1)}$:

$$\Delta(T^{(n+1)})_{\{n+1,n,n-1\},\{1,2,3\}} = \det \begin{bmatrix} a_{n-1} & b_{n-1} & t_{n-1,3} \\ a_n & b_n & t_{n3} \\ a_{n+1} & b_{n+1} & x_3 \end{bmatrix}.$$

We can compute $\Delta(T^{(n+1)})_{\{n+1,n,n-1\},\{1,2,3\}}$ by the Laplace expansion along the third row. By the assumption on sequences $(a_n)_n$ and $(b_n)_n$ we have $\det \begin{bmatrix} a_{n-1} & b_{n-1} \\ a_n & b_n \end{bmatrix} = 1$; hence, there exists a unique integer $x_3$ such that $\Delta(T^{(n+1)})_{\{n+1,n,n-1\},\{1,2,3\}} = 1$; take this $x_3$ and put $t_{n+1,3} = x_3$.

Consider now

$$\Delta(T^{(n+1)})_{\{n+1,n,n-1,n-2\},\{1,2,3,4\}} = \det \begin{bmatrix} a_{n-2} & b_{n-2} & t_{n-2,3} & t_{n-2,4} \\ a_{n-1} & b_{n-1} & t_{n-1,3} & t_{n-1,4} \\ a_n & b_n & t_{n3} & t_{n4} \\ a_{n+1} & b_{n+1} & t_{n+1,3} & x_4 \end{bmatrix}.$$

Note that the only unknown entry in this minor is $x_4$. We again use the Laplace expansion along the bottom row. By the induction hypothesis we know that

$$\det \begin{bmatrix} a_{n-1} & b_{n-2} & t_{n-2,3} \\ a_{n-1} & b_{n-1} & t_{n-1,3} \\ a_n & b_n & t_{n3} \end{bmatrix} = \Delta(T^{(n)})_{\{n,n-1,n-2\},\{1,2,3\}} = 1;$$

hence, there is a unique $x_4 \in \mathbb{Z}$ such that $\Delta(T^{(n+1)})_{\{n+1,n,n-1,n-2\},\{1,2,3,4\}} = 1$. Put $t_{n+1,4} = x_4$. We proceed in the same fashion to obtain $x_5, \ldots, x_{n+1}$ in order to achieve the above condition (5) for $T^{(n+1)}$. Since $T^{(n+1)}$ is symmetric, all its essential minors with the lower-right corner in the last column are also equal to 1. By the induction hypothesis $T^{(n)}$ is totally positive, so all essential minors of $T^{(n+1)}$ with the lower-right corner in the $i$th row and $j$th column such that $i, j \leq n$ are also positive. Thus, we can apply the above mentioned result of M. Gasca and J.M. Peña (Theorem 9 in [67]) to conclude that $T^{(n+1)}$ is totally positive. In particular, the integers $x_3, x_4, \ldots, x_n, x_{n+1}$ that we have computed along the way are all positive. This completes the induction step. $\square$

**Example 3.2.16.** Let us take $a_n = 1$ and $b_n = n$, for all $n \in \mathbb{N}$. Clearly, the sequences $(a_n)_n$ and $(b_n)_n$ defined in this way satisfy the conditions from Theorem 3.2.15. Thus, Theorem 3.2.15 gives us a totally positive matrix

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ 1 & 3 & 6 & 10 & 15 & 21 & \\ 1 & 4 & 10 & 20 & 35 & 56 & \\ 1 & 5 & 15 & 35 & 70 & 126 & \\ 1 & 6 & 21 & 56 & 126 & 252 & \\ \vdots & \vdots & & & & & \end{bmatrix}$$

By the construction, the coefficients of $T$ in the first two rows and columns are determined in advance. One can prove that all other coefficients of $T$, those that must be computed by an inductive procedure as described in the preceding proof, are given by

$$t_{i,j+1} = t_{ij} + t_{i-1,j+1}, \quad \forall i \geq 3, \, \forall j \geq 2. \tag{13}$$

This means that $T$ is in fact a well known Pascal matrix; i.e. that $t_{ij}$'s are given by $t_{ij} = \binom{i+j-2}{j-1}$ for all $i, j \geq 1$. A verification of (13) serves as an alternative proof of Theorem 3.2.15 with this special choice of $(a_n)_n$ and $(b_n)_n$. The key observation is the equality that one obtains by subtracting each row in $\Delta(T^{(n+1)})_{\{n+1,n,\ldots,n-j+1\},\{1,2,\ldots,j+1\}}$ from the next one and then using (13):

$$\Delta(T^{(n+1)})_{\{n+1,n,\ldots,n-j+1\},\{1,2,\ldots,j+1\}} = \begin{vmatrix} 1 & t_{n-j+1,2} & t_{n-j+1,3} & \ldots & t_{n-j+1,j+1} \\ 0 & & & & \\ 0 & & \multicolumn{3}{c}{\Delta(T^{(n+1)})_{\{n+1,n,\ldots,n-j+2\},\{1,2,\ldots,j\}}} \\ \vdots & & & & \\ 0 & & & & \end{vmatrix}.$$

We omit the details.

Another example of an infinite totally positive symmetric matrix is obtained by a different choice of sequences $(a_n)_n$ and $(b_n)_n$.

**Example 3.2.17.** Let $a_n = n$ and $b_n = 3n - 1$, for all $n \in \mathbb{N}$. Evidently, these two sequences satisfy the required conditions; namely, $b_1 = a_2$ and $a_n b_{n+1} - b_n a_{n+1} = 1$ for all $n \in \mathbb{N}$. An application of Theorem 3.2.15 gives us a totally positive matrix

$$T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\ 2 & 5 & 8 & 11 & 14 & 17 & 20 & 23 & \ldots \\ 3 & 8 & 14 & 21 & 29 & 38 & 48 & 59 & \ldots \\ 4 & 11 & 21 & 35 & 54 & 79 & \cdot & \cdot & \\ 5 & 14 & 29 & 54 & 94 & \cdot & \cdot & \cdot & \\ 6 & 17 & 38 & 79 & \cdot & \cdot & \cdot & \cdot & \\ 7 & 20 & 48 & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 8 & 23 & 59 & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \vdots & \vdots & \vdots & & & & & & \end{bmatrix}$$

**Remark 3.2.18.** Obviously, by choosing suitable sequences $(a_n)_n$ and $(b_n)_n$ one can generate in the same fashion many other totaly positive symmetric matrices with integer coefficients.

It is also clear from the proof of Proposition 3.2.15 that, by applying a similar inductive procedure, one can construct any infinite totally positive matrix (not necessarily symmetric) with coefficients merely in $\mathbb{R}^+$, with a prescribed first column or the first row.

We can now provide, using Theorem 3.2.14 and the preceding two examples, further examples of full spark frames for $H_N$ of arbitrary length. To do that, we only need to fix some $K \in \mathbb{N}$, and choose arbitrary set of indices $I = \{i_1, i_2, \ldots, i_N\}$, $J = \{j_1, j_2, \ldots, j_K\}$. Then we can take a totally positive matrix $T$ from Example 3.2.16 or Example 3.2.17, its submatrix $T_{I,J} \in M_{NK}$ and apply Theorem 3.2.14. In this way we obtain a full spark frame for $H_N$ consisting of $N + K$ elements.

**Example 3.2.19.** Denote by $(x_n)_{n=1}^N$ the canonical basis for $H_N$. Take arbitrary $K \in \mathbb{N}$ and the upper left $N \times K$ corner of the matrix from Example 3.2.16. An application of Theorem

3.2.14 gives us a full spark frame $(x_n)_{n=1}^{N+K}$ for $H_N$ whose members are represented in the basis $(x_n)_{n=1}^{N}$ by the matrix

$$F_{NK} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 2 & 3 & \cdots & K \\ 0 & 0 & 1 & \cdots & 0 & 1 & 3 & t_{33} & \cdots & t_{3K} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 4 & t_{43} & \cdots & t_{4K} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & N & t_{N3} & \cdots & t_{NK} \end{bmatrix}$$

with

$$t_{ij} = \binom{i+j-2}{j-1}, \quad i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, K.$$

We end the section with a general theorem concerning infinite matrices from Examples 3.2.16 and 3.2.17. It turns out that these matrices are just two representatives of an infinite family of infinite symmetric totally positive matrices. This is the content of the following theorem which we include without proof.

**Theorem 3.2.20.** *Let $d$ by any non-negative real number and let $T^d = (t_{ij}(d))_{i,j=1}^{\infty}$ be an infinite matrix defined by*

$$t_{ij}(d) = (1 + (i-1)d)\,(1 + (j-1)d) + \binom{i+j-2}{j-1} - 1, \ \forall i, j \in \mathbb{N}. \tag{14}$$

*Then $T^d$ is an infinite symmetric totally positive matrix whose all essential $k \times k$ minors, for all $k \geq 2$, are equal to 1.*

*In particular, for each $n$ in $\mathbb{N}$, $T^{d,n} = (t_{ij}(d))_{i,j=1}^{n}$ is a real symmetric totally positive matrix with the Cholesky decomposition*

$$T^{d,n} = L^{d,n} \left(L^{d,n}\right)^{tr}, \tag{15}$$

*where $(\cdot)^{tr}$ denotes the transpose and*

$$L^{d,n} = (l_{ij}(d))_{i,j=1}^{n}, \ \ l_{ij}(d) = \begin{cases} 1 + (i-1)d & \text{for } j = 1 \\ \binom{i-1}{j-1} & \text{for } j > 1 \end{cases}, \ \ \forall n \in \mathbb{N}. \tag{16}$$

**Remark 3.2.21.** Note that we have, for every $d > 0$,

$$T^d = \begin{bmatrix} 1 & 1+d & 1+2d & 1+3d & \cdots \\ 1+d & (1+d)(1+d)+1 & (1+d)(1+2d)+2 & (1+d)(1+3d)+3 & \cdots \\ 1+2d & (1+2d)(1+d)+2 & (1+2d)(1+2d)+5 & (1+2d)(1+3d)+9 & \cdots \\ 1+3d & & & & \\ 1+4d & & & & \\ \vdots & & & & \end{bmatrix} \tag{17}$$

If we substitute $d = 0$ and $d = 1$, we get matrices form Examples 3.2.16 and 3.2.17, respectively.

It is also useful to note that

$$t_{2j}(d) = (1+d)(1+(j-1)d) + \binom{j}{j-1} - 1 = 1 + d + (j-1)(1+d+d^2), \ \ \forall j \in \mathbb{N},$$

so the coefficients in the second row and in the second column of $T^d$ make up an arithmetic sequence.

**Remark 3.2.22.** Since, for all $d \geq 0$ and $n \in \mathbb{N}$, $T^{d,n}$ is a real, symmetric, and positive-definite matrix, it has a unique Cholesky decomposition. Observe that (16) shows that, for a fixed $n \in \mathbb{N}$, all lower triangular factors $L^{d,n}$ differ only in the first column. Taking, for example, $n = 6$ we have

$$L^{0,6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}, \quad L^{d,6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1+d & 1 & 0 & 0 & 0 & 0 \\ 1+2d & 2 & 1 & 0 & 0 & 0 \\ 1+3d & 3 & 3 & 1 & 0 & 0 \\ 1+4d & 4 & 6 & 4 & 1 & 0 \\ 1+5d & 5 & 10 & 10 & 5 & 1 \end{bmatrix}, \ \ \forall d > 0.$$

We note that it is well known that $L^{0,6}$ is the lower triangular Cholesky factor of the Pascal matrix of order 6 (and analogously for every $n$ in $\mathbb{N}$).

*Concluding remarks.* The notion of the spark of a matrix is introduced in [61]. The motivation for the definition is the observation that matrices whose spark is large enough are naturally equipped to distinguish sparse signals. To see this, consider a full spark $N \times M$ matrix $T$ and observe that $v \in H_M$, $\|v\|_0 < N, v \neq 0 \Rightarrow Tv \neq 0$. In particular, from this one concludes that $v_1, v_2 \in H_M$, $\|v_1\|_0, \|v_2\|_0 < \frac{N}{2}$, $v_1 \neq v_2 \Rightarrow Tv_1 \neq Tv_2$.

Chebotarëv's theorem was first used in this circle of problems in [32] for sparse signal processing.

Theorems 3.2.14 and 3.2.15 and the subsequent results first appeared in [6].

**Exercise 3.2.23.** Let $(x_n)_n$ be a frame for a Hilbert space $H$ with the lower frame bound $A$. If $\|x_m\| < \sqrt{A}$ for some index $m$, then the set $\{m\}$ satisfies the minimal redundancy condition for $(x_n)_n$.

**Exercise 3.2.24.** Let $(x_n)_n$ be a Parseval frame for a Hilbert space $H$. Show that the following conditions are equivalent:

(a) $(x_n)_n$ is 1-robust;

(b) $\|x_n\| < 1$, for all $n$.

**Exercise 3.2.25.** ([24], [89]) Let $(x_n)_n$ be a Parseval frame for a Hilbert space $H$ that is not an ONB. Suppose that $\|x_i\| = 1$ for some $i$. Show that there exist an index $j$ such that $\|x_j\| < 1$ and a Parseval frame $(x'_n)_n$ for $H$ with the properties $\|x'_i\| < 1$, $\|x'_j\| < 1$, and $x'_n = x_n$ for all $n \neq i, j$. Hint. Take a real number $\varphi$ such that $0 < \varphi < \frac{\pi}{2}$ and define

$$x'_n = \begin{cases} \cos\varphi\, x_i + \sin\varphi\, x_j, & \text{if } n = i; \\ -\sin\varphi\, x_i + \cos\varphi\, x_j, & \text{if } n = j; \\ x_n, & \text{if } n \neq i, j. \end{cases}$$

**Exercise 3.2.26.** Prove equality (13) for the matrix $T$ from Example 3.2.16.

# 4 Frames in wavelet theory

## 4.1 Shift-invariant spaces

For a function $f$ on $\mathbb{R}^N$ and $a \in \mathbb{R}^N$ we define the translation of $f$ by $a$ as the function $T_a f$ defined by $T_a f(x) = f(x-a)$, $x \in \mathbb{R}^N$. In this chapter we shall restrict ourselves to integer translations $T_k$, $k \in \mathbb{Z}^N$. It is easy to see that $\{T_k : k \in \mathbb{Z}^N\}$ is a group of unitary operators on $L^2(\mathbb{R}^N)$.

**Definition 4.1.1.** *A closed subspace $V$ of $L^2(\mathbb{R}^N)$ is said to be shift-invariant (or a shift-invariant space, SIS) if $V$ is invariant under the action of all $T_k$, $k \in \mathbb{Z}^N$, i.e. if*

$$f \in V \Longrightarrow T_k f \in V, \ \forall k \in \mathbb{Z}^N.$$

*Given $f \in L^2(\mathbb{R}^N)$, we denote by $\langle f \rangle$ the smallest SIS that contains $f$;*

$$\langle f \rangle = \overline{span}\{T_k f : k \in \mathbb{Z}^N\}.$$

*Such spaces which are closed subspaces of $L^2(\mathbb{R}^N)$ that are generated as shift-invariant spaces by a single function are called principal shift-invariant spaces.*

A major role in the study of shift-invariant spaces is played by the Fourier transform. Recall that the Fourier transform $\mathcal{F}f = \hat{f}$ of a function $f \in L^1(\mathbb{R}^N)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \ \xi \in \mathbb{R}^N. \tag{1}$$

The Plancherel formula

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle, \ \forall f, g \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \tag{2}$$

enables us to extend $\mathcal{F}$ from $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ to a unitary operator on $L^2(\mathbb{R}^N)$.

It is convenient to note the following useful (and well known) formula:

$$\widehat{T_k f}(\xi) = e^{-2\pi i \langle k, \xi \rangle} \hat{f}(\xi), \ \forall k \in \mathbb{Z}^N. \tag{3}$$

Given $f \in L^2(\mathbb{R}^N)$, the principal shift-invariant space $\langle f \rangle$ is generated, as a closed subspace of $L^2(\mathbb{R}^N)$, by the sequence $(T_k f)_{k \in \mathbb{Z}^N}$. A natural question arises: can we characterize those $f$ for which the sequence $(T_k f)_{k \in \mathbb{Z}^N}$ is an ONB/Riesz basis/frame for $\langle f \rangle$? A related question is the following one: given $f$, can we find $g \in \langle f \rangle$ such that $\langle f \rangle = \langle g \rangle$ and that the sequence $(T_k g)_{k \in \mathbb{Z}^N}$ has "nicer" properties than the original sequence $(T_k f)_{k \in \mathbb{Z}^N}$?

We denote by $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$ the $N$-dimensional torus. By $L^p(\mathbb{T}^N)$ we denote the space of all $\mathbb{Z}^N$-periodic functions (i.e., $f$ is 1-periodic in each variable) such that

$$\int_{C^N} |f(x)|^p dx < \infty,$$

where $C^N$ denotes the standard unit cube $[-\frac{1}{2}, \frac{1}{2})^N$ in $\mathbb{R}^N$. In fact, we shall freely identify $L^p(\mathbb{T}^N)$ with $L^p(C^N)$.

For $f, g \in L^2(\mathbb{R}^N)$ we denote by $[f, g]$ the bracket product which is the function defined a.e. by

$$[f,g](x) = \sum_{k \in \mathbb{Z}^N} f(x+k)\overline{g(x+k)}, \quad x \in C^N. \tag{4}$$

Clearly (see Corollary 4.4.17), we have $[f, g] \in L^1(\mathbb{T}^N)$. In particular, for any $\varphi \in L^2(\mathbb{R}^N)$, we denote by $\sigma_\varphi$ the function defined a.e. by

$$\sigma_\varphi(\xi) = [\hat{\varphi}, \hat{\varphi}](\xi) = \sum_{k \in \mathbb{Z}^N} |\hat{\varphi}(\xi+k)|^2, \quad x \in C^N. \tag{5}$$

We begin our study with a simple result concerning orthogonality of principal shift-invariant spaces.

**Proposition 4.1.2.** *Let $f, g \in L^2(\mathbb{R}^N)$. Then $\langle f \rangle \perp \langle g \rangle$ if and only if $[\hat{f}, \hat{g}] = 0$ a.e.*

**Proof.**

$$\begin{aligned}
\langle f \rangle \perp \langle g \rangle \quad &\Leftrightarrow \quad \langle T_l f, T_k g \rangle = 0, \ \forall l, k \in \mathbb{Z}^N \\
&\Leftrightarrow \quad \langle f, T_{k-l} g \rangle = 0, \ \forall l, k \in \mathbb{Z}^N \\
&\Leftrightarrow \quad \langle f, T_k g \rangle = 0, \ \forall k \in \mathbb{Z}^N \\
&\overset{(2)}{\Leftrightarrow} \quad \langle \hat{f}, \widehat{T_k g} \rangle = 0, \ \forall k \in \mathbb{Z}^N \\
&\Leftrightarrow \quad \int_{\mathbb{R}^N} \hat{f}(\xi)\overline{\widehat{T_k g}(\xi)} d\xi = 0, \ \forall k \in \mathbb{Z}^N \\
&\Leftrightarrow \quad \int_{C^N} \sum_{l \in \mathbb{Z}^N} \hat{f}(\xi+l)\overline{\widehat{T_k g}(\xi+l)} d\xi = 0, \ \forall k \in \mathbb{Z}^N \\
&\overset{(3)}{\Leftrightarrow} \quad \int_{C^N} \sum_{l \in \mathbb{Z}^N} \hat{f}(\xi+l) e^{2\pi i \langle k, \xi+l \rangle} \overline{\hat{g}(\xi+l)} = 0, \ \forall k \in \mathbb{Z}^N \\
&\Leftrightarrow \quad \int_{C^N} [\hat{f}, \hat{g}](\xi) e^{2\pi i \langle k, \xi \rangle} d\xi = 0, \ \forall k \in \mathbb{Z}^N.
\end{aligned}$$

The last equality tells us that all Fourier coefficients of the function $[\hat{f}, \hat{g}]$ with respect to the ONB $(e^{-2\pi i \langle k, \xi \rangle})_{k \in \mathbb{Z}^N}$ of $L^2(\mathbb{T}^N)$ vanish; thus, by the uniqueness theorem (see Theorem 4.4.20 and [81], Corollary 13.26), $[\hat{f}, \hat{g}] = 0$ a.e. $\square$

We proceed by considering the situation in which, for $\varphi \in L^2(\mathbb{R}^N)$, the system $(T_k\varphi)_{k \in \mathbb{Z}^N}$ makes up a Parseval frame for $\langle \varphi \rangle$. As before, we denote by $U : \langle \varphi \rangle \to \ell^2(\mathbb{Z}^N)$ the corresponding analysis operator. Here and throughout this chapter we denote by $(e_k)_{k \in \mathbb{Z}^N}$ the canonical basis of $\ell^2(\mathbb{Z}^N)$. Recall that $U^* e_k = T_k \varphi$ for all $k$.

**Proposition 4.1.3.** *Let $(T_k\varphi)_{k \in \mathbb{Z}^N}$ be a Bessel sequence. Then for each $\mathbb{Z}^N$-periodic function $m \in L^2(\mathbb{T}^N)$ we have $m\hat{\varphi} \in L^2(\mathbb{R}^N)$. (Here and in the sequel we understand that $m$ is extended by $\mathbb{Z}^N$-periodicity to the function $m$ on $\mathbb{R}^N$.) Moreover, if $(T_k\varphi)_{k \in \mathbb{Z}^N}$ is a frame for $\langle \varphi \rangle$, then*

$$\langle \varphi \rangle = \{ f \in L^2(\mathbb{R}^N) : \hat{f} = m\hat{\varphi}, \ m \in L^2(\mathbb{T}^N) \}. \tag{6}$$

**Proof.** Suppose that $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a Bessel sequence and denote by $U$ its analysis operator. Take any $m \in L^2(\mathbb{T}^N)$. Since the system $(e^{-2\pi i\langle k,\xi\rangle})_{k\in\mathbb{Z}^N}$ is an ONB for $L^2(\mathbb{T}^N)$, we have

$$m(\xi) = \sum_{k\in\mathbb{Z}^N} \mu_k e^{-2\pi i\langle k,\xi\rangle}. \tag{7}$$

In particular, we know that $\mu = (\mu_k)_{k\in\mathbb{Z}^N} \in \ell^2(\mathbb{Z}^N)$ and

$$\|m\|^2 = \|\mu\|^2 = \sum_{k\in\mathbb{Z}^N} |\mu_k|^2. \tag{8}$$

Let $f = U^*\mu = \sum_{k\in\mathbb{Z}^N} \mu_k T_k\varphi \in \langle\varphi\rangle$. Since this series converges in norm, by applying the Fourier transform we obtain $\hat{f} \in L^2(\mathbb{R}^N)$ that is given by

$$\hat{f}(\xi) = \sum_{k\in\mathbb{Z}^N} \mu_k e^{-2\pi i\langle k,\xi\rangle}\hat{\varphi}(\xi) = \left(\sum_{k\in\mathbb{Z}^N} \mu_k e^{-2\pi i\langle k,\xi\rangle}\right)\hat{\varphi}(\xi) = m(\xi)\hat{\varphi}(\xi). \tag{9}$$

This proves not only the first statement, but also the inclusion

$$\{f \in L^2(\mathbb{R}^N) : \hat{f} = m\hat{\varphi},\, m \in L^2(\mathbb{T}^N)\} \subseteq \langle\varphi\rangle.$$

Suppose now, additionally, that $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a frame for $\langle\varphi\rangle$. Then $U^* : \ell^2(\mathbb{Z}^N) \to \langle\varphi\rangle$ is a surjection; thus, for each $f$ in $\langle\varphi\rangle$ there exists $\mu = (\mu_k)_{k\in\mathbb{Z}^N} \in \ell^2(\mathbb{Z}^N)$ such that

$$f = U^*\mu = U^*\left(\sum_{k\in\mathbb{Z}^N} \mu_k e_k\right) = \sum_{k\in\mathbb{Z}^N} \mu_k U^* e_k = \sum_{k\in\mathbb{Z}^N} \mu_k T_k \varphi_k.$$

Applying the Fourier transform we get

$$\hat{f}(\xi) = \sum_{k\in\mathbb{Z}^N} \mu_k e^{-2\pi i\langle k,\xi\rangle}\hat{\varphi}(\xi) = \left(\sum_{k\in\mathbb{Z}^N} \mu_k e^{-2\pi i\langle k,\xi\rangle}\right)\hat{\varphi}(\xi). \tag{10}$$

Since the system $(e^{-2\pi i\langle k,\xi\rangle})_{k\in\mathbb{Z}^N}$ is an ONB for $L^2(\mathbb{T}^N)$ and $\mu = (\mu_k)_{k\in\mathbb{Z}^N}$ belongs to $\ell^2(\mathbb{Z}^N)$, we know from Lemma 1.1.4 that the function $m$ defined by

$$m(\xi) = \sum_{k\in\mathbb{Z}^N} \mu_k e^{-2\pi i\langle k,\xi\rangle} \tag{11}$$

is a well defined element of $L^2(\mathbb{T}^N)$. Hence we can rewrite (10) in the form

$$\hat{f}(\xi) = m(\xi)\hat{\varphi}(\xi). \tag{12}$$

. ◻

**Remark 4.1.4.** In the sequel we shall freely identify Hilbert spaces $\ell^2(\mathbb{Z}^N)$ and $L^2(\mathbb{T}^N)$ using the unitary operator $\mu = (\mu_k)_{k\in\mathbb{Z}^N} \mapsto \sum_{k\in\mathbb{Z}^N} \mu_k e^{-2\pi i\langle k,\xi\rangle}$.

**Remark 4.1.5.** Suppose that for $\varphi \in L^2(\mathbb{R}^N)$ the sequence $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a frame for $\langle\varphi\rangle$. Each function $m \in L^2(\mathbb{T}^N)$ that satisfies (12) is called a filter for $f$. Observe that for each $f$ in $\langle\varphi\rangle$ there exists a unique $\mu_0 \in \mathrm{R}(U)$ such that $f = U^*\mu_0$ i.e., $\hat{f} = m_0\hat{\varphi}$ (here we have used the identification $\mu_0 = m_0$ established in the preceding remark). This function $m_0$ is called the minimal filter for $f$ since, obviously, we have $\|\mu_0\| \leq \|\mu\|$ and hence $\|m_0\| \leq \|m\|$ for every $\mu$ such that $U^*\mu = f$.

If $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\varphi\rangle$ we claim that

$$m_0 = [\hat{f}, \hat{\varphi}]. \tag{13}$$

To see this, observe that the Fourier coefficients of $[\hat{f}, \hat{\varphi}]$ with respect to the ONB $(e^{-2\pi i\langle k,\xi\rangle})_{k\in\mathbb{Z}^N}$ of $L^2(\mathbb{T}^N)$ are

$$\begin{aligned}
\int_{C^N} \sum_{l\in\mathbb{Z}^N} \hat{f}(\xi+l)\overline{\hat{\varphi}(\xi+l)}e^{2\pi i\langle k,\xi\rangle}d\xi &= \int_{C^N} \sum_{l\in\mathbb{Z}^N} \hat{f}(\xi+l)\overline{\hat{\varphi}(\xi+l)}e^{2\pi i\langle k,\xi+l\rangle}d\xi \\
&= \int_{\mathbb{R}^N} \hat{f}(\xi)\overline{\hat{\varphi}(\xi)}e^{2\pi i\langle k,\xi\rangle}d\xi \\
&= \langle \hat{f}, \widehat{T_k\varphi}\rangle \\
&= \langle f, T_k\varphi\rangle.
\end{aligned}$$

On the other hand, we know that $\mu_0 = Uf$ (because $U^*Uf = f$, $Uf \in \mathrm{R}(U)$, and $\mu_0$ is the only element in $\mathrm{R}(U)$ with the property $U^*\mu_0 = f$); thus, $\mu_0 = (\langle f, T_k\varphi\rangle)_{k\in\mathbb{Z}^N}$ and hence the Fourier coefficients of $m_0$ with respect to $(e^{-2\pi i\langle k,\xi\rangle})_{k\in\mathbb{Z}^N}$ are also $\langle f, T_k\varphi\rangle$, $k \in \mathbb{Z}^N$.

In the following proposition we give another description of minimal filters for Parseval generators.

**Proposition 4.1.6.** *Suppose that $\varphi \in L^2(\mathbb{R}^N)$ is such that the sequence $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\varphi\rangle$. Put $\Omega = supp(\sigma_\varphi) = \{\xi : \sigma_\varphi(\xi) \neq 0\}$. Then*

$$R(U) = \{m \in L^2(\mathbb{T}^N) : m(\xi) = 0, \text{ for a.e. } \xi \notin \Omega\}. \tag{14}$$

*In particular, for each $f \in \langle\varphi\rangle$, the minimal filter $m_0$ is characterized among all filters for $f$ by the property $m_0(\xi) = 0$, for a.e. $\xi \notin \Omega$.*

**Proof.** Since $\sigma_\varphi$ is a $\mathbb{Z}^N$-periodic function, the set $\Omega$ is also $\mathbb{Z}^N$-periodic.

Observe that each $m \in \mathrm{R}(U)$ is of the form

$$m = Uf = \sum_{k\in\mathbb{Z}^N} \langle f, T_k\varphi\rangle e^{-2\pi i\langle k,\xi\rangle}.$$

Take any $\xi \notin \Omega$. Then we have $\sigma_\varphi(\xi) = 0$ which implies $\hat{\varphi}(\xi+k) = 0$ for all $k$ in $\mathbb{Z}^N$. Equality (13) now implies $m(\xi) = 0$ for a.e. $\xi$.

To prove the opposite inclusion in (14), suppose that $m(\xi) = 0$ for all $\xi \notin \Omega$. We want to show that $m \in \mathrm{R}(U) = \mathrm{N}(U^*)^\perp$. Let $m = m_1 + m_2$ with $m_1 \in \mathrm{R}(U)$ and $m_2 \in \mathrm{N}(U^*)$. By the first part of the proof we have $m_1(\xi) = 0$ for a.e. $\xi \notin \Omega$. Thus, we also have $m_2(\xi) = 0$ for a.e. $\xi \notin \Omega$. Since $m_2 \in \mathrm{N}(U^*)$, we know that $U^*m_2 = 0$ which means that $m_2(\xi)\hat{\varphi}(\xi) = 0$ a.e. By the $\mathbb{Z}^N$-periodicity of $m_2$ this implies $m_2(\xi)\hat{\varphi}(\xi+k) = 0$ a.e. for all $k$. Now, if $\xi \in \Omega$, we know that $\sigma_\varphi(\xi) \neq 0$, so we must have $\hat{\varphi}(\xi+k) \neq 0$ for at least one $k$. The preceding equality now implies $m_2(\xi) = 0$. Hence, $m_2 = 0$ a.e. and $m = m_1 \in \mathrm{R}(U)$. □

We can now describe those $\varphi \in L^2(\mathbb{R}^N)$ which have the property that the system $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\varphi\rangle$.

**Theorem 4.1.7.** *Let $\varphi \in L^2(\mathbb{R}^N)$. Then the system $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\varphi\rangle$ if and only if there exists a $\mathbb{Z}^N$-periodic set $\Omega \subseteq \mathbb{R}^N$ such that $\sigma_\varphi = \chi_\Omega$ a.e. In particular, $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is an ONB for $\langle\varphi\rangle$ if and only if $\sigma_\varphi(\xi) = 1$ for a.e. $\xi \in \mathbb{R}^N$ i.e., if and only if $\Omega \overset{0}{=} \mathbb{R}^N$.*

**Proof.** Let $(T_k\varphi)_{k\in\mathbb{Z}^N}$ be a Parseval frame for $\langle\varphi\rangle$. Put $\Omega = \{\xi : \sigma_\varphi(\xi) \neq 0\}$. Recall from Remark 4.1.5 that

$$\hat{f}(\xi) = [\hat{f}, \hat{\varphi}](\xi)\hat{\varphi}(\xi), \ a.e. \ \forall f \in \langle\varphi\rangle \tag{15}$$

and, in particular,

$$\hat{\varphi}(\xi) = \sigma_\varphi(\xi)\hat{\varphi}(\xi), \ a.e. \tag{16}$$

For any $\xi \in \Omega$ we have $\sigma_\varphi(\xi) \neq 0$; thus, there exists $k \in \mathbb{Z}^N$ such that $\hat{\varphi}(\xi+k) \neq 0$. Now equality (16) implies $\sigma_\varphi(\xi+k) = 1$ which gives us, since $\sigma_\varphi$ is $\mathbb{Z}^N$-periodic, $\sigma_\varphi(\xi) = 1$.

To prove the converse, suppose that $\varphi$ has the property $\sigma_\varphi = \chi_\Omega$ a.e. for some $\mathbb{Z}^N$-periodic set $\Omega$. Take any $f \in L^2(\mathbb{R}^N)$. Then we have

$$\begin{aligned}
\int_{C^N} [\hat{f}, \hat{\varphi}](\xi) e^{2\pi i\langle k,\xi\rangle} d\xi &= \int_{C^N} \sum_{l\in\mathbb{Z}^N} \hat{f}(\xi+l)\overline{\hat{\varphi}(\xi+l)} e^{2\pi i\langle k,\xi+l\rangle} d\xi \\
&= \int_{\mathbb{R}^N} \hat{f}(\xi)\overline{\hat{\varphi}(\xi)e^{-2\pi i\langle k,\xi\rangle}} d\xi \\
&= \langle \hat{f}, \widehat{T_k\varphi}\rangle \\
&= \langle f, T_k\varphi\rangle.
\end{aligned}$$

This implies that

$$\left\|[\hat{f}, \hat{\varphi}]\right\|^2_{L^2(\mathbb{T}^N)} = \sum_{k\in\mathbb{Z}^N} |\langle f, T_k\varphi\rangle|^2. \tag{17}$$

We now continue our computation:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^N} |\langle f, T_k\varphi\rangle|^2 &= \left\|[\hat{f},\hat{\varphi}]\right\|^2_{L^2(\mathbb{T}^N)} \\
&= \int_{C^N} \left|\sum_{k\in\mathbb{Z}^N} \hat{f}(\xi+k)\overline{\hat{\varphi}(\xi+k)}\right|^2 d\xi \\
&\leq \int_{C^N} \left(\sum_{k\in\mathbb{Z}^N} |\hat{f}(\xi+k)|^2\right)\left(\sum_{k\in\mathbb{Z}^N} |\hat{\varphi}(\xi+k)|^2\right) d\xi \\
&= \int_{C^N} \left(\sum_{k\in\mathbb{Z}^N} |\hat{f}(\xi+k)|^2\right) \sigma_\varphi(\xi) d\xi \\
&= \sum_{k\in\mathbb{Z}^N} \int_{C^N} |\hat{f}(\xi+k)|^2 \sigma_\varphi(\xi) d\xi \\
&= \sum_{k\in\mathbb{Z}^N} \int_{C^N} |\hat{f}(\xi+k)|^2 \sigma_\varphi(\xi+k) d\xi \\
&= \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 \chi_\Omega(\xi) d\xi \\
&\leq \|\hat{f}\|^2 \\
&= \|f\|^2.
\end{aligned}$$

This shows that $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a Bessel sequence in $L^2(\mathbb{R}^N)$; thus, its analysis operator $U : L^2(\mathbb{R}^N) \to \ell^2(\mathbb{Z}^N)$ is well defined and bounded. To finish the proof we now only need to show that

$$\|Uf\| = \|f\|, \ \ \forall f \in \langle\varphi\rangle. \tag{18}$$

To prove (18), it suffices to obtain the same equality for all functions from span $\{T_k\varphi : k \in \mathbb{Z}^N\}$ which is a dense set in $\langle\varphi\rangle$.

Take any $f = \sum_{k\in F} \alpha_k T_k\varphi \in \text{span}\,\{T_k\varphi : k \in \mathbb{Z}^N\}$ where $F$ is a finite subset of $\mathbb{Z}^N$. By applying the Fourier transform we get

$$\hat{f}(\xi) = \left(\sum_{k\in F} \alpha_k e^{-2\pi i\langle k,\xi\rangle}\right)\hat{\varphi}(\xi).$$

After denoting

$$m(\xi) = \sum_{k\in F} \alpha_k e^{-2\pi i\langle k,\xi\rangle}$$

we can write

$$\hat{f}(\xi) = m(\xi)\hat{\varphi}(\xi).$$

We now compute

$$\begin{aligned}
\left\|[\hat{f},\hat{\varphi}]\right\|^2_{L^2(\mathbb{T}^N)} &= \int_{C^N} \left|[\hat{f},\hat{\varphi}](\xi)\right|^2 d\xi \\
&= \int_{C^N} \left|\sum_{k\in\mathbb{Z}^N} \hat{f}(\xi+k)\overline{\hat{\varphi}(\xi+k)}\right|^2 d\xi \\
&= \int_{C^N} \left|\sum_{k\in\mathbb{Z}^N} m(\xi+k)|\hat{\varphi}(\xi+k)|^2\right|^2 d\xi \\
&= \int_{C^N} |m(\xi)\sigma_\varphi(\xi)|^2\, d\xi \quad (\text{since } \sigma_\varphi(\xi)^2 = \sigma_\varphi(\xi)) \\
&= \int_{C^N} |m(\xi)|^2\, \sigma_\varphi(\xi) d\xi \\
&= \int_{C^N} |m(\xi)|^2 \sum_{k\in\mathbb{Z}^N} |\hat{\varphi}(\xi+k)|^2 d\xi \\
&= \int_{C^N} \sum_{k\in\mathbb{Z}^N} |m(\xi+k)|^2\, |\hat{\varphi}(\xi+k)|^2 d\xi \\
&= \sum_{k\in\mathbb{Z}^N} \int_{C^N} |m(\xi+k)|^2\, |\hat{\varphi}(\xi+k)|^2 d\xi \\
&= \int_{\mathbb{R}^N} |m(\xi)\hat{\varphi}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 d\xi \\
&= \|\hat{f}\|^2 \\
&= \|f\|^2.
\end{aligned}$$

This, together with (17), gives us the desired equality.

Let us now prove the second statement. If the measure of the complement of the set $\Omega$ is greater than zero, equality (14) from Proposition 4.1.6 shows us that the analysis operator $U$ is not a surjection and hence $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is not a basis. If $\Omega$ is equal to $\mathbb{R}^N$ up to a set of measure zero, then the same argument shows that the analysis operator $U$ is surjective; thus, $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is an ONB. Alternatively, the same conclusion follows by a simple calculation:

$$\langle T_k\varphi, T_l\varphi\rangle = \langle \varphi, T_{l-k}\varphi\rangle = \langle \hat{\varphi}, \widehat{T_{l-k}\varphi}\rangle = \int_{C^N} \sigma_\varphi(\xi) e^{2\pi i\langle l-k,\xi\rangle} d\xi = \delta_{kl}.$$

□

Consider now an arbitrary $\varphi \in L^2(\mathbb{R}^N)$ and $\langle\varphi\rangle$. As before, for each $f \in \operatorname{span}\{T_k f : k \in \mathbb{Z}^N\}$ we have a finite subset $F$ of $\mathbb{Z}^N$ and scalars $\alpha_k$, $k \in F$, such that $f = \sum_{k\in F} \alpha_k T_k\varphi$. This implies

$$\hat{f}(\xi) = \left(\sum_{k\in F} \alpha_k e^{-2\pi i\langle k,\xi\rangle}\right) \hat{\varphi}(\xi).$$

So, if we put

$$t(\xi) = \sum_{k\in F} \alpha_k e^{-2\pi i\langle k,\xi\rangle}$$

we can write

$$\hat{f}(\xi) = t(\xi)\hat{\varphi}(\xi). \tag{19}$$

Observe now that

$$\begin{aligned}
\|f\|^2 &= \|\hat{f}\|^2 \\
&= \int_{\mathbb{R}^N} |t(\xi)|^2 |\hat{\varphi}(\xi)^2 d\xi \\
&= \sum_{k\in\mathbb{Z}^N} \int_{C^N} |t(\xi+k)|^2 |\hat{\varphi}(\xi+k)^2 d\xi \\
&= \int_{C^N} \sum_{k\in\mathbb{Z}^N} |t(\xi)|^2 |\hat{\varphi}(\xi+k)^2 d\xi \\
&= \int_{C^N} |t(\xi)|^2 \sigma_\varphi(\xi) d\xi.
\end{aligned}$$

This shows that the map $\widetilde{W_\varphi}$ that assigns to $f \in \operatorname{span}\{T_k f : k \in \mathbb{Z}^N\}$ the unique trigonometric polynomial $t$ such that $\hat{f} = t\hat{\varphi}$ is an isometry between $\operatorname{span}\{T_k f : k \in \mathbb{Z}^N\}$ and the space $P_\varphi$ of all trigonometric polynomials endowed with the norm

$$\|t\|_{L^2(\mathbb{T}^N,\sigma_\varphi)} = \left(\int_{C^N} |t(\xi)|^2 \sigma_\varphi(\xi) d\xi\right)^{\frac{1}{2}}.$$

Thus, $\widetilde{W_\varphi}$ has a unique extension to a unitary operator $W_\varphi$ between $\langle\varphi\rangle$ and the space $L^2(\mathbb{T}^N, \sigma_\varphi)$ consisting of all $\mathbb{Z}^N$-periodic functions $s$ satisfying $\|s\|_{L^2(\mathbb{T}^N,\sigma_\varphi)} < \infty$. Let us note that

$$\|f\| = \|W_\varphi f\| = \|t\|_{L^2(\mathbb{T}^N,\sigma_\varphi)}, \ \ \forall f \in \langle\varphi\rangle. \tag{20}$$

**Proposition 4.1.8.** *For each $\varphi \in L^2(\mathbb{R}^N)$ there exists $\psi \in \langle\varphi\rangle$ such that $\langle\varphi\rangle = \langle\psi\rangle$ and that the system $(T_k\psi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi\rangle$.*

**Proof.** Let $\Omega = \{\xi : \sigma_\varphi(\xi) \neq 0\}$. Consider the function $s$ defined by

$$s(\xi) = \begin{cases} \frac{1}{\sqrt{\sigma_\varphi(\xi)}}, & \xi \in \Omega \\ 0, & \xi \notin \Omega \end{cases} \tag{21}$$

Since $s \in L^2(\mathbb{T}^N, \sigma_\varphi)$, by the preceding discussion there exists a unique function $\psi \in \langle\varphi\rangle$ such that

$$\hat{\psi} = s\hat{\varphi}. \tag{22}$$

Clearly, we have

$$\sigma_\psi(\xi) = \sum_{k\in\mathbb{Z}^N} |\hat{\psi}(\xi+k)|^2 = \chi_\Omega(\xi) \sum_{k\in\mathbb{Z}^N} \frac{1}{\sigma_\varphi(\xi)} |\hat{\varphi}(\xi+k)|^2 = \chi_\Omega(\xi).$$

Thus, by Theorem 4.1.7, the system $(T_k\psi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi\rangle$. Also, since $\langle\varphi\rangle$ is shift-invariant and contains $\psi$, we conclude that $\langle\varphi\rangle \supseteq \langle\psi\rangle$. Let us now take any $f \in \langle\varphi\rangle$. Then

$$\begin{aligned}
\langle f, T_k\psi\rangle &= \langle \hat{f}, \widehat{T_k\psi}\rangle \\
&\overset{(19),(22)}{=} \int_{\mathbb{R}^N} t(\xi)\hat{\varphi}(\xi)\overline{s(\xi)\hat{\varphi}(\xi)}e^{2\pi i\langle k,\xi\rangle}d\xi \\
&= \int_{C^N} t(\xi)\overline{s(\xi)}e^{2\pi i\langle k,\xi\rangle}\sum_{l\in\mathbb{Z}^N}|\hat{\varphi}(\xi+l)|^2 d\xi \\
&= \int_{C^N} t(\xi)\overline{s(\xi)}\sigma_\varphi(\xi)e^{2\pi i\langle k,\xi\rangle}d\xi.
\end{aligned}$$

This shows us that $(\langle f, T_k\psi\rangle)_{k\in\mathbb{Z}^N}$ is the sequence of the Fourier coefficients of the function $t(\xi)\overline{s(\xi)}\sigma_\varphi(\xi)$. From this we conclude that

$$\sum_{k\in\mathbb{Z}^N}|\langle f, T_k\psi\rangle|^2 = \int_{C^N}|t(\xi)|^2|s(\xi)|^2\sigma_\varphi(\xi)^2 d\xi = \int_{C^N}|t(\xi)|^2\sigma_\varphi(\xi)d\xi \overset{(20)}{=} \|f\|^2$$

which tells us that $(T_k\psi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\varphi\rangle$. □

**Theorem 4.1.9.** *Suppose that $V \neq \{0\}$ is a closed subspace of $L^2(\mathbb{R}^N)$. Then $V$ is shift-invariant if and only if there exists a sequence of functions $(\varphi_j)_{j=1}^\infty$ in $V$ such that, for each $j$, the system $(T_k\varphi_j)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\varphi_j\rangle$ and $V = \oplus_{j=1}^\infty\langle\varphi_j\rangle$.*

*Remark.* All but a finite number of the $\varphi_j$ can be the zero function; in this case $\langle\varphi_j\rangle = \{0\}$. We always assume that the $\varphi_j$ are ordered so that the non-zero ones are listed at the beginning.

**Proof.** Choose a non-zero $\varphi \in V$. Applying Proposition 4.1.8 we obtain $\psi \in \langle\varphi\rangle$ such that $\langle\varphi\rangle = \langle\psi\rangle$ and that the system $(T_k\psi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi\rangle$. We let $\varphi_1 = \psi$ and consider the orthogonal complement of $\langle\varphi_1\rangle$ in $V$. We now apply the same argument to the shift-invariant space (see Exercise 4.1.13) $V \ominus \langle\varphi_1\rangle$. Continuing in this fashion we obtain the desired conclusion. A rigorous argument uses Zorn's lemma and separability of $V$. □

Let us now consider a shift-invariant space $V$ in $L^2(\mathbb{R}^N)$ and the decomposition $V = \oplus_{j=1}^\infty\langle\varphi_j\rangle$ from the preceding theorem. Applying Theorem 4.1.7 we can find, for each $j$, a $\mathbb{Z}^N$-periodic set $\Omega_j$ such that $\sigma_{\varphi_j}(\xi) = \chi_{\Omega_j}(\xi)$[2].

Fix $j$ and $\xi$ and consider the vector $L_j(\xi)$ in $\ell^2(\mathbb{Z}^N)$ defined by

$$L_j(\xi) = (\widehat{\varphi_j}(\xi+k))_{k\in\mathbb{Z}^N}. \tag{23}$$

Observe that

$$\|L_j(\xi)\|^2 = \sum_{k\in\mathbb{Z}^N}|\widehat{\varphi_j}(\xi+k)|^2 = \sigma_{\varphi_j}(\xi) = \chi_{\Omega_j}(\xi) \in \{0,1\}. \tag{24}$$

[2]In order to avoid to repeatedly add the expression "a.e." we tacitly assume that we only choose $\xi$ in a subset of $\mathbb{R}^N$ whose complement has measure 0 and, for all such $\xi$ all related properties we invoke are valid.

The orthogonality of the spaces $\langle\varphi_j\rangle$ and Proposition 4.1.2 give us

$$\langle L_j(\xi), L_{j'}(\xi)\rangle = 0 \ \text{ for } j \neq j'. \tag{25}$$

Let

$$L(\xi) = \overline{\text{span}}\,\{L_j(\xi) : j \in \mathbb{N}\}. \tag{26}$$

It is evident from (24) and (25) that the sequence $(L_j(\xi))_{j=1}^{\infty}$ is a Parseval frame for $L(\xi)$ (even if $L(\xi) = \{0\}$).

**Definition 4.1.10.** *Let $V = \oplus_{j=1}^{\infty}\langle\varphi_j\rangle$ be a decomposition of a shift-invariant space $V$ as in Theorem 4.1.9. The dimension function $dim_V$ of $V$ is defined by*

$$dim_V(\xi) = dim\, L(\xi), \ \ \xi \in \mathbb{R}^N. \tag{27}$$

**Remark 4.1.11.** Suppose that $V = \oplus_{j=1}^{\infty}\langle\varphi_j\rangle$ is a decomposition of a shift-invariant space $V$ as in Theorem 4.1.9; let $\sigma_{\varphi_j}(\xi) = \chi_{\Omega_j}(\xi)$, $j \in \mathbb{N}$. Since $(L_j(\xi))_{j=1}^{\infty}$ is a Parseval frame for $L(\xi)$, we have

$$\dim L(\xi) \overset{\text{(by Exercise 2.1.26)}}{=} \sum_{j=1}^{\infty} \|L_j(\xi)\|^2 \overset{(24)}{=} \sum_{j=1}^{\infty} \chi_{\Omega_j}(\xi). \tag{28}$$

In fact, the first equality above follows simply from the fact that the sequence $(L_j(\xi))_{j=1}^{\infty}$ is almost - up to some zero-vectors - an ONB for $L(\xi)$; nevertheless, it is always enjoyable to invoke the beautiful statement of Exercise 2.1.26 :) ). Anyhow, we conclude from the preceding equality that

$$\dim_V(\xi) = \sum_{j=1}^{\infty} \chi_{\Omega_j}(\xi). \tag{29}$$

However, one should note that a decomposition from Theorem 4.1.9 is not unique (see Exercise 4.1.14) and hence, at the moment, our definition of the dimension function depends on the decomposition under consideration.

**Remark 4.1.12.** It is evident from (28) that $\dim L(\xi) = \dim L(\xi + k)$ for all $\xi \in \mathbb{R}^N$ and $k$ in $\mathbb{Z}^N$. However, this can be seen directly. Namely, given $k \in Z^N$, we see that $L_j(\xi+k) = U_k^* L_j(\xi)$ for all $\xi$ and $j$, where $U_k \in \mathbb{B}(\ell^2(\mathbb{Z}^N))$ is a suitable unitary operator. (For example, if we take $N = 1$ and $k > 0$, then $U_k = S^k$, where $S$ is the bilateral shift.) Consequently, we have $L(\xi + k) = U_k^* L(\xi)$, so these two spaces, being unitarily equivalent, must have the same dimension.

*Concluding remarks.* For more results and details we refer the reader to [27], [118], [21] and the references therein. Theorem 4.1.7 is first proved in [25]. Theorem 4.1.9 is borrowed from [27]. In Theorem 10.19 in [81] various characterizations of the shift-invariant space generated by a single function are collected. Among other characterizations, this theorem contains a description of all functions $g \in L^2(\mathbb{R})$ for which the system $(T_k g)_{k\in\mathbb{Z}}$ makes up a Schauder basis for $\langle g\rangle$ ([99]).

The Appendix at the end of this chapter contains some useful technical results which are stated in the form as in [73].

**Exercise 4.1.13.** Let $V_1$ and $V$ be shift-invariant subspaces of $L^2(\mathbb{R}^N)$. Show that $V_2 = V \ominus V_1$ is also shift invariant.

**Exercise 4.1.14.** Show that any principal shift-invariant space can be decomposed (as in Theorem 4.1.9) in any number of mutually orthogonal spaces of the form $\langle\psi\rangle$ with the property that the system $(T_k\psi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi\rangle$. Hint. Suppose that $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\varphi\rangle$, let $\sigma_\varphi(\xi) = \chi_\Omega(\xi)$, a.e. Put $S = \Omega \cap C^N$. Take a disjoint union $S = S_1 \cup S_2$ such that $S_1$ and $S_2$ both have positive measure and put $m_i = \chi_{S_1}$, $i = 1, 2$. Consider $f_1, f_2 \in \langle\varphi\rangle$ defined by $\hat{f}_i = m_1\hat{\varphi}$, $i = 1, 2$.

**Exercise 4.1.15.** Let $V \leq L^2(\mathbb{R}^N)$ be a shift-invariant space of the form $V = \oplus_{j=1}^{\infty}\langle\phi_j\rangle$. For any $\xi \in \mathbb{R}^N$ and $j \in \mathbb{N}$ consider the sequence $K_j(\xi) = (\hat{\phi}(\xi + k))_{k\in\mathbb{Z}}$. Show that $K_j(\xi)$ belongs to $\ell^2(\mathbb{Z}^N)$ for all $\xi$ and $j$. Let $K(\xi) := \overline{\operatorname{span}}\{K_j(\xi) : j \in \mathbb{N}\} \leq \ell^2(\mathbb{Z}^N)$. Prove that $\dim_V(\xi) = \dim K(\xi)$ for a.e. $\xi$. *Remark.* Observe that the functions $\phi_j$ are not necessarily Parseval generators for the spaces $\langle\phi_j\rangle$.

## 4.2 The spectral function

We proceed our study of shift-invariant spaces. Let $V = \oplus_{j=1}^{\infty}\langle\varphi_j\rangle$ be a decomposition of a shift-invariant space $V$ as in Theorem 4.1.9 which means that, for each $j$, the sequence $(T_k\varphi_j)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\varphi_j\rangle$. Recall that this implies that there exists a sequence $(\Omega_j)_j$ of $Z^N$-periodic sets such that $\sigma_{\varphi_j} = \chi_{\Omega_j}$ a.e. for all $j$. Using this, we have introduced the dimension function $\dim_V$ by the formula $\dim_V(\xi) = \sum_{j=1}^{\infty}\chi_{\Omega_j}(\xi)$. Finally, we shall need the subspaces $L(\xi)$ of $\ell^2(\mathbb{Z}^N)$ that are generated by sequences $(L_j(\xi))_j$, where $L_j(\xi) = (\widehat{\varphi_j}(\xi+k))_{k\in\mathbb{Z}^N}$, for every $\xi$.

Denote additionally by $P(\xi) \in \mathbb{B}(\ell^2(\mathbb{Z}^N))$ the orthogonal projection onto $L(\xi)$.

**Definition 4.2.1.** *Let $V = \oplus_{j=1}^{\infty}\langle\varphi_j\rangle$ be a decomposition of a shift-invariant space $V$ as in Theorem 4.1.9. Denote by $(e_k)_{k\in\mathbb{Z}^N}$ the canonical basis for $\ell^2(\mathbb{Z}^N)$. The spectral function $\sigma_V$ of $V$ is defined by*

$$\sigma_V(\xi) = \|P(\xi)e_0\|^2,\ \ \xi \in \mathbb{R}^N. \tag{30}$$

It will be useful to obtain an alternative formula for the spectral function $\sigma_V$. First we need a lemma.

**Lemma 4.2.2.** *Let $M$ i $L$ be closed subspaces of a Hilbert space $H$ for which there exists a unitary operator $U \in \mathbb{B}(H)$ such that $L = U^*M$. Denote by $P_M$ and $P_L$ the orthogonal projections to $M$ and $L$, respectively. Then we have $P_L = U^*P_MU$.*

**Proof.** We leave the proof as an exercise. □

**Remark 4.2.3.** Let $V$ be a shift-invariant space with a decomposition as in Theorem 4.1.9. Consider, as before, the subspaces $L(\xi)$ and the corresponding orthogonal projections $P(\xi)$. Given $\xi \in C^N$ and $k \in \mathbb{Z}^N$, we know from Remark 4.1.12 that $L(\xi+k) = U_k^*L(\xi)$, where $U$ is a unitary operator with the property $U_ke_0 = e_k$. Using (30) and Lemma 4.2.2 we now conclude that

$$\sigma_V(\xi+k) = \|P(\xi+k)e_0\|^2 = \|U_k^*P(\xi)U_ke_0\|^2 = \|P(\xi)e_k\|^2,\ \forall\xi \in C^N,\ \forall k \in \mathbb{Z}^N. \tag{31}$$

**Remark 4.2.4.** Let $V$ ba a shift-invariant space with a decomposition $V = \oplus_{j=1}^{\infty}\langle\varphi_j\rangle$ from Theorem 4.1.9. Let us keep the notation from the preceding considerations. Then we have

$$\sigma_V(\xi) = \sum_{j=1}^{\infty}|\widehat{\varphi_j}(\xi)|^2,\ \ \forall\xi \in \mathbb{R}^N \tag{32}$$

and

$$\dim_V(\xi) = \sum_{k\in\mathbb{Z}^N}\sigma_V(\xi+k),\ \ \forall\xi \in \mathbb{R}^N,\ \forall k \in \mathbb{Z}^N. \tag{33}$$

To prove (32), recall that $(L_j(\xi))_{j=1}^{\infty}$ is a Parseval frame for $L(\xi)$. Now we compute.

$$\begin{aligned}
\sigma_V(\xi) &= \|P(\xi)e_0\|^2 \\
&= \sum_{j=1}^{\infty} |\langle P(\xi)e_0, L_j(\xi)\rangle|^2 \\
&= \sum_{j=1}^{\infty} |\langle e_0, P(\xi)L_j(\xi)\rangle|^2 \\
&= \sum_{j=1}^{\infty} |\langle e_0, L_j(\xi)\rangle|^2 \\
&= \sum_{j=1}^{\infty} |\hat{\varphi}_j(\xi)|^2.
\end{aligned}$$

Let us now prove (33).

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^N} \sigma_V(\xi+k) &\overset{(32)}{=} \sum_{k\in\mathbb{Z}^N} \sum_{j=1}^{\infty} |\hat{\varphi}_j(\xi+k)|^2 \\
&= \sum_{j=1}^{\infty} \sigma_{\varphi_j}(\xi) \\
&= \sum_{j=1}^{\infty} \chi_{\Omega_j}(\xi) = \dim_V(\xi).
\end{aligned}$$

Alternatively, (33) can also be obtained using (31). For all $\xi \in C^N$ we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^N} \sigma_V(\xi+k) &\overset{(31)}{=} \sum_{k\in\mathbb{Z}^N} \|P(\xi)e_k\|^2 \\
&= \sum_{k\in\mathbb{Z}^N} \langle P(\xi)e_k, P(\xi)e_k\rangle \\
&= \sum_{k\in\mathbb{Z}^N} \langle P(\xi)e_k, e_k\rangle \\
&= \operatorname{tr}(P(\xi)) \\
&= \dim(\mathrm{R}(P(\xi))) \\
&= \dim L(\xi) \\
&\overset{(27)}{=} \dim_V(\xi).
\end{aligned}$$

However, one should keep in mind that our definition of the spectral function (like the definition of the dimension function) is formulated in terms of a decomposition of $V$ as in Theorem 4.1.9. Since decomposition of this type is not unique, it is now time for the following theorem.

**Theorem 4.2.5.** *Let $V$ be a shift-invariant space. The definitions of the dimension function $\dim_V$ and the spectral function $\sigma_V$ do not depend on the choice of a decomposition of $V$ into the orthogonal sum of principal shift-invariant spaces.*

**Proof.** Let $V = \oplus_{j=1}^{\infty}\langle\varphi_j\rangle$, where the sequence $(T_k\varphi_j)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\varphi_j\rangle$, for every $j \in \mathbb{N}$. Denote by $P \in \mathbb{B}(L^2(\mathbb{R}^N))$ the orthogonal projection to $V$.

Since the sequence $(T_k\varphi_j)_{k\in\mathbb{Z}^N,\, j\in\mathbb{N}}$ is a Parseval frame for $V$, we have for each $f \in L^2(\mathbb{R}^N)$

$$Pf = \sum_{k\in\mathbb{Z}^N}\sum_{j=1}^{\infty}\langle Pf, T_k\varphi_j\rangle T_k\varphi_j = \sum_{k\in\mathbb{Z}^N}\sum_{j=1}^{\infty}\langle f, T_k\varphi_j\rangle T_k\varphi_j$$

and hence

$$\langle Pf, f\rangle = \sum_{k\in\mathbb{Z}^N}\sum_{j=1}^{\infty}\langle f, T_k\varphi_j\rangle\langle T_k\varphi_j, f\rangle. \tag{34}$$

We shall apply the preceding equality to the function $\hat{f} = \chi_{C^N+m}$, where $m$ is a fixed (but arbitrary) element from $\mathbb{Z}^N$. In the computation that follows we shall use the fact that $\widehat{\varphi_j}\chi_{C^N+m}$ belongs to $L^2(C^N+m)$ and that the sequence $(e^{2\pi i\langle k,\xi\rangle})_{k\in\mathbb{Z}^N}$ is an ONB for $L^2(C^N+m)$.

$$\begin{aligned}
\langle Pf, f\rangle &\overset{(34)}{=} \sum_{k\in\mathbb{Z}^N}\sum_{j=1}^{\infty}|\langle T_k\varphi_j, f\rangle|^2 \\
&\overset{(2)}{=} \sum_{k\in\mathbb{Z}^N}\sum_{j=1}^{\infty}|\langle \widehat{T_k\varphi_j}, \hat{f}\rangle|^2 \\
&= \sum_{j=1}^{\infty}\left(\sum_{k\in\mathbb{Z}^N}\left|\left\langle \widehat{\varphi_j}\chi_{C^N+m}, e^{2\pi i\langle k,\xi\rangle}\right\rangle\right|^2\right) \\
&= \sum_{j=1}^{\infty}\left\|\widehat{\varphi_j}\chi_{C^N+m}\right\|^2 \\
&= \sum_{j=1}^{\infty}\int_{C^N+m}|\widehat{\varphi_j}(\xi)|^2\,d\xi \\
&= \int_{C^N+m}\left(\sum_{j=1}^{\infty}|\widehat{\varphi_j}(\xi)|^2\right)d\xi.
\end{aligned}$$

If we now take another decomposition of $V$, say $V = \oplus_{j=1}^{\infty}\langle\psi_j\rangle$, where the sequence $(T_k\psi_j)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi_j\rangle$, for every $j\in\mathbb{N}$, repeating the same computation we get

$$\int_{C^N+m}\left(\sum_{j=1}^{\infty}|\widehat{\varphi_j}(\xi)|^2\right)d\xi = \langle Pf, f\rangle = \int_{C^N+m}\left(\sum_{j=1}^{\infty}|\widehat{\psi_j}(\xi)|^2\right)d\xi.$$

From this we conclude that

$$\sum_{j=1}^{\infty}|\widehat{\varphi_j}(\xi)|^2 = \sum_{j=1}^{\infty}|\widehat{\psi_j}(\xi)|^2 \text{ for a.e. } \xi \in C^N+m.$$

Since $m$ was arbitrary, this implies

$$\sum_{j=1}^{\infty} |\widehat{\varphi_j}(\xi)|^2 = \sum_{j=1}^{\infty} |\widehat{\psi_j}(\xi)|^2 \text{ a.e.}$$

Equality (32) now shows that the definition $\sigma_V(\xi)$ does not depend on the decomposition under consideration. Invoking (33) we obtain the same conclusion concerning $\dim_V(\xi)$. □

The following proposition provides more useful properties of the spectral function.

**Proposition 4.2.6.** *The spectral function has the following properties:*

*(a)* $\sigma_{L^2(\mathbb{R}^N)} = 1$ *a.e.*

*(b) If* $(V_n)_n$ *is a sequence of mutually orthogonal shift-invariant spaces and* $V = \oplus_{n=1}^{\infty} V_n$, *then* $\sigma_V = \sum_{n=1}^{\infty} \sigma_{V_n}$.

*(c) If* $V$ *and* $W$ *are shift-invariant and* $V \leq W$, *then* $\sigma_V \leq \sigma_W$.

*(d)* $0 \leq \sigma_V \leq 1$, *for each shift-invariant space* $V$. *Moreover,* $\sigma_V = 0$ *if and only if* $V = \{0\}$ *and* $\sigma_V = 1$ *if and only if* $V = L^2(\mathbb{R}^N)$.

**Proof.** (a) Consider the sequence $(\varphi_n)_{n\in\mathbb{Z}^N}$ defined by $\widehat{\varphi_n} = \chi_{C^N+n}$, $n \in \mathbb{Z}^N$. Observe that

$$\widehat{T_k\varphi_n}(\xi) = e^{-2\pi i\langle k,\xi\rangle}\widehat{\varphi_n}(\xi) = e^{-2\pi i\langle k,\xi\rangle}\chi_{C^N+n}(\xi), \ \ \forall k, n \in \mathbb{Z}^N.$$

Thus, the system $(\widehat{T_k\varphi_n})_{k,n\in\mathbb{Z}^N}$ makes up an ONB for $L^2(\mathbb{R}^N)$. Since the Fourier transform is a unitary operator, this implies that $(T_k\varphi_n)_{k,n\in\mathbb{Z}^N}$ is an ONB for $L^2(\mathbb{R}^N)$. In particular, we have

$$L^2(\mathbb{R}^n) = \oplus_{n\in\mathbb{Z}^N}\langle\varphi_n\rangle$$

and the sequence $(T_k\varphi_n)_{k\in\mathbb{Z}^N}$ is a Parseval frame (in fact, an ONB) for $\langle\varphi_n\rangle$. Hence, by definition, we have

$$\sigma_{L^2(\mathbb{R}^N)}(\xi) = \sum_{n\in\mathbb{Z}^N} |\widehat{\varphi_n}(\xi)|^2 = \sum_{n\in\mathbb{Z}^N} \chi_{C^N+n}(\xi) = 1.$$

(b) We can decompose each $V_n$ as in Theorem 4.1.9 and apply (32) to each $V_n$ and to $V$.

(c) Clearly, $W \ominus V$ is also shift invariant; therefore, (c) follows from (b).

(d) The first assertion follows from (30). It is clear that $\sigma_V = 0$ if and only if $V = \{0\}$. We also know from (a) that $\sigma_{L^2(\mathbb{R}^N)} = 1$. It remains to discuss the possibility $\sigma_V = 1$. Let $W = V^\perp$. Now we have $L^2(\mathbb{R}^N) = V \oplus W$ and we conclude from (b) that $\sigma_W = 0$. Thus, by (c), $W = \{0\}$. □

**Corollary 4.2.7.** *If* $(V_n)_n$ *is a sequence of mutually orthogonal shift-invariant spaces and* $V = \oplus_{n=1}^{\infty} V_n$, *then* $dim_V = \sum_{n=1}^{\infty} dim_{V_n}$.

**Proof.** Equality (33) from Remark 4.2.4 and Proposition 4.2.6 (b). □

The following refinement of Theorem 4.1.9 is useful in applications, in particular in developing generalized mutiresolution technique in wavelet theory.

**Theorem 4.2.8.** *(The canonical decomposition, [27]) Let $V$ be a shift-invariant space. Then there exists a sequence $(\psi_j)_{j=1}^{\infty}$ in $V$ such that, for each $j$, the system $(T_k\psi_j)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi_j\rangle$, $V = \oplus_{j=1}^{\infty}\langle\psi_j\rangle$, and that $\sigma_{\psi_j} = \chi_{E_j}$ a.e., where the sets $E_j$ satisfy*

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots.$$

*Remark.* As before, we allow the possibility that all but a finite number of the $\psi_j$ can be the zero function.

**Proof.** We start with a decomposition of $V$ as in Theorem 4.1.9: $V = \oplus_{j=1}^{\infty}\langle\varphi_j\rangle$ with $\sigma_{\varphi_j} = \chi_{\Omega_j}$, for every $j \in \mathbb{N}$. Recall that $\dim_V(\xi) = \sum_{j=1}^{\infty}\chi_{\Omega_j}(\xi)$.

Here we again neglect sets of measure zero. Denote as before by $C^N$ the unit cube in $\mathbb{R}^N$ and put

$$S_j = \Omega_j \cap C^N,\ \ j \in \mathbb{N}.$$

We also introduce the sets

$$E_0 = \mathbb{R}^N,\ \ E_m = \{\xi \in \mathbb{R}^N : \dim_V(\xi) \geq m\},\ \ T_m = E_m \cap C^N,\ \ m \in \mathbb{N}.$$

In particular, $T_1$ is the support of $\dim_V$ in the unit cube. We now decompose $T_1$ into a disjoint union. Put

$$T_{11} = S_1,\ T_{12} = S_2 \setminus S_1,\ T_{13} = S_3 \setminus (S_1 \cup S_2),\ T_{1j} = S_j \setminus (S_1 \cup \ldots \cup S_{j-1}),\ \ j \in \mathbb{N}.$$

Then, clearly, we have

$$T_1 = \cup_{j=1}^{\infty} T_{1j}. \tag{35}$$

We now introduce the functions

$$m_{1j} = \chi_{T_{1j}},\ \ \widehat{\psi_{1j}} = m_{1j}\widehat{\varphi_j},\ \ j \in \mathbb{N}.$$

One should observe that the functions $m_{1j}$ belong to $L^2(\mathbb{T}^N)$, but in the second equality above that defines functions $\psi_{ij}$ we understand (as in Proposition 4.1.3) that $m_{1j}$ is extended by $\mathbb{Z}^N$-periodicity to the function $m_{1j}$ on $\mathbb{R}^N$. Notice that by Proposition 4.1.3 we have $\psi_{1j} \in \langle\varphi_j\rangle$, for every $j$ in $\mathbb{N}$. Moreover, we claim that

$$\psi_1 := \psi_{11} \oplus \psi_{12} \oplus \psi_{13} \oplus \ldots \in \oplus_{j=1}^{\infty}\langle\varphi_j\rangle = V. \tag{36}$$

Indeed, since we have

$$\varphi_1 \oplus \varphi_2 \oplus \varphi_3 \oplus \ldots \in \oplus_{j=1}^{\infty}\langle\varphi_j\rangle = V,$$

we know that $\sum_{j=1}^{\infty}\|\varphi_j\|^2 < \infty$. Since $\|\psi_{1j}\|^2 \leq \|\varphi_j\|^2$ for each $j$, we have

$$\sum_{j=1}^{\infty}\|\psi_{1j}\|^2 \leq \sum_{j=1}^{\infty}\|\varphi_j\|^2 < \infty$$

and this is enough to conclude that the function $\psi_1$ defined in (36) does belong to $V$.

Consider now the subspace $\langle\psi_1\rangle \leq V$. We claim that $\psi_1$ is in fact a Parseval generator for $\langle\psi_1\rangle$. Indeed, since the sets $T_{1j}$ are pairwise disjoint, we have

$$\begin{aligned}
\sigma_{\psi_1}(\xi) &= \sum_{k\in\mathbb{Z}^n} |\widehat{\psi_{11}}(\xi+k) + \widehat{\psi_{12}}(\xi+k) + \dots|^2 \\
&= \sum_{k\in\mathbb{Z}^n} |m_{11}(\xi+k)\widehat{\varphi_1}(\xi+k) + m_{12}(\xi+k)\widehat{\varphi_2}(\xi+k) + \dots|^2 \\
&= \sum_{k\in\mathbb{Z}^n} |\chi_{T_{11}}(\xi+k)\widehat{\varphi_1}(\xi+k) + \chi_{T_{12}}(\xi+k)\widehat{\varphi_2}(\xi+k) + \dots|^2 \\
&= \sum_{j=1}^{\infty} \chi_{(T_{1j}+\mathbb{Z}^N)}(\xi) \\
&= \chi_{\cup_{j=1}^{\infty}(T_{1j}+\mathbb{Z}^N)}(\xi) \\
&= \chi_{(T_1+\mathbb{Z}^N)}(\xi) \\
&= \chi_{E_1}(\xi).
\end{aligned}$$

We now let $V_1 = V \ominus \langle\psi_1\rangle$, i.e. $V = \langle\psi_1\rangle + V_1$. Observe that $V_1$ is shift-invarant and that, by Corollary 4.2.7, we have

$$\dim_V = \chi_{E_1} + \dim_{V_1}.$$

We now apply the preceding argument to the shift-invariant space $V_1$. Inductively we obtain a sequence $(\psi_j)_{j=1}^{\infty}$ of principal shift-invariant subspaces of $V$ such that, for each $j$, the sequence $(T_k\psi_j)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi_j\rangle$ and $\sigma_{\psi_j} = \chi_{E_j}$. Thus, we have

$$\oplus_{j=1}^{\infty}\langle\psi_j\rangle \leq V$$

and

$$\dim_{\oplus_{j=1}^{\infty}\langle\psi_j\rangle} = \sum_{j=1}^{\infty}\chi_{E_j} = \sum_{j=1}^{\infty}\chi_{\Omega_j} = \dim_V. \tag{37}$$

This is enough to conclude that

$$\oplus_{j=1}^{\infty}\langle\psi_j\rangle = V,$$

since the orthogonal complement of $\oplus_{j=1}^{\infty}\langle\psi_j\rangle$ in $V$ is a shift-invariant space which, by Corollary 4.2.7 and equality (37), has the dimension function that is equal to 0 a.e.

□

We end the section with another property of the spectral function which will play an important role in our study of multiresolution analysis in wavelet theory.

Let $A$ be an invertible $N \times N$ matrix with integer coefficients; $A \in M_N(\mathbb{Z})$. Let $d = |\det A| \in \mathbb{N}$. We define the dilation operator $D_A$ on $L^2(\mathbb{R}^N)$ by

$$D_A f(x) = \sqrt{d} f(Ax). \tag{38}$$

It is easy to see that $D_A$ is a unitary operator. It does not commute with translations $T_k$, $k \in \mathbb{Z}^N$, but we have the following commutation relations:

$$T_k (D_A)^j = (D_A)^j T_{A^j k}, \;\; \forall k \in \mathbb{Z}^n, \forall j \in \mathbb{Z}. \tag{39}$$

It is immediate from the preceding formula that $D_A$ preserves shift-invariance: if $V$ is a shift-invariant space, then $D_A(V)$ is also shift invariant.

It is also useful to note that

$$\widehat{(D_A)^j f}(\xi) = \frac{1}{d^{j/2}} \hat{f}(B^{-j}\xi), \ \ \forall j \in \mathbb{Z}, \tag{40}$$

where $B = A'$ is the transpose of $A$.

For $a \in \mathbb{R}^N$ we define the modulation by $a$ as the operator on $L^2(\mathbb{R}^N)$ that is given by

$$M_a f(x) = e^{2\pi i \langle a, x\rangle} f(x). \tag{41}$$

Her we have the following commutation relations:

$$T_k M_a = e^{-2\pi i \langle k, a\rangle} M_a T_k, \ \ \forall a, k, \tag{42}$$

from which we see that $M_a$ also preserves shift-invariance.

Finally, we note

$$\widehat{T_k f} = M_{-k}\hat{f}, \ \ \widehat{M_a f} = T_a \hat{f}, \ \ \forall k, a \in \mathbb{R}^N. \tag{43}$$

We now state the result that gives us the spectral function of shift-invariant spaces that arise by applying dilations and modulations to such spaces.

**Theorem 4.2.9.** *Let $V \subseteq L^2(\mathbb{R}^N)$ be a shift-invariant space. If $A$ is an invertible $N \times N$ matrix with integer coefficients then $D_A(V)$ is shift-invariant and*

$$\sigma_{D_A(V)}(\xi) = \sigma_V(B^{-1}\xi), \tag{44}$$

*where $B = A'$.*

*Likewise, for any $a \in \mathbb{R}^n$, $M_a(V)$ is shift-invariant and*

$$\sigma_{M_a(V)}(\xi) = \sigma_V(\xi - a). \tag{45}$$

For the proof we refer the reader to [28] or [106]; however, we note that a crucial part of the proof is the content of Exercise 4.2.14.

Take again an invertible $N \times N$ matrix $A$ with integer coefficients; let $B = A'$ and $d = |\det A| = |\det B|$. It is well known that $\mathbb{Z}^N / B\mathbb{Z}^N$ is a group of order $d$ (see Exercise 4.2.13). In the following corollary we will make use of a set of $d$ representatives of different cosets of $\mathbb{Z}^N / B\mathbb{Z}^N$.

**Corollary 4.2.10.** *Let $V \subseteq L^2(\mathbb{R}^N)$ be a shift-invariant space. Let $A$ be an invertible $N \times N$ matrix with integer coefficients with $d = |\det A| = |\det B|$, where $B = A'$. Take any set $\{\alpha_0, \alpha_1, \ldots, \alpha_{d-1}\}$ of $d$ representatives of different cosets of $\mathbb{Z}^N / B\mathbb{Z}^N$. Then*

$$dim_{D_A(V)}(\xi) = \sum_{j=0}^{d-1} dim_V(B^{-1}(\xi + \alpha_j)). \tag{46}$$

**Proof.**

$$\begin{aligned}
\dim_{D_A(V)}(\xi) &\overset{(33)}{=} \sum_{k\in\mathbb{Z}^N} \sigma_{D_A(V)}(\xi+k) \\
&\overset{(44)}{=} \sum_{k\in\mathbb{Z}^N} \sigma_V(B^{-1}\xi+B^{-1}k) \\
&= \sum_{j=0}^{d-1}\sum_{k\in\mathbb{Z}^N} \sigma_V(B^{-1}\xi+B^{-1}(\alpha_j+Bk)) \\
&= \sum_{j=0}^{d-1}\left(\sum_{k\in\mathbb{Z}^N} \sigma_V(B^{-1}\xi+B^{-1}\alpha_j+k)\right) \\
&\overset{(33)}{=} \sum_{j=0}^{d-1} \dim_V(B^{-1}(\xi+\alpha_j)).
\end{aligned}$$

□

*Concluding remarks.* The results of this section are taken from [106] and [28]. The canonical decomposition (as in Theorem 4.2.8 is first proved in [27]), but the proof presented here is different in the sense that it does not make use of the range function.

**Exercise 4.2.11.** Verify formulae (39), (40), (42), and (43).

**Exercise 4.2.12.** Prove Lemma 4.2.2

**Exercise 4.2.13.** Let $A$ be an invertible $N\times N$ matrix with integer coefficients, let $d = |\det A|$. Prove that $\mathbb{A}^{-1}\mathbb{Z}^N/\mathbb{Z}^N$ and $\mathbb{Z}^N/A\mathbb{Z}^N$ are isomorphic groups of order $d$.

**Exercise 4.2.14.** Suppose that $(\psi)_n$ is a sequence of functions in a shift-invariant space $V \subseteq L^2(\mathbb{R}^N)$ such that the system $(T_k\psi_n)_{k\in\mathbb{Z}^N,n\in\mathbb{N}}$ is a Parseval frame for $V$. Let $A \in M_N(\mathbb{Z})$ be an invertible matrix, let $d = |\det A|$. Take any set $\{m_0, m_1, \ldots, m_{d-1}\}$ of $d$ representatives of different cosets of $\mathbb{Z}^N/A\mathbb{Z}^N$. Prove that the system $(T_k(D_A T_{m_j}\psi_n))_{k\in\mathbb{Z}^N,\, n\in\mathbb{N},\, j\in\{0,1,\ldots,m-1\}}$ is a Parseval frame for $D_A(V)$. ([106], Lemma 2.5.)

## 4.3 Semi-orthogonal Parseval wavelets

**Definition 4.3.1.** *We say that a function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if the system*

$$(\psi_{j,k})_{j,k\in\mathbb{Z}}, \ \ \psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j \cdot -k), \ \ j,k \in \mathbb{Z} \tag{47}$$

*is an ONB for $L^2(\mathbb{R})$. More generally, $\psi$ is said to be a Parseval wavelet if $(\psi_{j,k})_{j,k\in\mathbb{Z}}$ is a Parseval frame for $L^2(\mathbb{R})$.*

Let us denote by $D \in \mathbb{B}(L^2(\mathbb{R}))$ the dyadic dilation operator that is defined by

$$Df(x) = \sqrt{2}f(2x), \ \ f \in L^2(\mathbb{R}), \tag{48}$$

and let

$$T_k f(x) = f(x-k), \ \ f \in L^2(\mathbb{R}). \tag{49}$$

Then we see that

$$\psi_{j,k} = D^j T_k \psi, \ \ j,k \in \mathbb{Z};$$

thus, $\psi$ is an orthonormal (Parseval) wavelet if the system $(D^j T_k\psi)_{j,k\in\mathbb{Z}}$ is an ONB (a Parseval frame) for $L^2(\mathbb{R})$.

More generally, one can define wavelets not only in $L^2(\mathbb{R})$ with dilation factors other than 2, but also in $L^2(\mathbb{R}^N)$, $N \in \mathbb{N}$. In $N$ dimensions one takes a matrix $A \in M_N(\mathbb{Z})$ and consider the corresponding dilation operator defined by (38): $D_A f(x) = \sqrt{d} f(Ax)$, $f \in L^2(\mathbb{R}^N)$, where $d = |\det A|$. For technical reasons we require that $A$ is an expansive matrix which means that all eigenvalues of $A$, both real and complex, have absolute value greater than 1. Notice that this implies that $d \geq 2$.

For simplicity we shall denote $D_A$ by $D$ whenever the dilation matrix $A$ is fixed and clear from the context. We also need translations $T_k \in \mathbb{B}(L^2(\mathbb{R}^N))$, $k \in \mathbb{Z}^N$. Now, again, we say that $\psi \in L^2(\mathbb{R}^N)$ is an orthonormal wavelet (resp. a Parseval wavelet) if the system

$$(D^j T_k \psi)_{j\in\mathbb{Z},\, k\in\mathbb{Z}^N} \tag{50}$$

is an ONB (resp. a Parseval frame) for $L^2(\mathbb{R}^N)$.

**Example 4.3.2.** The Haar wavelet is the function $\psi$ defined by $\psi(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$.

It is relatively easy to see that $(D^j T_k\psi)_{j,k\in\mathbb{Z}}$ is an orthonormal system (see e.g. [51], p. 73). For the proof of the spanning property we refer to [57] (the reader may also consult [81] or [117]).

**Example 4.3.3.** The Shannon wavelet is the function $\psi$ defined by $\hat{\psi} = \chi_{[-1,-\frac{1}{2})\cup[\frac{1}{2},1)}$. One can show that the Shannon function is an orthonormal wavelet by using the following characterization theorem (but see also Exercise 4.3.20).

**Theorem 4.3.4.** *Let $A \in M_N(\mathbb{Z})$ be an expansive matrix and let $B = A'$. A function $\psi \in L^2(\mathbb{R}^N)$ is a Parseval wavelet if and only if the following two conditions are satisfied:*

*(a)* $\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^j\xi)|^2 = 1$ *a.e.;*

*(b)* $\sum_{j=0}^{\infty} \hat{\psi}(B^j\xi)\overline{\hat{\psi}(B^j(\xi+q))} = 0$ *a.e.,* $\forall q \in \mathbb{Z}^N \setminus B\mathbb{Z}^N$*.*

*In particular,* $\psi$ *is an orthonormal wavelet if and only if in addition to (a) and (b)* $\psi$ *satisfies* $\|\psi\| = 1$.

For the proof, which is omitted, we refer the reader to [83]. Here we just mention that the last assertion trivially follows. Namely, since $D_A$ and $T_k$ are unitary operators, the hypothesis $\|\psi\| = 1$ implies that $\|D^jT_k\psi\| = 1$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^N$. Then one applies the simple observation (cf. Exercise 3.2.24) that an element of a Parseval frame with the norm equal to 1 is orthogonal to all other frame members.

We now turn to the multiresolution analysis which is the most prominent concept in constructing wavelets. In what follows we fix an expansive matrix $A \in M_N(\mathbb{Z})$. As before, let $B = A'$ and $d = |\det A|$. We denote by $D$ the operator induced by $A$: $Df(x) = \sqrt{d}f(Ax)$, $f \in L^2(\mathbb{R}^N)$.

We will also fix a set $\{\alpha_0, \alpha_1, \ldots, \alpha_{d-1}\}$ of $d$ representatives of different cosets of $\mathbb{Z}^N/B\mathbb{Z}^N$ (see Exercise 4.2.13). If we denote $\beta_i = B^{-1}\alpha_i$ for $i = 0, 1, \ldots, d-1$, then $\{\beta_0, \beta_1, \ldots, \beta_{d-1}\}$ is a set of $d$ representatives of different cosets of $\mathbb{B}^{-1}Z^N/\mathbb{Z}^N$.

First we need to introduce a class of wavelets in between orthonormal and Parseval wavelets.

**Definition 4.3.5.** *A Parseval wavelet* $\psi \in L^2(\mathbb{R}^N)$ *is said to be semi-orthogonal if* $D^{j_1}T_k\psi \perp D^{j_2}T_l\psi$ *for all* $j_1 \neq j_2$ *in* $\mathbb{Z}$ *and all* $k, l$ *from* $\mathbb{Z}^N$*.*

Note that semi-orthogonality simply means that the spaces $D^j\langle\psi\rangle$, $j \in \mathbb{Z}$, are mutually orthogonal.

**Theorem 4.3.6.** *Let* $\psi \in L^2(\mathbb{R}^N)$ *be a semi-orthogonal Parseval wavelet. Let*

$$V_0 = \overline{span}\,\{D^jT_k\psi : j < 0, k \in \mathbb{Z}^N\} \tag{51}$$

*and*

$$V_j = D^jV_0, \ \ j \in \mathbb{Z}. \tag{52}$$

*Then the sequence* $(V_j)_{j\in\mathbb{Z}}$ *of closed subspaces of* $L^2(\mathbb{R}^N)$ *has the following properties:*

*(a)* $V_{j+1} = DV_j$, $\forall j \in \mathbb{Z}$*;*

*(b)* $V_j \subseteq V_{j+1}$, $\forall j \in \mathbb{Z}$*;*

*(c)* $\cap_{j\in\mathbb{Z}}V_j = \{0\}$, $\overline{\cup_{j\in\mathbb{Z}}V_j} = L^2(\mathbb{R}^N)$*;*

*(d)* $V_0$ *is shift-invariant.*

**Proof.** Since $V_0$ is closed and $D$ is a unitary operator, all $V_j$'s are obviously closed. Also, we see from (51) and (52) that

$$V_j = \overline{\text{span}}\,\{D^{j'}T_k\psi : j' < j, k \in \mathbb{Z}^N\}, \ \ \forall j \in \mathbb{Z}, \tag{53}$$

which immediately implies (a) and (b). Since $L^2(\mathbb{R}^N)$ is generated by all $D^jT_k\psi$, we also have $\overline{\cup_{j\in\mathbb{Z}}V_j} = L^2(\mathbb{R}^N)$.

Observe also that the assumed semi-orthogonality together with (53) implies

$$V_j = \oplus_{j'<j} D^{j'}\langle\psi\rangle, \;\; \forall j \in \mathbb{Z}. \tag{54}$$

Let $W_j := V_{j+1} \ominus V_j$, $j \in \mathbb{Z}$. Obviously, (54) implies $W_j = D^j\langle\psi\rangle$ for each $j$ and, in particular, $W_0 = \langle\psi\rangle$. Now we see that $f \in \cap_{j\in\mathbb{Z}}V_j$ implies $f \in V_{j+1} = V_j \oplus W_j$, for each $j$, which gives us $f \in W_j^{\perp}$ for all $j$. Thus, $f \perp D^jT_k\psi$, for all $j$ and $k$ and hence, since $\psi$ is a Parseval wavelet, $f = 0$.

It remains to prove (d). First observe that $V_0$ is invariant for all $T_k$ if and only if $V_0^{\perp}$ is invariant for all $T_k^*$. Since $T_k^* = T_{-k}$, for each $k$, we conclude that $V_0$ is shift-invariant if and only if $V_0^{\perp}$ is shift-invariant. Note that

$$f \in V_0^{\perp} \iff f \perp D^jT_k\psi, \;\; \forall j < 0, \forall k \in \mathbb{Z}^N.$$

This, together with the fact that the system $(D^jT_k\psi)_{j\in\mathbb{Z},k\in\mathbb{Z}^N}$ is a Parseval frame for $L^2(\mathbb{R}^N)$, gives us

$$f \in V_0^{\perp} \iff \|f\|^2 = \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^N} |\langle f, D^jT_k\psi\rangle|^2. \tag{55}$$

Consider now arbitrary $f \in V_0^{\perp}$ and $k_0 \in \mathbb{Z}^N$. Since $T_{k_0}$ is unitary, we have

$$\begin{aligned}
\|T_{k_0}f\|^2 &= \|f\|^2 \\
&\overset{(55)}{=} \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^N} |\langle f, D^jT_k\psi\rangle|^2 \\
&= \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^N} |\langle T_{k_0}f, T_{k_0}D^jT_k\psi\rangle|^2 \\
&\overset{(39)}{=} \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^N} |\langle T_{k_0}f, D^jT_{A^jk_0+k}\psi\rangle|^2
\end{aligned}$$

Observe that $A^jk_0 + k \in \mathbb{Z}^N$, for each $j \geq 0$ and for all $k$. Since $(D^jT_k\psi)_{j\in\mathbb{Z},k\in\mathbb{Z}^N}$ is a Parseval frame for $L^2(\mathbb{R}^N)$, this equality implies that all other frame coefficients vanish. In particular, we have $\langle T_{k_0}f, D^jT_k\psi\rangle = 0$ for all $j < 0$ and all $k$ in $\mathbb{Z}^N$. Thus, $T_{k_0}f \in V_0^{\perp}$. $\square$

**Definition 4.3.7.** *A sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^N)$ is said to be a generalized multiresolution analysis (GMRA) if the following conditions are satisfied:*

*(a)* $V_{j+1} = DV_j$, $\forall j \in \mathbb{Z}$;

*(b)* $V_j \subseteq V_{j+1}$, $\forall j \in \mathbb{Z}$;

*(c)* $\cap_{j\in\mathbb{Z}}V_j = \{0\}$, $\overline{\cup_{j\in\mathbb{Z}}V_j} = L^2(\mathbb{R}^N)$;

*(d)* $V_0$*, that is called the core space, is shift-invariant.*

Clearly, this definition is motivated by the preceding theorem: we can now say that each semi-orthogonal Parseval frame generates a GMRA. In fact, much more is true.

**Remark 4.3.8.** Suppose that $(V_j)_{j\in\mathbb{Z}}$ is a GMRA in $L^2(\mathbb{R}^N)$. As in the preceding proof we introduce the orthogonal complements $W_j := V_{j+1} \ominus V_j$, $j \in \mathbb{Z}$. Observe that $V_{j+1} = DV_j$ implies $DW_{j+1} = DW_j$ and, in particular, $W_j = D^j W_0$. Moreover, we also have

$$L^2(\mathbb{R}^N) = \oplus_{j\in\mathbb{Z}} W_j = \oplus_{j\in\mathbb{Z}} D^j W_0. \tag{56}$$

Further, since $V_0$ is shift-invariant, it follows that $V_1 = DV_0$ is shift invariant and then, using Exercise 4.1.13, we conclude that $W_0$ is also shift-invariant. If $W_0$ is a principal shift-invariant space, i.e. if there exists a function $\psi \in W_0$ such that $(T_k)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $W_0$, then (56) tells us that this function is a semi-orthogonal Parseval wavelet. When this is the case, we say that $\psi$ is a wavelet associated with $(V_j)_{j\in\mathbb{Z}}$.

**Definition 4.3.9.** *We say that a GMRA $(V_j)_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R}^N)$ is admissible if $W_0$ is a principal shift-invariant space.*

Observe that, by Theorem 4.2.8, a GMRA $(V_j)_{j\in\mathbb{Z}}$ is admissible if and only if $\dim_{W_0}(\xi) \in \{0,1\}$ a.e. By Remark 4.3.8 semi-orthogonal Parseval wavelets can be constructed from admissible GMRA's. In fact, by putting together Theorem 4.3.6 and Remark 4.3.8 we conclude that all semi-orthogonal Parseval wavelets arise from admissible GMRA's. More precisely, semi-orthogonal Parseval wavelets are Parseval generators of the orthogonal complements $W_0$ of $V_0$ in $V_1$ in admissible GMRA's $(V_j)_{j\in\mathbb{Z}}$.

Two questions now naturally arise. First, how to recognize admissible GMRA's among all GMRA's and, secondly, how to construct admissible GMRA's?. In what follows we provide answers to these questions.

**Definition 4.3.10.** *The dimension function of a Parseval wavelet $\psi \in L^2(\mathbb{R}^N)$ is defined by*

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}^N} \left|\hat{\psi}(B^j(\xi+k))\right|^2. \tag{57}$$

**Remark 4.3.11.** For each $\psi \in L^2(\mathbb{R}^N)$ (not necessarily a Parseval wavelet) we have

$$\begin{aligned}
\|D_\psi\|_{L^1(\mathbb{T}^N)} &= \int_{C^N} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}^N} \left|\hat{\psi}(B^j(\xi+k))\right|^2 d\xi \\
&= \sum_{j=1}^{\infty} \int_{\mathbb{R}^N} \left|\hat{\psi}(B^j\xi)\right|^2 d\xi \\
&= \sum_{j=1}^{\infty} \frac{1}{d^j} \|\hat{\psi}\|^2 \\
&= \frac{1}{d-1} \|\psi\|^2
\end{aligned}$$

which shows that $D_\psi(\xi)$ is finite for a.e. $\xi$.

**Definition 4.3.12.** *For a Parseval wavelet $\psi \in L^2(\mathbb{R}^N)$ and $\xi$ in $\mathbb{R}^N$ we define the sequence of vectors $(v_j(\xi))_{j=1}^{\infty}$ by*

$$v_j(\xi) = (\hat{\psi}(B^j(\xi + k))_{k \in \mathbb{Z}^N}. \tag{58}$$

**Remark 4.3.13.** Observe that Remark 4.3.11 shows that $v_j(\xi) \in \ell^2(\mathbb{Z}^N)$ and $D_\psi(\xi) = \sum_{j=1}^{\infty} \|v_j(\xi)\|^2$ for a.e. $\xi$. We also note that $\|v_j(\xi + l)\| = \|v_j(\xi)\|$ for all $j \in \mathbb{N}$, $\xi \in \mathbb{R}^N$, and $l \in \mathbb{Z}^N$.

**Proposition 4.3.14.** *Let $\psi \in L^2(\mathbb{R}^N)$ be a semi-orthogonal Parseval wavelet. Put $V_0 = \overline{span}\,\{D^j T_k \psi : j < 0, k \in \mathbb{Z}^N\}$. Then*

$$dim_{V_0}(\xi) = dim\,(\overline{span}\{v_j(\xi) : j \in \mathbb{N}\}) \quad a.e. \tag{59}$$

**Proof.** We first claim that

$$V_0 = \overline{\mathrm{span}}\,\{T_k D^{-j}\psi : k \in \mathbb{Z}^N, j \in \mathbb{N}\}. \tag{60}$$

To prove this, first observe that

$$V_0 = \overline{\mathrm{span}}\,\{D^{-j} T_k \psi : j \in \mathbb{N}, k \in \mathbb{Z}^N\} \supseteq \overline{\mathrm{span}}\,\{D^{-j}\psi : j \in \mathbb{N}\}.$$

In particular, we have $V_0 \supseteq \{D^{-j}\psi : j \in \mathbb{N}\}$. Since $V_0$ is shift-invariant, this implies $V_0 \supseteq \mathrm{span}\,\{T_k D^{-j}\psi : k \in \mathbb{Z}^N, j \in \mathbb{N}\}$ and since $V_0$ is closed, from this we obtain $V_0 \supseteq \overline{\mathrm{span}}\,\{T_k D^{-j}\psi : k \in \mathbb{Z}^N, j \in \mathbb{N}\}$.

The opposite inclusion we obtain in the following way:

$$\begin{aligned}
\overline{\mathrm{span}}\,\{T_k D^{-j}\psi : k \in \mathbb{Z}^N, j \in \mathbb{N}\} &\overset{(39)}{=} \overline{\mathrm{span}}\,\{D^{-j} T_{A^{-j}k}\psi : k \in \mathbb{Z}^N, j \in \mathbb{N}\} \\
&\supseteq \overline{\mathrm{span}}\,\{D^{-j} T_{k'}\psi : k' \in \mathbb{Z}^N, j \in \mathbb{N}\} \\
&= V_0.
\end{aligned}$$

Thus, we have proved (60). This in fact means that we can write $V_0 = \oplus_{j=1}^{\infty} \langle D^{-j}\psi \rangle$ (observe that $\langle T_k D^{-j}\psi, T_l D^{-m}\psi \rangle = 0$ for all $k, l, j, m$; this follows from equality (39) and assumed semi-orthogonality of $\psi$). Using Exercise 4.1.15 we now conclude that

$$\mathrm{dim}_{V_0}(\xi) = \mathrm{dim}\,(\overline{\mathrm{span}}\,\{(\widehat{D^{-j}\psi}(\xi + k))_{k \in \mathbb{Z}^N} : j \in \mathbb{N}\}) \quad \text{a.e.}$$

The proof is now completed by observing that

$$(\widehat{D^{-j}\psi}(\xi + k))_{k \in \mathbb{Z}^N} \overset{(40)}{=} d^{\frac{j}{2}} (\hat{\psi}(B^j(\xi + k)))_{k \in \mathbb{Z}^N} = d^{\frac{j}{2}} v_j(\xi), \ \forall j \in \mathbb{N}.$$

$\square$

**Theorem 4.3.15.** *Let $\psi \in L^2(\mathbb{R}^N)$ be a semi-orthogonal Parseval wavelet and $V_0 = \overline{span}\,\{D^j T_k \psi : j < 0, k \in \mathbb{Z}^N\}$. Then*

$$dim_{V_0}(\xi) = D_\psi(\xi), \quad for\ a.e.\ \xi. \tag{61}$$

**Proof.** We claim that

$$v_p(\xi) = \sum_{j=1}^{\infty} \langle v_p(\xi), v_j(\xi)\rangle v_j(\xi), \quad \text{for a.e. } \xi,\ \forall p \in \mathbb{N}. \tag{62}$$

Observe that this equality is all what we need. Namely, Exercise 2.1.26 now implies that $\dim(\overline{\mathrm{span}}\{v_j(\xi) : j \in \mathbb{N}\}) = \sum_{j=1}^{\infty} \|v_j(\xi)\|^2$ a.e. Hence, by the preceding proposition, we have $\dim_{V_0}(\xi) = \sum_{j=1}^{\infty} \|v_j(\xi)\|^2$ a.e. On the other hand, we know form Remark 4.3.13 that $D_\psi(\xi) = \sum_{j=1}^{\infty} \|v_j(\xi)\|^2$ a.e.

Thus, we only need to prove (62) (which is not quite easy).

We start by noticing that

$$\sum_{k\in\mathbb{Z}^n} \hat{\psi}(B^j(\xi+k))\overline{\hat{\psi}(\xi+k)} = 0, \quad \text{a.e.}, \forall j \in \mathbb{N}. \tag{63}$$

To see this, let $V_j = D^j V_0$ and $W_j = V_{j+1} \ominus V_j$, $j \in \mathbb{Z}$. Recall from Theorem 4.3.6 that $W_0 = \langle\psi\rangle$, $W_j = D^j W_0$, for all $j$, and $L^2(\mathbb{R}^N) = \oplus_{j\in\mathbb{Z}} W_j$. Therefore, for each $j$ in $\mathbb{N}$, we have $D^{-j}W_0 \perp W_0$, i.e. $D^{-j}\langle\psi\rangle \perp \langle\psi\rangle$. In particular, we conclude that $D^{-j}\psi \perp \langle\psi\rangle$ and, since $\langle\psi\rangle$ is shift-invariant, this implies $T_k D^{-j}\psi \perp \langle\psi\rangle$, for all $k$ in $\mathbb{Z}^N$. Thus, we have $\langle D^{-j}\psi\rangle \perp \langle\psi\rangle$. By Proposition 4.1.2 we now have $[\widehat{D^{-j}\psi}, \hat{\psi}] = 0$ a.e. It remains to observe that $\widehat{D^{-j}\psi}(\xi) = d^{\frac{j}{2}}\hat{\psi}(B^j\xi)$ for a.e. $\xi$ and for all $j$ in $\mathbb{N}$.

Next we claim that the series

$$\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^N} \hat{\psi}(B^p(\xi+k))\overline{\hat{\psi}(B^j(\xi+k))}\hat{\psi}(B^j\xi), \quad p \in \mathbb{N}, \tag{64}$$

converges absolutely. To see this, first observe that, since $(D^jT_k\psi)_{j\in\mathbb{Z},k\in\mathbb{Z}^N}$ is a Parseval frame for $L^2(\mathbb{R}^N)$ and the spaces $D^j\langle\psi\rangle$, $j \in \mathbb{Z}$, are mutually orthogonal, the sequence $(T_k\psi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi\rangle$. From this we conclude, using Theorem 4.1.7, that $\sigma_\psi = \chi_E$ a.e. for some measurable $Z^N$-periodic set $E$. We now compute:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^N}\left|\hat{\psi}(B^p(\xi+k))\overline{\hat{\psi}(B^j(\xi+k))}\right| &\le \left(\sum_{k\in\mathbb{Z}^N}\left|\hat{\psi}(B^p(\xi+k))\right|^2\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}^N}\left|\hat{\psi}(B^j(\xi+k))\right|^2\right)^{\frac{1}{2}} \\
(\text{ since } B^p\mathbb{Z}^N \subset \mathbb{Z}^N) \quad &\le \left(\sum_{k'\in\mathbb{Z}^N}\left|\hat{\psi}(B^p\xi+k'))\right|^2\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}^N}\left|\hat{\psi}(B^j(\xi+k))\right|^2\right)^{\frac{1}{2}} \\
&= \sigma_\psi(B^p\xi)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}^N}\left|\hat{\psi}(B^j(\xi+k))\right|^2\right)^{\frac{1}{2}} \\
(\text{ since } \sigma_\psi \le 1 \text{ a.e.}) \quad &\le \left(\sum_{k\in\mathbb{Z}^N}\left|\hat{\psi}(B^j(\xi+k))\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

From this we obtain, for each $p \in \mathbb{N}$,

$$
\begin{aligned}
\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^N}\left|\hat\psi(B^p(\xi+k))\overline{\hat\psi(B^j(\xi+k))}\hat\psi(B^j\xi)\right| &\le \sum_{j=1}^{\infty}\left(\sum_{k\in\mathbb{Z}^N}\left|\hat\psi(B^j(\xi+k))\right|^2\right)^{\frac12}\left|\hat\psi(B^j\xi)\right| \\
&\le \left(\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^N}\left|\hat\psi(B^j(\xi+k))\right|^2\right)^{\frac12}\left(\sum_{j=1}^{\infty}\left|\hat\psi(B^j\xi)\right|^2\right)^{\frac12} \\
(\text{ by Theorem 4.3.4 (a)} \quad &\le \left(\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^N}\left|\hat\psi(B^j(\xi+k))\right|^2\right)^{\frac12} \\
&= D_\psi(\xi)^{\frac12} \\
&< \infty \text{ a.e.}
\end{aligned}
$$

Let us denote the sum of series (64) by $G_p(\xi)$. By interchanging the order of summations we obtain

$$G_p(\xi) = \sum_{k\in\mathbb{Z}^N}\hat\psi(B^p(\xi+k))\sum_{j=1}^{\infty}\overline{\hat\psi(B^j(\xi+k))}\hat\psi(B^j\xi). \tag{65}$$

We now observe that we can add to this sum the term that corresponds to $j=0$ since this term is by (63) equal to 0. Thus, we have

$$G_p(\xi) = \sum_{k\in\mathbb{Z}^N}\hat\psi(B^p(\xi+k))\sum_{j=0}^{\infty}\overline{\hat\psi(B^j(\xi+k))}\hat\psi(B^j\xi). \tag{66}$$

Another observation is in order. If $k \in \mathbb{Z}^N/B\mathbb{Z}^N$ the corresponding term is equal to 0 by Theorem 4.3.4 (b). Therefore we need to take into account only the terms in which $k$ is of the form $k = Bk'$ and hence (66) can be rewritten as

$$G_p(\xi) = \sum_{k\in\mathbb{Z}^N}\hat\psi(B^p(\xi+Bk))\sum_{j=0}^{\infty}\overline{\hat\psi(B^j(\xi+Bk))}\hat\psi(B^j\xi). \tag{67}$$

If we now replace $\xi$ with $B\xi$ in (67) and compare the result to (65) written with $p+1$ instead of $p$ we see that

$$G_p(B\xi) = G_{p+1}(\xi).$$

By induction we then obtain

$$G_p(\xi) = G_1(B^{p-1}\xi), \quad \text{a.e., } \forall p \in \mathbb{N}. \tag{68}$$

Finally, we have

$$\begin{aligned}
G_1(\xi) &= \sum_{k\in\mathbb{Z}^N} \hat\psi(B(\xi+k)) \sum_{j=1}^{\infty} \overline{\hat\psi(B^j(\xi+k))}\hat\psi(B^j\xi) \quad (j-1=j' \to j)\\
&= \sum_{k\in\mathbb{Z}^N} \hat\psi(B(\xi+k)) \sum_{j=0}^{\infty} \overline{\hat\psi(B^j(B\xi+Bk))}\hat\psi(B^jB\xi) \ \ \text{(using Theorem 4.3.4 (b))}\\
&= \sum_{k\in\mathbb{Z}^N} \hat\psi(B\xi+k) \sum_{j=0}^{\infty} \overline{\hat\psi(B^j(B\xi+k))}\hat\psi(B^jB\xi)\\
&= \sum_{k\in\mathbb{Z}^N} \hat\psi(B\xi+k)\overline{\hat\psi(B\xi+k)}\hat\psi(B\xi) + \sum_{k\in\mathbb{Z}^N} \hat\psi(B\xi+k) \sum_{j=1}^{\infty} \overline{\hat\psi(B^j(B\xi+k))}\hat\psi(B^jB\xi)\\
&\overset{(63)}{=} \sum_{k\in\mathbb{Z}^N} \hat\psi(B\xi+k)\overline{\hat\psi(B\xi+k)}\hat\psi(B\xi)\\
&= \hat\psi(B\xi)\sigma_\psi(B\xi) \ \ (\text{since } \sigma_\psi = \chi_E)\\
&= \hat\psi(B\xi).
\end{aligned}$$

Using (68), from this we obtain

$$G_p(\xi) = \hat\psi(B^p\xi) \text{ a.e.}$$

which we rewrite explicitly:

$$\hat\psi(B^p\xi) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^N} \hat\psi(B^p(\xi+k))\overline{\hat\psi(B^j(\xi+k))}\hat\psi(B^j\xi), \ \ p\in\mathbb{N}. \tag{69}$$

We now fix $k_0 \in \mathbb{Z}^N$ and rewrite (69) with $\xi + k_0$ instead of $\xi$: on the left hand side we get the $k_0$th component of $v_p(\xi)$, while the right hand side becomes the $k_0$th component of $\sum_{j=1}^{\infty}\langle v_p(\xi), v_j(\xi)\rangle v_j(\xi)$. □

**Theorem 4.3.16.** *A GMRA $(V_j)_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R}^N)$ is admissible if and only if the following two conditions are satisfied:*

$$\mathit{dim}_{V_0}(\xi) < \infty, \quad a.e., \tag{70}$$

$$\sum_{i=0}^{d-1} \mathit{dim}_{V_0}(\xi+\beta_i) - \mathit{dim}_{V_0}(B\xi) \le 1, \quad a.e. \tag{71}$$

**Proof.** Suppose that $(V_j)_{j\in\mathbb{Z}}$ is admissible. This means (see Remark 4.3.8) that there exists a function $\psi \in W_0$ such that $(T_k\psi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi\rangle$ and, consequently, that $\psi$ is a semi-orthogonal Parseval wavelet. Now Remark 4.3.11 and Theorem 4.3.15 imply (70).

Consider now $V_1 = D(V_0)$. Recall from Corollary 4.2.7 that the dimension function of shift-invariant spaces is additive on orthogonal sums: since $V_1 = V_0 \oplus W_0$, this gives us

$$\dim_{V_1} = \dim_{V_0} + \dim_{W_0}.$$

Now (70) allows us to rewrite this equality as

$$\dim_{W_0} = \dim_{V_1} - \dim_{V_0},$$

which, for technical reasons, we write with the argument $B\xi$ instead of $\xi$:

$$\dim_{W_0}(B\xi) = \dim_{V_1}(B\xi) - \dim_{V_0}(B\xi).$$

By Proposition 4.1.8, $\dim_{W_0} = \chi_\Omega$ a.e. for some measurable $\mathbb{Z}^N$-periodic set $\Omega$. Thus, from the preceding equality we obtain

$$\dim_{V_1}(B\xi) - \dim_{V_0}(B\xi) \leq 1, \quad \text{a.e.}$$

Recall now from Corollary 4.2.10 that we have

$$\dim_{D(V_0)}(\xi) = \sum_{i=0}^{d-1} \dim_{V_0}(B^{-1}\xi + \beta_i).$$

Taking into account that $D(V_0) = V_1$ the last equality combined with the preceding inequality gives us precisely (71).

Conversely, suppose that $(V_j)_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R}^N)$ is a GMRA with properties (70) and (71). Arguing precisely as in the first part of the proof (again, the role of (70) should be recognized) we conclude that $\dim_{W_0} \leq 1$ a.e. It is easy to conclude that $\dim_{W_0} = 0$ a.e. is impossible; therefore, there exist a measurable set $\Omega$ such that $\Omega \cap C^N$ has positive measure and $\dim_{W_0} = \chi_\Omega$. Theorem 4.2.8 now implies that there exists a function $\psi \in W_0$ such that the sequence $(T_k\psi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle\psi\rangle$. $\square$

*Concluding remarks.* The material presented in this section is oriented toward theoretical aspects of wavelet theory with the emphasis on the role played by frames in the (generalized) multiresolution technique. Our approach is influenced very much by the work of G. Weiss and his collaborators. However, applications of wavelets is why such reproducing systems are invented and the interested reader is urged to consult Daubechies' book [57].

The MRA concept is present in the theory from the very beginning. GMRA's entered into the theory only after [21] and [101]. Basically, a GMRA $(V_j)_{j\in\mathbb{Z}}$ is called an MRA if the core space $V_0$ is singly generated as a shift-invariant space. In fact, in a coarser sense, one requires more; i.e. that there is a function $\varphi \in V_0$ such that the sequence $(T_k\varphi)_{k\in\mathbb{R}^N}$ is an ONB for $V_0$. Our next section is devoted to MRA's. One should note (which is clearly visible from Corollary 4.4.1) that there is a real "added value" when working with MRA's only in the case $d = 2$.

Theorem 4.3.16 is first proved in [9], but see also [28].

**Exercise 4.3.17.** Let $(c_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} c_n = 0$. Show that the sequence $(T_{c_n})_n$ converges to the identity operator in the strong operator topology of $L^2(\mathbb{R})$.

**Exercise 4.3.18.** ([56]) Show that there does not exist a function $\psi \in L^2(\mathbb{R})$ for which the system $(T_k D^j \psi)_{k,j\in\mathbb{Z}}$ is orthonormal, where $D$ is the dyadic dilation operator on $L^2(\mathbb{R})$. Hint. Consider $\|TD^{-n}\psi - D^{-n}\psi\|$ and use the result of the preceding exercise.

**Exercise 4.3.19.** Let $(X_j)_{j\in\mathbb{Z}}$ be an increasing sequence of closed subspaces of a Hilbert space $H$. For each $j$ in $\mathbb{Z}$ denote by $P_j \in \mathbb{B}(H)$ the orthogonal projection to $X_j$. Show that $\overline{\cup_{j\in\mathbb{Z}}} X_j = H$ if and only if $\lim_{j\to\infty} I$ and $\cap_{j\in\mathbb{Z}} X_j = \{0\}$ if and only if $\lim_{j\to-\infty} 0$, where both limits are in the sense of the strong operator topology.

**Exercise 4.3.20.** ([81], Ex. 12.4) Let $E \subseteq \mathbb{R}^N$ be a measurable set. We say that $E$ is a tiling domain by integer translations if $((E+k))_{k\in\mathbb{Z}^N}$ is, up to a set of measure zero, a partition of $\mathbb{R}^N$. Analogously, given the expansive matrix $A \in M_n(\mathbb{Z})$, we say that $E$ is a tiling domain (by $A$-dilations) if $(A^j(E))_{j\in\mathbb{Z}}$ is, up to a set of measure zero, a partition of $\mathbb{R}^N$.

(a) Prove that if $E$ is a tiling domain by integer translations then the sequence $(e^{2\pi i\langle k,x\rangle})_{k\in\mathbb{Z}^N}$ is an ONB for $L^2(E)$.

(b) Prove: $E$ is a tiling domain by integer translations and by by $A$-dilations if and only if the function $\psi \in L^2(\mathbb{R}^N)$ defined by $\hat{\psi}$ is an orthonormal wavelet with respect to the dilation operator $D = D_A$.

(c) Characterize those measurable sets for which the function $\psi \in L^2(\mathbb{R}^N)$ defined by $\hat{\psi}$ is a Parseval wavelet with respect to the dilation operator $D = D_A$.

**Exercise 4.3.21.** Consider the function $\hat{\varphi} = \chi_{[-\frac{1}{2},\frac{1}{2})}$ and observe that $\varphi(x) = \frac{\sin \pi x}{\pi x}$. Let $V_0 = \langle\varphi\rangle$ and $V_j = D^j V_0$, $j \in \mathbb{Z}$, where $D \in \text{Ł}^2(\mathbb{R})$ is the dyadic dilation operator. Show that $(V_j)_{j\in\mathbb{Z}}$ is a GMRA, in fact an MRA (see the concluding remarks above).

**Exercise 4.3.22.** Let $\varphi = \chi_{[0,1)}$, $V_0 = \langle\varphi\rangle$, and $V_j = D^j V_0$, $j \in \mathbb{Z}$, where $D \in \text{Ł}^2(\mathbb{R})$ is the dyadic dilation operator. Show that $(V_j)_{j\in\mathbb{Z}}$ is an MRA.

## 4.4 Orthonormal wavelets and multiresolution analysis

It is convenient to restate Theorem 4.3.16 that characterizes admissible GMRA's $(V_j)_{j\in\mathbb{Z}}$ using the canonical decomposition of the core space $V_0$.

**Corollary 4.4.1.** *Let $(V_j)_{j\in\mathbb{Z}}$ be a GMRA in $L^2(\mathbb{R}^N)$. Suppose that $V_0 = \oplus_{n=1}^{\infty}\langle\varphi_n\rangle$ where $\sigma_{\varphi_n} = \chi_{\Omega_n}$ a.e. for all $n$ in $\mathbb{N}$ and*

$$\Omega_1 \supseteq \Omega_2 \supseteq \Omega_3 \supseteq \dots,$$

*so that we have*

$$dim_{V_0}(\xi) = \sum_{n=1}^{\infty} \chi_{\Omega_n}(\xi), \quad a.e.$$

*with Then $(V_j)_{j\in\mathbb{Z}}$ is admissible if and only if the following two conditions are satisfied:*

$$\sum_{n=1}^{\infty} \chi_{\Omega_n}(\xi) < \infty, \quad a.e., \tag{72}$$

$$\sum_{i=0}^{d-1}\sum_{n=1}^{\infty} \chi_{\Omega_n}(\xi+\beta_i) - \sum_{n=1}^{\infty} \chi_{\Omega_n}(B\xi) \le 1, \quad a.e. \tag{73}$$

A number of corollaries and comments is now in order.

**Remark 4.4.2.** (a) Condition (72) is equivalent to $|\cap_{n=1}^{\infty}\Omega_n| = 0$, where $|S|$ denotes the Lebesgue measure of the set $S$. Of, course, if $V_0$ is finitely generated as a shift-invariant space (we then tacitly assume that $\Omega_n = \emptyset$, for all $n > M$, for some $M \in \mathbb{N}$), (72) is authomatically fulfilled.

(b) A GMRA admits an orthonormal wavelet if and only if

$$\sum_{i=0}^{d-1}\sum_{n=1}^{\infty} \chi_{\Omega_n}(\xi+\beta_i) - \sum_{n=1}^{\infty} \chi_{\Omega_n}(B\xi) = 1, \quad \text{a.e.}; \tag{74}$$

this follows immediately from the proof of Theorem 4.3.16 and the second assertion of Theorem 4.1.7

**Definition 4.4.3.** *A multiresolution analysis (MRA) is a GMRA $(V_j)_{j\in\mathbb{Z}^N}$ whose core space $V_0$ is singly generated by an orthonormal generator. A function $\varphi \in V_0$ such that $V_0 = \langle\varphi\rangle$ and that the sequence $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is an ONB for $V_0$ is called a scaling function.*

**Remark 4.4.4.** (a) The concept of an MRA preceded that of a GMRA. Observe that for an MRA the left hand side of (73) reduces to $d-1$. Thus, an MRA is admissible if and only $d = 2$. This explains why the MRA concept was that successful in the classical dyadic case on the real line: each MRA with the dyadic dilations is admissible and, moreover, the resulting wavelets are necessarily orthonormal. Obviously, the same is true in $\mathbb{R}^N$ as long as we work with dilation matrices such that $d = 2$.

(b) If, on the other hand, we work with dilation factors $d$ greater than 2, an MRA can not be admissible. However, it should be noted that GMRA's with singly generated core spaces can produce wavelets even when $d > 2$; a necessary and sufficient condition is

$$\sum_{i=0}^{d-1} \chi_\Omega(\xi + \beta_i) - \chi_\Omega(B\xi) \leq 1, \quad \text{a.e.} \tag{75}$$

However, if $d > 2$ and $V_0 = \langle \varphi \rangle$ is the core space of the GMRA under consideration, (75) can be satisfied only if $\varphi$ is only a Parseval and not orthonormal generator of $V_0$. In other words, if $d > 2$ and $\sigma_\varphi = \chi_\Omega$ a.e. a necessary condition for (75) is $|\mathbb{R}^N \setminus \Omega| > 0$.

Here we state a general theorem that characterizes functions which generate GMRAs whose core spaces are principal shift-invariant spaces.

**Theorem 4.4.5.** *Let $\varphi \in L^2(\mathbb{R}^N)$, $V_0 = \langle \varphi \rangle$, and $V_j = D^j V_0$, $j \in \mathbb{Z}$. Then $(V_j)_{j\in\mathbb{Z}}$ is a GMRA such that the sequence $(T_k\varphi)_{k\in\mathbb{Z}^N}$ is a Parseval frame for $\langle \varphi \rangle$ if and only if the following conditions are satisfied:*

*(a) $\sigma_\varphi = \chi_\Omega$ a.e. for some $\mathbb{Z}^N$-periodic set $\Omega$;*

*(b) there exists a measurable $\mathbb{Z}^N$-periodic function $m_0 \in L^2(\mathbb{T}^N)$ such that $\hat{\varphi}(B\xi) = m_0(\xi)\hat{\varphi}(\xi)$ for a.e. $\xi$;*

*(c) $\lim_{j\to\infty} |\hat{\varphi}(B^{-j}\xi) = 1$ for a.e. $\xi$.*

We omit the proof of this theorem. For the proof concerned with scaling functions (i.e. orthonormal gnerators) in the dyadic case on the real line we refer the reader to Theorem 7.5.2 in [84]. One should mention that in applications one prefers to work with scaling functions as smooth as possible. We refer the reader to Sections 12.5 and 12.6 in [81] for more details concerning construction of scaling functions.

The proof if Theorem 4.4.5 in $N$ dimensions for $d = 2$ can be found in [15] (Theorem 3.7). The general proof can be obtained by an easy adaptation of these standard arguments. Here we only mention that the above condition (b) corresponds to the equality $V_1 = D(V_0)$, while (c) reflects the equality $L^2(\mathbb{R}^N) = \overline{\cup_{j\in\mathbb{Z}^N} V_j}$.

We now restrict our discussion to dyadic wavelets on the real line. Observe that in this case our dilation operator $D$ is defined by $Df(x) = \sqrt{2}f(2x)$, $f \in L^2(\mathbb{R})$. Also, note that $\{\beta_0 = 0, \beta_1 = \frac{1}{2}\}$ is a set of $d = 2$ representatives of different cosets in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Here we demonstrate a technique for construction of wavelets from multiresolution analyses. The following theorem describes all wavelets that arise from a given MRA.

**Theorem 4.4.6.** *Suppose that $\varphi$ is a scaling function for an MRA $(V_j)_{j\in\mathbb{Z}}$. Let $m_0$ be a measurable 1-periodic function $m_0 \in L^2(\mathbb{T})$ such that $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$ for a.e. $\xi$ (from Theorem 4.4.5 (b)). Then a function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet associated with $(V_j)_{j\in\mathbb{Z}}$ if and only if $\psi$ is of the form*

$$\hat{\psi}(2\xi) = e^{-2\pi i\xi} s(2\xi)\overline{m_0(\xi + \frac{1}{2})}\hat{\varphi}(\xi) \tag{76}$$

*where $s$ is a unimodular (i.e. $|s(\xi)| = 1$) 1-periodic measurable function.*

**Proof.** Let $V_1 = D(V_0)$ and $V_1 = V_0 \oplus W_0$. It is clear from our preceding considerations that $W_0$ is singly generated as a shift-invariant space and, moreover, that wavelets associated with $(V_j)_{j\in\mathbb{Z}}$ are in fact those functions $\psi \in W_0$ such that the sequence $(T_k\psi)_{k\in\mathbb{Z}}$ is an ONB for $W_0$. It can be proved (see Exercise 4.4.11) that

$$V_1 = \{f \in \mathbb{L}^2(\mathbb{R}) : \hat{f}(2\xi) = t(\xi)\hat{\varphi}(\xi) : t \in L^2(\mathbb{T})\}.$$

In particular, as we already know from Theorem 4.4.5, there exists a function $m_0 \in L^2(\mathbb{T})$ (that is called the low pass filter) such that

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi).$$

Since $(T_k\varphi)_{k\in\mathbb{Z}}$ is an ONB for $V_0$, Theorem 4.1.7 implies that

$$\sigma_\varphi(\xi) = 1 \quad \text{a.e.}$$

It turns out (see Exercise 4.4.11 and Exercise 4.4.12) that this implies

$$|m_0(\xi)|^2 + |m_0(\xi + \frac{1}{2})|^2 = 1 \quad \text{a.e.} \tag{77}$$

Furthermore, we conclude from Proposition 4.1.2 and Exercise 4.4.12 that a function $\psi \in V_1$, where $\hat{\psi}(2\xi) = t(\xi)\hat{\varphi}(\xi)$, with $t \in L^2(\mathbb{T})$, belongs to $W_0$ if and only if

$$t(\xi)\overline{m_0(\xi)} + t(\xi + \frac{1}{2})\overline{m_0(\xi + \frac{1}{2})} = 0 \quad \text{a.e.} \tag{78}$$

Now we see from (78) that $\psi \in W_0$ is equivalent to

$$\left(t(\xi), t(\xi + \frac{1}{2})\right) \perp \left(m_0(\xi), m_0(\xi + \frac{1}{2})\right) \quad \text{a.e.} \tag{79}$$

On the other hand, we also have

$$\left(\overline{m_0(\xi + \frac{1}{2})}, -\overline{m_0(\xi)}\right) \perp \left(m_0(\xi), m_0(\xi + \frac{1}{2})\right) \quad \text{a.e.} \tag{80}$$

Since $\big(m_0(\xi), m_0(\xi + \frac{1}{2})\big)$ is, by (77), a non-trivial vector, we conclude from (79) and (80) that

$$\left(t(\xi), t(\xi + \frac{1}{2})\right) = -\lambda(\xi + \frac{1}{2})\left(\overline{m_0(\xi + \frac{1}{2})}, -\overline{m_0(\xi)}\right) \quad \text{a.e.} \tag{81}$$

where $\lambda$ is an appropriate (necessarily 1-periodic) function. The preceding equality with $\xi$ replaced with $\eta + \frac{1}{2}$ gives us

$$\left(t(\eta + \frac{1}{2}), t(\eta)\right) = -\lambda(\eta + 1)\left(\overline{m_0(\eta)}, -\overline{m_0(\eta + \frac{1}{2})}\right) \quad \text{a.e.} \tag{82}$$

which is equivalent to

$$\left(t(\xi), t(\xi + \frac{1}{2})\right) = \lambda(\xi)\left(\overline{m_0(\xi + \frac{1}{2})}, -\overline{m_0(\xi)}\right) \quad \text{a.e.} \tag{83}$$

From (81) and (83) we now have

$$\lambda(\xi) = -\lambda(\xi + \frac{1}{2}) \quad \text{a.e.} \tag{84}$$

Let

$$s(\xi) = e^{\pi i \xi} \lambda(\frac{\xi}{2}).$$

Then (84) shows us that $s$ is a 1-periodic function and we have

$$\lambda(\xi) = e^{-2\pi i \xi} s(2\xi).$$

Thus, we obtain from (83) that

$$t(\xi) = e^{-2\pi i \xi} s(2\xi) \overline{m_0(\xi + \frac{1}{2})} \quad \text{a.e.}$$

In this way we have proved that a function $\psi$ belongs to $W_0$ if and only if $\psi$ is given by

$$\hat{\psi}(2\xi) = e^{-2\pi i \xi} s(2\xi) \overline{m_0(\xi + \frac{1}{2})} \hat{\varphi}(\xi) \quad \text{a.e.}$$

where $s$ is a 1-periodic function. The proof is now finished by a simple observation that a function $\psi$ is an orthonormal generator of $\langle \psi \rangle$ if and only if the function $s$ is unimodular. $\square$

**Example 4.4.7.** The Haar wavelet from Example 4.3.2 is an MRA wavelet arising from the MRA whose scaling function is $\varphi = \chi_{[0,1)}$. See Exercise 4.3.22. For more details we refer the reader to [84], pp 59-60 or to [117], p 186.

**Example 4.4.8.** The Shannon wavelet from Example 4.3.3 is an MRA wavelet arising from the MRA whose scaling function $\hat{\varphi} = \chi_{[-\frac{1}{2}, \frac{1}{2})}$.

Obviously, this function satisfies the conditions from Theorem 4.4.5; it turns out that $m_0 = \chi_{[-\frac{1}{4}, \frac{1}{4}) + \mathbb{Z}}$. Now we apply (76) from Theorem 4.4.6 to obtain associated wavelets. In particular, we can choose the function $s$ which is defined by $s(\xi) = e^{2\pi i \xi}$ for $\xi \in [-\frac{1}{4}, \frac{1}{4})$ and then extended to a unimodular 1-periodic measurable function. With choice of $s$ (76) gives us the wavelet $\psi$ that is defined by $\hat{\psi} = \chi_{[-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1)}$. Compare Exercise 4.3.21

**Remark 4.4.9.** It is clear from Definition 4.4.3 and Theorem 4.3.15 that the dimension function $D_\psi$ of each MRA wavelet (i.e. a wavelet that arises from an MRA) satisfies $D_\psi = 1$ a.e. In fact, the equation $D_\psi = 1$ a.e. characterizes all orthonormal wavelets which arise from MRA's.

However, there are orthonormal wavelets which are not associated with MRA's which means that such wavelets come from GMRA'a with core spaces generated by at least 2 generators. The simplest non-MRA wavelet is the Journé wavelet which is defined by

$$\hat{\psi} = \chi_S, \;\; S = \left[-\frac{16}{7}, -2\right) \cup \left[-\frac{1}{2}, -\frac{2}{7}\right) \cup \left[\frac{2}{7}, \frac{1}{2}\right) \cup \left[2, \frac{16}{7}\right).$$

It can be seen that in this case $D_\psi$ takes values 1 and 2 on sets of positive measure which indicates, by Theorem 4.3.15, that the underlying GMRA has the core space $V_0$ that is generated as a shift-invariant space with two functions.

One approach to non-MRA wavelets can be based on a study of dimension function of orthonormal wavelets. Here we state without proof the key result from [29] that characterizes dimension functions of orthonormal wavelets.

**Theorem 4.4.10.** *Let $D : \mathbb{R} \to \mathbb{N} \cup \{0\}$ be a 1-periodic function that is integrable on the unit interval $[-\frac{1}{2}, -\frac{1}{2}]$. Then $D$ is a dimension function of an orthonormal wavelet if and only if the following conditions are satisfied:*

*(a)* $\liminf_{j\to\infty} D(2^{-j}\xi) \geq 1$ *a.e.*

*(b)* $D(\xi) + D(\xi + \frac{1}{2}) - D(2\xi) = 1$ *a.e.*

*(c)* $\sum_{k\in\mathbb{Z}} \chi_\Delta(\xi + k) \geq D(\xi)$ *a.e. where* $\Delta = \{\xi \in \mathbb{R} : D(2^{-j}\xi) \geq 1,\ j \in \mathbb{N} \cup \{0\}\}$.

It should be noted that the original result from [29] is proved in $N$ dimensions and for arbitrary expansive matrices with integer coefficients. Another remarkable fact obtained in [29] is that for each natural number $M$ there exists a dimension function whose essential maximum is equal to $M$. In addition to that, the first concrete example of an essentially unbounded dimension function is provided.

Related results can be found in [12], [7] and [8]. In [7] more properties od dimension functions are obtained as well as a method for construction of dimension functions. In [12] (see also [7]) the first concrete example of an orthonormal wavelet with essentially unbounded dimension function is constructed.

At the end we mention that there are also many examples of wavelets on the real line with dilation factors other than 2. Here we mention a series of examples from [16] in $L^2(\mathbb{R})$ with the dilation operator $D$ defined by $Df(x) = \sqrt{d}f(dx)$, $d \in \mathbb{N}$, $d \geq 2$.

Let $d \geq 3$. Take any $k$ such that $1 \leq k \leq d-1$ and the set

$$W = \left[\frac{d(k-d)}{d^2-1}, \frac{k-d}{d^2-1}\right) \bigcup \left[\frac{k}{d^2-1}, \frac{dk-1}{d^2-1}\right) \bigcup \left[\frac{d(dk-1)}{d^2-1}, \frac{d^2k}{d^2-1}\right)$$

Then the function $\psi$ defined by

$$\hat{\psi} = \chi_W$$

is an orthonormal wavelet. It should be mentioned that for $k = 1$ this reduces to the series of examples first presented in [56].

In fact, [16] provides a method for construction (in $N$ dimensions, for arbitrary expansive matrix with integer coefficients) of all sets $S \subset \mathbb{R}^N$ such that the function $\varphi$ defined by $\hat{\varphi} = \chi_S$ is a Parseval generator of a singly generated admissible GMRA (i.e. such that the sequence $(V_j)_{j\in\mathbb{Z}}$, where $V_j = D^jV_0$, $j \in \mathbb{Z}$, $V_0 = \langle\varphi\rangle$ is an admissible GMRA). In a sense, this result with its consequences complements those from [9]. [16] also contains more interesting examples of so called WSF wavelets (i.e. those of the form $\hat{\psi} = \chi_W$, where $W$ is a measurable set) on the real line for all dilation factors $d \in \mathbb{R}$, $d > 2$.

*Concluding remarks.* For the general theory of wavelets and many more aspects we refer the reader to [57], [84], and [117].

**Exercise 4.4.11.** Suppose that $(V_j)_{j\in\mathbb{Z}}$ is a dyadic MRA in $L^2(\mathbb{R})$ with a scaling function $\varphi$. Show that

$$V_1 = \{f \in L^2(\mathbb{R}) : \hat{f}(2\xi) = t(\xi)\hat{\varphi}(\xi) : t \in L^2(\mathbb{T})\}$$

**Exercise 4.4.12.** Suppose that $(V_j)_{j\in\mathbb{Z}}$ is a dyadic MRA in $L^2(\mathbb{R})$ with a scaling function $\varphi$. Let $m_0 \in Ł^2(\mathbb{T})$ be the low pass filter, i.e. the function with the property

$$\hat{\varphi}(\xi) = m_0(\xi)\hat{\varphi}(\xi).$$

Show that

$$|m_0(\xi)|^2 + |m_0(\xi + \frac{1}{2})|^2 = 1 \quad \text{a.e.}$$

Furthermore, if $f_1$ and $f_2$ are any two functions from $V_1$ such that $\hat{f}_i(2\xi) = t_i(\xi)\hat{\varphi}(\xi)$ with $t_i \in L^2(\mathbb{T})$, $i = 1, 2$, show that

$$[\hat{f}_1, \hat{f}_2](2\xi) = t_1(\xi)\overline{t_2(\xi)} + t_1(\xi + \frac{1}{2})\overline{t_2(\xi + \frac{1}{2})} \quad \text{a.e.}$$

## Appendix

**A.1 Tonelli's and Fubini's theorem.** Let $\mu$ and $\nu$ be positive Borel measures on $\mathbb{R}^N$ and let $\mu \times \nu$ be their product measure on $\mathbb{R}^{2N}$.

**Theorem 4.4.13.** *(Tonelli.) If $f \geq 0$ is a measurable function on $\mathbb{R}^{2N}$ then*

$$\begin{aligned}\int_{\mathbb{R}^{2N}} f(x,\xi)d(\mu\times\nu) &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} f(x,\xi)d\mu(x)\right)d\nu(\xi)\\ &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} f(x,\xi)d\nu(\xi)\right)d\mu(x) \qquad (85)\end{aligned}$$

*In particular, either these three integrals are finite and equal or they are all infinite.*

**Theorem 4.4.14.** *(Fubini.) If $f \in L^1(\mathbb{R}^{2N}, \mu\times\nu)$, then (85) holds. Furthermore, for almost all $\xi \in \mathbb{R}^N$ the function $x \mapsto f(x,\xi)$ is in $L^1(\mathbb{R}^N,\mu)$ and for almost all all $x \in \mathbb{R}^N$ the function $\xi \mapsto f(x,\xi)$ is in $L^1(\mathbb{R}^N,\nu)$. If $\varphi$ and $\psi$ are defined by*

$$\varphi(x) = \int_{\mathbb{R}^N} f(x,\xi)d\nu(\xi), \quad \psi(\xi) = \int_{\mathbb{R}^N} f(x,\xi)d\mu(x),$$

*then $\varphi \in L^1(\mathbb{R}^N,\mu)$ and $\psi \in L^1(\mathbb{R}^N,\nu)$.*

In applications one chooses an appropriate order of integration, verifies that the iterated integral is finite and then applies Fubini's theorem.

In these notes we use only Lebesgue measure $dx$ and the discrete (counting) measure $\nu = \sum_{k\in\mathbb{Z}^N}\delta_k$. The integration with respect to $\nu$ is just the summation: $\int_{\mathbb{R}^N} f(x)d\nu(x) = \sum_{k\in\mathbb{R}^N} f(k)$. Therefore Fubini's theorem implies the following special cases regarding the interchange of sums and integrals.

**Corollary 4.4.15.** *Let $(f_k)_{k\in\mathbb{Z}^N}$ be a sequence in $L^1(\mathbb{R}^N)$ such that $\sum_{k\in\mathbb{Z}^N}\|f_k\|_1 < \infty$. Then*

$$\int_{\mathbb{R}^N}\left(\sum_{k\in\mathbb{Z}^N} f_k(x)\right)dx = \sum_{k\in\mathbb{Z}^N}\int_{\mathbb{R}^N} f_k(x)dx.$$

**Corollary 4.4.16.** *Let $(c_{kn})_{k,n\in\mathbb{Z}^N}$ be a sequence of scalars such that $\sum_{k,n\in\mathbb{Z}^N}|c_{kn}| < \infty$. Then*

$$\sum_{(k,n)\in\mathbb{Z}^{2N}} c_{kn} = \sum_{k\in\mathbb{Z}^N}\left(\sum_{n\in\mathbb{Z}^N} c_{kn}\right) = \sum_{n\in\mathbb{Z}^N}\left(\sum_{k\in\mathbb{Z}^N} c_{kn}\right).$$

**A.2 Periodization.** The following periodization trick has numerous applications.

**Corollary 4.4.17.** *If $f \in L^1(\mathbb{R}^N)$ then for all $a > 0$,*

$$\int_{\mathbb{R}^N} f(x)dx = \int_{[0,a)^N}\left(\sum_{k\in\mathbb{Z}^N} f(x+ka)\right)dx.$$

**Proof.** The translated cubes $ka + [0,a)^N$ form a partition of $\mathbb{R}^N$. Thus

$$\int_{\mathbb{R}^N} f(x)dx = \sum_{k\in\mathbb{Z}^N} \int_{ka+[0,a)^N} f(x)dx = \int_{[0,a)^N} \left( \sum_{k\in\mathbb{Z}^N} f(x+ka) \right) dx.$$

Since $f \in L^1(\mathbb{R}^N)$, the sum and the integral can be interchanged by Fubini's theorem. □

**A.3 Double sums revisited.** The unconditional convergence is good enough.

**Lemma 4.4.18.** *Suppose that $x_{kn}$, $k, n \in \mathbb{Z}^N$, and $x$ are vectors in a Banach space $X$ such that $\sum_{(k,n)\in\mathbb{Z}^{2N}} x_{kn}$ converges unconditionally to $x$. Then the partial sum $s_{k,N} = \sum_{|n|\leq N} x_{kn}$ converges to some element $y_k \in B$ for each $k \in \mathbb{Z}^N$, and $x = \sum_{k\in\mathbb{Z}^N} y_k$ with unconditional convergence.*

*Likewise, $\sum_{|k|\leq K} x_{kn}$ converges to some element $z_n \in B$ for each $n \in \mathbb{Z}^N$, and $x = \sum_{n\in\mathbb{Z}^N} z_n$ with unconditional convergence.*

*Thus, the order of summation can be interchanged in the double sum.*

**A.4 Integration by parts.** We will make use of the following formula for integration by parts.

**Lemma 4.4.19.** *Suppose that $f, g \in L^2(\mathbb{R})$ are such that $f'g, g'f \in L^1(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} f'(x)g(x)dx = -\int_{-\infty}^{\infty} f(x)g'(x)dx.$$

**A.5 The uniqueness theorem.** For $f \in L^1(\mathbb{T})$ we define the Fourier coefficients by

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx, \;\; n \in \mathbb{Z}.$$

Note that the sequence $(e^{2\pi inx})_{n\in\mathbb{Z}}$ is an ONB for the Hilbert space $L^2(\mathbb{T})$ that is contained in $L^1(\mathbb{T})$; thus, we have

$$f \in L^2(\mathbb{T}), \;\; \hat{f}(n) = 0, \forall n \in \mathbb{Z}, \implies f = 0.$$

Although the trigonometric system is not a basis for $L^1(\mathbb{T})$, the Fourier transform $f \mapsto (\hat{f}(n))_{n\in\mathbb{Z}}$ is injective on $L^1(\mathbb{T})$.

**Theorem 4.4.20.** *If $f \in L^1(\mathbb{T})$ is such that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ a.e.*

# 5 Gabor frames

## 5.1 The short-time Fourier transform

Given $f \in L^2(\mathbb{R})$, the knowledge of the values $f(x)$ for all $x \in \mathbb{R}$ determines, theoretically, all properties of $f$ and also of $\hat{f}$, because the Fourier transform is a unitary operator on $L^2(\mathbb{R})$. However, it is very difficult to obtain properties of $\hat{f}$ by looking only at $f$. In time-frequency analysis we study representations that combine the features of both $f$ and $\hat{f}$. (Recall that in signal analysis $f(x)$ describes temporal behavior, and $\hat{f}(\xi)$ describes the frequency behavior; accordingly, we say that $x$ is time variable, while $\xi$ is frequency variable.)

The ideal time-frequency representation would provide direct information about the frequencies $\xi$ occuring at any given time $x$. In other words, one wants to describe something that can be called instantaneous frequency spectrum. However, it is evident already from the definition of the Fourier transform that in order to obtain $\hat{f}(\xi)$ we need to know all values $f(x)$. But there is a deeper obstruction to the concept of instantaneous frequency: a collection of inequalities that involve both $f$ and $\hat{f}$ and are called uncertainty principles. In a very row form the uncertainty principle states that a function $f$ and its Fourier transform $\hat{f}$ cannot be supported on arbitrarily small sets.

We shall state and prove later in this chapter the Heisenberg-Pauli-Weil uncertainty principle and also the Balian-Low theorem. Here we state without proof the uncertainty principle of Donoho and Stark.

**Definition 5.1.1.** *A function $f \in L^2(\mathbb{R})$ is $\epsilon$-concentrated on a measurable set $T \subseteq \mathbb{R}$ if*

$$\left( \int_{T^c} |f(x)|^2 dx \right)^{\frac{1}{2}} \leq \epsilon \|f\|.$$

If $\epsilon < \frac{1}{2}$ then most of $f$ is concentrated on $T$. If $\epsilon = 0$ then $f$ is essentially supported in $T$.

**Theorem 5.1.2.** *([62]) Suppose that $f \in L^2(\mathbb{R})$, $f \neq 0$, is $\epsilon_T$-concentrated on $T \subseteq \mathbb{R}$ and $\hat{f}$ is $\epsilon_\Omega$-concentrated on $\Omega \subseteq \mathbb{R}$. Then*

$$|T| \cdot |\Omega| \geq (1 - \epsilon_T - \epsilon_\Omega)^2.$$

**Corollary 5.1.3.** *Let $f \in L^2(\mathbb{R})$. If supp$(f) \subseteq T$ and supp$(\hat{f}) \subseteq \Omega$, then*

$$|T| \cdot |\Omega| \geq 1.$$

Suppose now we want to determine the instantaneous frequency spectrum of $f$ at $x$. In order to do that, we need to record $f$ at least over a short period $[x - \delta, x]$. This can be done by considering the function $f_{x,\delta} := f \cdot g_{x,\delta}$, where $g_{x,\delta}$ is some window-function supported on $[x - \delta, x]$, e.g. $g_{x,\delta} = \chi_{[x-\delta,x]}$. Then we take the Fourier transform $\widehat{f_{x,\delta}}$ of $f_{x,\delta}$ and we interpret the support of $\widehat{f_{x,\delta}}$ as the local frequency spectrum. But by the uncertainty principle the support of $\widehat{f_{x,\delta}}$ cannot be small. Moreover, by Corollary 5.1.3, as $\delta \to \infty$, the Lebesgue measure of the support of $\widehat{f_{x,\delta}}$ will tend to infinity.

Despite this fundamental obstacle there are still some useful forms of time-frequency analysis. Here we briefly describe the short-time Fourier transform. The idea behind is implicit in the preceding discussion.

Before we introduce the formal definition, we need to recall fundamental operators and their basic properties. For $x, \xi \in \mathbb{R}$ and $a > 0$ we define

$$T_x f(t) = f(t-x) \text{ translation by } x, \tag{1}$$

$$M_\xi f(t) = e^{2\pi i \xi t} f(t) \text{ modulation by } \xi, \tag{2}$$

$$D_a f(t) = \sqrt{a} f(at) \text{ dilation by } a. \tag{3}$$

Recall that the Fourier transforms of these basic operators are given by

$$\widehat{T_x f} = M_{-x}\hat{f}, \tag{4}$$

$$\widehat{M_\xi f} = T_\xi \hat{f}, \tag{5}$$

$$\widehat{D_a f} = D_{\frac{1}{a}} \hat{f}. \tag{6}$$

Formula (5) explains why modulations are also called frequency shifts. We also have the following commutation relations:

$$T_x M_\xi f(t) = e^{2\pi i \xi (t-x)} f(t-x), \tag{7}$$

$$M_\xi T_x f(t) = e^{2\pi i \xi t} f(t-x), \tag{8}$$

$$M_\xi T_x = e^{2\pi i \xi x} T_x M_\xi, \tag{9}$$

$$T_x D_a f(t) = \sqrt{a} f(at - ax), \tag{10}$$

$$D_a T_x f(t) = \sqrt{a} f(at - x), \tag{11}$$

$$T_x D_a = D_a T_{ax}, \tag{12}$$

$$D_a M_\xi f(t) = \sqrt{a} e^{2\pi i \xi a t} f(at), \tag{13}$$

$$M_\xi D_a f(t) = \sqrt{a} e^{2\pi i \xi t} f(at), \tag{14}$$

$$D_a M_\xi = M_{a\xi} D_a, \tag{15}$$

**Definition 5.1.4.** *Fix a function $g \in L^2(\mathbb{R})$ (called the window function). Then the short time Fourier transform of a function $f$ with respect to $g$ is defined as*

$$V_g f(x,\omega) = \int_{-\infty}^{\infty} f(t)\overline{g(t-x)} e^{-2\pi i t \omega} dt, \;\; x, \omega \in \mathbb{R}. \tag{16}$$

Suppose for the moment that $g$ is compactly supported with its support centered at the origin (for example, one may take $g = \chi_{[-\frac{1}{2},\frac{1}{2}]}$). Then $V_g f(x,\cdot)$ is the Fourier transform of a segment of $f$ centered in a neighborhood of $x$. As $x$ varies, the window slides along the $x$-axis. For this reason the short time Fourier transform is sometimes called the sliding Fourier transform. With some reserves, $V_g(f,\omega)$ can be thought of as a measure for the amplitude of the frequency band near $\omega$ at time $x$. To avoid artificial discontinuities one usually chooses a smooth cut-off function as a window $g$.

The short time Fourier transform $V_g f$ is linear in $f$ and conjugate linear in $g$. Our first lemma provides more properties of $V_g f$ that can be obtained directly from the definition.

**Lemma 5.1.5.** *Let $f, g \in L^2(\mathbb{R})$ Then $V_g f$ is uniformly continuous on $\mathbb{R}^2$ and*

$$V_g f(x,\omega) = \widehat{fT_x\overline{g}}(\omega) = \langle f, M_\omega T_x g\rangle = \langle \hat{f}, T_\omega M_{-x}\hat{g}\rangle. \tag{17}$$

**Proof.** The uniform continuity of $V_g f$ follows from the facts

$$\lim_{x\to 0}\|T_x f - f\| = 0, \ \ \forall f \in L^2(\mathbb{R}) \tag{18}$$

and

$$\lim_{x\to 0}\|M_\omega f - f\| = \lim_{x\to 0}\|T_\omega \hat{f} - \hat{f}\| = 0, \ \ \forall f \in L^2(\mathbb{R}). \tag{19}$$

The equalities (17) follow directly from (16), (4), and (5). $\square$

**Theorem 5.1.6.** *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then $V_{g_j} f_j \in L^2(\mathbb{R}^2)$ for $j = 1, 2$ and*

$$\langle V_{g_1} f_1, V_{g_2} f_2\rangle = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}. \tag{20}$$

**Proof.** We first assume that the window functions $g_j$ are continuous compactly supported so that $f_j T_x\overline{g_j} \in L^2(\mathbb{R})$ for all $x$ in $\mathbb{R}$. Therefore

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_{g_1} f_1(x,\omega)\overline{V_{g_2} f_2(x,\omega)}d\omega dx &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\widehat{f_1 T_x\overline{g_1}}(\omega)\overline{\widehat{f_2 T_x\overline{g_2}}}(\omega)d\omega\right)dx \\
&= \int_{-\infty}^{\infty}\left\langle \widehat{f_1 T_x\overline{g_1}}, \widehat{f_2 T_x\overline{g_2}}\right\rangle dx \\
&= \int_{-\infty}^{\infty}\langle f_1 T_x\overline{g_1}, f_2 T_x\overline{g_2}\rangle\, dx \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f_1(t)\overline{f_2(t)g_1(t-x)}g_2(t-x)dt\right)dx
\end{aligned}$$

Here we have $f_1\overline{f_2} \in L^1(\mathbb{R}, dx)$ and $\overline{g_1}g_2 \in L^1(\mathbb{R}, dt)$, therefore Fubini's theorem allows us to interchange the order of integration. Continuing the preceding computation we now obtain

$$\langle V_{g_1} f_1, V_{g_2} f_2\rangle = \int_{-\infty}^{\infty} f_1(t)\overline{f_2(t)}\left(\int_{-\infty}^{\infty}\overline{g_1(t-x)}g_2(t-x)dx\right)dt = \langle f_1, f_2\rangle\langle g_2, g_1\rangle.$$

The extension to general $g_j$ is done by a standard density argument. (Here one should observe that there is nothing special in using continuous functions in the above computations; we could also work with $g_1, g_2 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.)

With $g_1$ continuous and compactly supported fixed, the mapping $g_2 \mapsto \langle V_{g_1} f_1, V_{g_2} f_2\rangle$ is a linear functional that coincides with $\langle f_1, f_2\rangle\langle g_2, g_1\rangle$ on the subspace consisting of all continuous compactly supported functions which is dense in $L^2(\mathbb{R})$. It is therefore bounded and extends to all $g \in L^2(\mathbb{R})$.

In the same way, for arbitrary $f_1, f_2, g_2 \in L^2(\mathbb{R})$, the conjugate linear functional $g_1 \mapsto \langle V_{g_1} f_1, V_{g_2} f_2\rangle$ coincides with $\langle f_1, f_2\rangle\langle g_2, g_1\rangle$ on the subspace consisting of all continuous compactly supported functions and extends to all of $L^2(\mathbb{R})$. $\square$

**Corollary 5.1.7.** *If $f, g \in L^2(\mathbb{R})$ then $\|V_g f\| = \|f\| \cdot \|g\|$. In particular, if $\|g\| = 1$ then $\|V_g f\| = \|f\|$ for all $f \in L^2(\mathbb{R})$.*

If $\|g\| = 1$ we see from the preceding corollary that $f$ is completely determined by $V_g f$ since $f \mapsto V_g f$ is an isometry and hence an injection. In particular, if $V_g f = 0$, i.e. if $V_g f(x, \omega) = 0$ for all $x, \omega$ (or, equivalently, if $\langle f, M_\omega T_x g\rangle = 0$, $\forall x, \omega$), then $f = 0$. In other words, we have $\overline{\operatorname{span}}\{M_\omega T_x g : x, \omega \in \mathbb{R}\} = L^2(\mathbb{R})$. However, there is still a question of how $f$ can be reconstructed from $V_g f$.

Suppose now that $F \in L^2(\mathbb{R}^2)$ and $g, h \in L^2(\mathbb{R})$ are given. We know from Theorem 5.1.6 that, for each $k \in L^2(\mathbb{R})$, the function $V_h k$ also belongs to $L^2(\mathbb{R}^2)$. Thus, the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,\omega)\overline{V_h k(x,\omega)}dxd\omega = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,\omega)\langle M_\omega T_x h, k\rangle dxd\omega$$

converges. Let

$$l(k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,\omega)\langle M_\omega T_x h, k\rangle dxd\omega. \tag{21}$$

Clearly, $l$ is a well defined conjugate linear functional on $L^2(\mathbb{R})$. We claim that $l$ is bounded. Indeed, applying the Cauchy-Schwarz inequality we obtain

$$|l(k)| \leq \|F\|\|V_k h\| \overset{(20)}{\leq} \|F\|\|h\|\|k\|, \;\; \forall h \in L^2(\mathbb{R}). \tag{22}$$

This implies that $l$ defines a unique element $\tilde{f} \in L^2(\mathbb{R})$ for which we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,\omega)\langle M_\omega T_x h, k\rangle dxd\omega = \langle \tilde{f}, k\rangle, \;\; \forall k \in L^2(\mathbb{R}). \tag{23}$$

In this (weak) sense we now write

$$\tilde{f} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,\omega) M_\omega T_x h \, dxd\omega. \tag{24}$$

**Theorem 5.1.8.** *(Inversion formula for the short time Fourier transform.) Suppose that $g, h \in L^2(\mathbb{R})$ are such that $\langle g, h\rangle \neq 0$. Then we have for each $f \in L^2(\mathbb{R})$*

$$f = \frac{1}{\langle h, g\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_g f(x,\omega) M_\omega T_x h \, dxd\omega. \tag{25}$$

**Proof.** Recall from Theorem 5.1.6 that $V_g f \in L^2(\mathbb{R}^2)$ for all $f \in L^2(\mathbb{R})$. By the preceding considerations the integral

$$\tilde{f} = \frac{1}{\langle h, g\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_g f(x,\omega) M_\omega T_x h \, dxd\omega$$

is well defined in the weak sense for all $f \in L^2(\mathbb{R})$. This means, by (23), that we have for all $k$ in $L^2(\mathbb{R})$

$$\begin{aligned}
\langle \tilde{f}, k\rangle &= \frac{1}{\langle h, g\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_g f(x,\omega)\langle M_\omega T_x h, k\rangle\, dx d\omega \\
&\overset{(17)}{=} \frac{1}{\langle h, g\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_g f(x,\omega)\overline{V_h k(x,\omega)} dx d\omega \\
&= \frac{1}{\langle h, g\rangle}\langle V_g f, V_h k\rangle \\
&\overset{(20)}{=} \frac{1}{\langle h, g\rangle}\langle f, k\rangle\overline{\langle g, h\rangle} \\
&= \langle f, k\rangle.
\end{aligned}$$

Thus, $\tilde{f} = f$. □

At the end we mention that there is also a strong version of the inversion formula for the short time Fourier transform which uses a nested sequence of compact sets $K_n \subseteq \mathbb{R}$ which exhaust $\mathbb{R}$. For the details we refer the reader to Theorem 3.2.4 in [73].

*Concluding remarks.* The material in this section is adapted from [73]. There the interested reader will find a much more elaborated discussion in $d$ dimensions and many more results. Here we restricted ourselves only to the basics to provide a motivation for the introduction of Gabor systems.

**Exercise 5.1.9.** Verify formulae (4) - (15).

**Exercise 5.1.10.** Prove formulae (18) and (19).

**Exercise 5.1.11.** Show that $g(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$ is infinitely differentiable function supported in $[0,1]$.

## 5.2 Basic properties of Gabor systems

The short time Fourier transform provides us with a "continuous expansion" of $f \in L^2(\mathbb{R})$ with respect to the uncountable system of functions $M_\omega T_x g,\ x, \omega \in \mathbb{R}$. However, since $L^2(\mathbb{R})$ is a separable Hilbert space, a series expansion with respect to a countable subset of time-frequency shifts should suffice to represent $f$. The first attempt towards a discrete representation of $f$ would be to replace the integral by a sum over a sufficiently dense lattice, writing $f$ as

$$f = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, M_{mb} T_{na} g \rangle M_{mb} T_{na} h \tag{26}$$

for some suitable window functions $g$ and $h$ from $L^2(\mathbb{R})$ and lattice parameters $a, b > 0$. This motivates the following definition.

**Definition 5.2.1.** *A Gabor system is a sequence in $L^2(\mathbb{R})$ of the form*

$$G(g, a, b) = (M_{mb} T_{na} g)_{m,n\in\mathbb{Z}} \tag{27}$$

*where $g \in L^2(\mathbb{R})$ and $a, b$ are fixed. We call $g$ the generator or the atom of the system (sometimes also the mother wavelet) and refer to $a, b$ as to lattice parameters. The sequence $G(g, a, b)$ is said to be a Gabor frame if $(M_{mb} T_{na} g)_{m,n\in\mathbb{Z}}$ makes up a frame for $L^2(\mathbb{R})$. Frames of this type are also called the Weyl-Heisenberg frames.*

More generally, an "irregular" Gabor system is a system of the form $G(g, \Lambda) = (M_\beta T_\alpha g)_{\alpha,\beta\in\Lambda}$, where $\Lambda$ is an arbitrary countable set of points in $\mathbb{R}^2$. Lattice Gabor systems have many nice properties and applications and are much easier to study than irregular Gabor systems, so here we focus on lattice systems.

**Remark 5.2.2.** Observe that the operators $M_{mb}$ and $T_{na}$ do not commute, so one can also consider systems of the form $(T_{na} M_{mb} g)_{m,n\in\mathbb{Z}}$; however we see from (9) that

$$|\langle f, M_{mb} T_{na} g \rangle|^2 = |\langle f, T_{na} M_{mb} g \rangle|^2, \ \ \forall m, n \in \mathbb{Z}.$$

Thus, $(M_{mb} T_{na} g)_{m,n\in\mathbb{Z}}$ is a frame if and only if $(T_{na} M_{mb} g)_{m,n\in\mathbb{Z}}$ is a frame.

The product $ab$ of the lattice parameters appears in many calculations involving Gabor systems. It turns out that the product $ab$ is important, rather than the individual values of $a$ and $b$. This is made visible in the next lemma.

**Lemma 5.2.3.** *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. Then, given $r > 0$, $G(g, a, b)$ is a frame for $L^2(\mathbb{R})$ if and only if $G(D_r g, \frac{a}{r}, br)$ is a frame for $L^2(\mathbb{R})$.*

**Proof.** Since $D_r$ is a unitary operator, $(M_{mb} T_{na} g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ if and only if $(D_r M_{mb} T_{na} g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ (with the same frame bounds). Now,

$$D_r M_{mb} T_{na} g \overset{(15)}{=} M_{mbr} D_r T_{an} g \overset{(12)}{=} M_{mbr} T_{\frac{a}{r} n} D_r g.$$

$\square$

**Remark 5.2.4.** Whenever $G(g,a,b)$ is a Bessel sequence we have a well defined and bounded analysis operator

$$U : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}\times\mathbb{Z}),\;\; Uf = (\langle f, M_{mb}T_{na}g\rangle)_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}.$$

We shall write $U_g^{a,b}$ when it is necessary to emphasize the dependence on $g, a, b$.

**Lemma 5.2.5.** *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. Suppose that $G(g,a,b)$ is a Bessel sequence. Then the corresponding frame operator $U^*U$ commutes with all $M_{mb}T_{na}$, $m, n \in \mathbb{Z}$.*

**Proof.** Let $f \in L^2(\mathbb{R})$ and $m, n \in \mathbb{Z}$. Then

$$\begin{aligned}
U^*UM_{mb}T_{na}f &= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\langle M_{mb}T_{na}f, M_{lb}T_{ka}g\rangle M_{lb}T_{ka}g\\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\langle f, T_{-na}M_{(l-m)b}T_{ka}g\rangle M_{lb}T_{ka}g\\
&\overset{(9)}{=} \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\langle f, e^{2\pi ina(l-m)b}M_{(l-m)b}T_{-na}T_{ka}g\rangle M_{lb}T_{ka}g\\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\langle f, e^{2\pi ina(l-m)b}M_{(l-m)b}T_{(k-n)a}g\rangle M_{lb}T_{ka}g
\end{aligned}$$

We now change indices ($k \to k' + n$, $l \to l' + m$) and continue our computation:

$$\begin{aligned}
U^*UM_{mb}T_{na}f &= \sum_{k'\in\mathbb{Z}}\sum_{l'\in\mathbb{Z}}\langle f, e^{2\pi inal'b}M_{l'b}T_{k'a}g\rangle M_{(l'+m)b}T_{(k'+n)a}g\\
&\overset{(9)}{=} \sum_{k'\in\mathbb{Z}}\sum_{l'\in\mathbb{Z}}\langle f, e^{2\pi inal'b}M_{l'b}T_{k'a}g\rangle e^{2\pi inal'b}M_{mb}T_{na}M_{l'b}T_{k'a}g\\
&= M_{mb}T_{na}U^*Uf.
\end{aligned}$$

□

**Proposition 5.2.6.** *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. Suppose that $G(g,a,b)$ is a frame. Then its canonical dual is also a Gabor frame $G((U^*U)^{-1}g, a, b)$. Moreover, the associated Parseval frame is $G((U^*U)^{-\frac{1}{2}}g, a, b)$.*

**Proof.** We know that the canonical dual of $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is $((U^*U)^{-1}M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$. However, $(U^*U)^{-1}$ commutes with all $M_{mb}T_{na}$, $m, n \in \mathbb{Z}$, since $U^*U$ does. For the second assertion recall that the associated Parseval frame is $((U^*U)^{-\frac{1}{2}}M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ and observe that $(U^*U)^{-\frac{1}{2}}$ also commutes with all $M_{mb}T_{na}$, $m, n \in \mathbb{Z}$. □

Gabor systems are named after the Nobel prize winner Dennis Gabor. In his paper [68] Gabor proposed using the system $G(\phi, 1, 1)$ generated by the Gaussian function $\phi(x) = e^{-\pi x^2}$. Gabor conjectured (it turned out incorrectly) that every function $f$ in $L^2(\mathbb{R})$ could be represented in the form

$$f = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_{mn}(f)M_mT_n\phi \tag{28}$$

for some scalars $c_{mn}(f)$ depending on $f$.

Von Neumann ([115]) had earlier claimed, without proof, that $G(\phi, 1, 1)$ is fundamental in $L^2(\mathbb{R})$. Von Neumann's claim was proved in 1970's ([22], [102]). However, the fact that $G(\phi, 1, 1)$ is fundamental does not imply the existence of representations of the form (28).

On the other hand, there are simple examples of Gabor frames. The simplest example is $G(\chi_{[0,1]}, 1, 1)$. To see this, observe that if we fix a particular $n \in \mathbb{Z}$, the sequence $(e^{2\pi imx}\chi_{[n,n+1]})_{m\in\mathbb{Z}}$ is an ONB for $L^2([n, n+1])$. Hence the Gabor system $G(\chi_{[0,1]}, 1, 1)$ is simply the union of ONB's for $L^2([n, n+1])$ over all $n \in \mathbb{Z}$ and therefore $G(\chi_{[0,1]}, 1, 1)$ is an ONB for $L^2(\mathbb{R})$.

However, this Gabor basis is not very usefull in practice. The generator $\chi_{[0,1]}$ is well localized in time in the sense that it is zero outside of a finite interval. However, it is discontinuous which implies that the expansion of a smooth function in the ONB $G(\chi_{[0,1]}, 1, 1)$ does not converge faster than the expansion of a discontinuous function. Moreover, the problem is that the Fourier transform of $\chi_{[0,1]}$ is

$$\widehat{\chi_{[0,1]}}(\xi) = e^{-\pi i\xi}\frac{\sin \pi\xi}{\pi\xi}; \tag{29}$$

thus, $\widehat{\chi_{[0,1]}}$ is not localized, decays only on the order $\frac{1}{|\xi|}$ and is even not integrable.

In general, we want to find Gabor frames generated by functions that are both smooth and well localized. In fact, we now know how to create Gabor frames with smooth and compactly supported generators. This was first done by Daubechies, Grossmann and Meyer ([59]) who reffered to this as "painless nonorthogonal expansions". We first state a lemma whose proof is a simple verification and hence omitted.

**Lemma 5.2.7.** *The map $V : L^2([0,1]) \to L^2([0,c])$, $c > 0$, $Vf(x) = \frac{1}{\sqrt{c}}f(\frac{x}{c})$ is a unitary operator. In particular, since $(e^{2\pi imx})_{m\in\mathbb{Z}}$ is an ONB for $L^2([0,1])$, it follows that $(\frac{1}{\sqrt{c}}e^{2\pi im\frac{x}{c}})_{m\in\mathbb{Z}}$ is an ONB for $L^2([0,c])$. The same applies for $L^2(I)$ where $I$ is any segment on the real line of length $c$.*

**Theorem 5.2.8.** *([59]) Suppose that $g \in L^2(\mathbb{R})$ is such that $\operatorname{supp} g \subseteq I = [0, \frac{1}{b}]$ for some $b > 0$ and that $a > 0$ is such that $ab \le 1$. Then $G(g, a, b)$ is a frame for $L^2(\mathbb{R})$ if and only if there exist constants $A, B > 0$ such that*

$$Ab \le \sum_{k\in\mathbb{Z}} |g(x - ak)|^2 \le Bb \quad \text{for a.e. } x. \tag{30}$$

*In this case $A$ and $B$ are frame bounds for $G(g, a, b)$. If $ab < 1$ then there exist $g$ supported in $[0, \frac{1}{b}]$ that satisfy condition (30) and are smooth as we like.*

**Proof.** Let $f$ be any continuous compactly supported function in $L^2(\mathbb{R})$. Fix $n \in \mathbb{Z}$ and observe that the function $fT_{na}\overline{g}$ is supported on $I + na$. Note that (30) implies that $g$ is bounded a.e. Hence, $fT_{na}\overline{g} \in L^2(I + na)$. Since by the preceding lemma $(\sqrt{b}e^{2\pi imbx})_{m\in\mathbb{Z}}$ is an

ONB for $L^2(I+na)$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)g(x-na)|^2 dx &= \int_{na}^{na+\frac{1}{b}} |f(x)T_{na}\overline{g}(x)|^2 dx \\
&= \|fT_{na}\overline{g}\|_{L^2([I+na])} \\
&= \sum_{m\in\mathbb{Z}} \left|\left\langle fT_{na}\overline{g}, \sqrt{b}e^{2\pi imbx}\right\rangle\right|^2 \\
&= b\sum_{m\in\mathbb{Z}} \left|\int_{na}^{na+\frac{1}{b}} f(x)\overline{g(x-na)}e^{-2\pi imbx}dx\right|^2 \\
&= b\sum_{m\in\mathbb{Z}} \left|\int_{na}^{na+\frac{1}{b}} f(x)\overline{e^{2\pi imbx}g(x-na)}dx\right|^2 \\
&= b\sum_{m\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 .
\end{aligned}$$

We now use Tonelli's theorem to interchange the sum and integral:

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 &= \frac{1}{b}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} |f(x)g(x-na)|^2 dx \\
&= \frac{1}{b}\int_{-\infty}^{\infty} |f(x)|^2 \left(\sum_{n\in\mathbb{Z}} |g(x-na)|^2\right) dx. \qquad (31)
\end{aligned}$$

Using (30) we now obtain

$$A\|f\|^2 \leq \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 \leq B\|f\|^2.$$

Since the subspace of all continuous compactly supported functions is dense in $L^2(\mathbb{R})$, this is enough to conclude that $G(g,a,b)$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A$ and $B$.

Suppose we have $0 < ab < 1$. Take any continuous function $g$ such that $g(x) > 0$ for all $x \in (0, \frac{1}{b})$ and $g(x) = 0$ for all $x \notin (0, \frac{1}{b})$. Because $a < \frac{1}{b}$, it follows that the $a$-periodic function $G_0(x) = \sum_{k\in\mathbb{Z}} |g(x-ak)|^2$ is continuous and strictly positive at every point. Hence, $0 < \inf G_0 \leq \sup G_0 < \infty$. There are many smooth functions that satisfy the above requirements (see Exercise 5.2.15). $\square$

**Remark 5.2.9.** The hypothesis $ab \leq 1$ was used only implicitly, already in the statement of the theorem. Observe that $a > \frac{1}{b}$ would imply $\sum_{k\in\mathbb{Z}} |g(x-ak)|^2 = 0$ on $[\frac{1}{b}, a)$. So in this case condition (30) cannot be satisfied.

**Remark 5.2.10.** We observe the following remarkable fact: if $ab = 1$, then any $g$ that is supported in $[0, \frac{1}{b}]$ and satisfies condiiton (30) must be discontinuous.

To see this, first notice that if $\operatorname{supp} g \subseteq [0, \frac{1}{b}] = [0, a]$, then $T_{na}g$ is supported in $[na, (n+1)a]$. If $g$ is continuous, the fact that $g(x) = 0$ for all $x \notin [0, a]$ forces $g(0) = g(a) = 0$. Since the intervals $[na, (n+1)a]$ overlap (pairwise) at at most one point, it follows that the function $G_0(x) = \sum_{n\in\mathbb{Z}} |g(x - na)|^2$ is continuous and has the property $G_0(na) = 0$, for every $n \in \mathbb{Z}$. But then $G_0$ cannot satisfy condition (30) and hence, by Theorem 5.2.8, $G(g, a, b)$ cannot be a frame.

**Remark 5.2.11.** Let $g$, $a$, and $b$ be as in Theorem 5.2.8 and suppose that

$$G_0(x) = \sum_{n\in\mathbb{Z}} |g(x - na)|^2 \tag{32}$$

satisfies (30). Denote by $U$ the analysis operator of the frame $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$. Take again $f$ continuous with compact support. By (31) we have

$$\langle U^*Uf, f\rangle = \|Uf\|^2 = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 = \frac{1}{b}\int_{-\infty}^{\infty} |f(x)|^2 G_0(x)dx.$$

By continuity of $U^*U$ we now conclude that

$$\langle U^*Uf, f\rangle = \frac{1}{b}\int_{-\infty}^{\infty} |f(x)|^2 G_0(x)dx, \ \ \forall f \in L^2(\mathbb{R}).$$

Consider now the operator $M_{\frac{1}{b}G_0}$ on $L^2(\mathbb{R})$ defined by $M_{\frac{1}{b}G_0}f = \frac{1}{b}G_0 f$. Obviously, $M_{\frac{1}{b}G_0}$ is a well defined, bounded, and self-adjoint operator on $L^2(\mathbb{R})$. Clearly, we have $\langle U^*Uf, f\rangle = \langle M_{\frac{1}{b}G_0}f, f\rangle$ for all $f$ from $L^2(\mathbb{R})$. Since both operators are self-adjoint, this is enough to conclude that $U^*U = M_{\frac{1}{b}G_0}$. We now observe that $(U^*U)^{-1} = M_{b\frac{1}{G_0}}$, i.e.

$$(U^*U)^{-1}f = b\frac{1}{G_0}f, \ \ \forall f \in L^2(\mathbb{R}).$$

Therefore the canonical dual of our frame $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is

$$\big((U^*U)^{-1}M_{mb}T_{na}g\big)_{m,n\in\mathbb{Z}} = \big(M_{mb}T_{na}(U^*U)^{-1}g\big)_{m,n\in\mathbb{Z}} = \left(M_{mb}T_{na}\frac{b}{G_0}g\right)_{m,n\in\mathbb{Z}}.$$

The reconstruction formula gives us

$$f = b\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle M_{mb}T_{na}\frac{1}{G_0}g, \ \ \forall f \in L^2(\mathbb{R}).$$

Note that

$$T_{na}(\frac{1}{G_0}g)(x) = \left(\sum_{k\in\mathbb{Z}} |g(x - ka - na)|^2\right)^{-1} g(x - na) = \frac{1}{G_0(x)}T_{na}g(x);$$

thus, the reconstruction formula can be rewritten in the form

$$f = \frac{b}{G_0}\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle M_{mb}T_{na}g, \ \ \forall f \in L^2(\mathbb{R})$$

and this formula is in fact what can be considered as the painless nonorthogonal expansion.

**Corollary 5.2.12.** *Suppose that $g$ is a continous function supported on an interval $I$ of length $L > 0$ which does not vanish in the interior of $I$. Then $G(g, a, b)$ is a frame for $L^2(\mathbb{R})$ for any $0 < a < L$ and $0 < b \leq \frac{1}{L}$.*

**Proof.** Take any $0 < a < L$ and $0 < b \leq \frac{1}{L}$. Since $\frac{1}{b} \geq L$, the support of $g$ is contained in an interval of length $\frac{1}{b}$. Consider $G_0$ defined by (32). The result will follow from Theorem 5.2.8 if we show that $G_0$ is bounded above and below. Since $g$ is compactly supported, the sum defining $G_0$ is in fact finite sum with at most $\frac{1}{ab}$ terms and therefore $G_0$ is bounded above since $g$ is continuous and compactly supported and hence bounded.

Now let $J$ be the subinterval of $I$ with the same center but with length $a$ (recall that $a < L$). Given $x \in \mathbb{R}$, there is always an $n \in \mathbb{Z}$ such that $x - na \in J$. Hence $\inf_{x\in\mathbb{R}} G_0(x) \geq \inf_{x\in J} |g(x)|^2 > 0$. $\square$

**Corollary 5.2.13.** *Assume that $ab < 1$, take $0 < \epsilon < \frac{a}{2}$ such that $a + 2\epsilon < \frac{1}{b}$, and choose a function $g \in L^2(\mathbb{R})$ such that $\operatorname{supp} g \subseteq [0, a + 2\epsilon]$, $g(x) = 1$ for $x \in [\epsilon, a + \epsilon]$, $g \in C^\infty(\mathbb{R})$, and $\|g\|_\infty = 1$. Then $G(g, a, b)$ is a frame for $L^2(\mathbb{R})$ with frame bounds $\frac{1}{b}$ and $\frac{2}{b}$.*

**Proof.** It is easy to verify that the assumptions on $\epsilon$ and $g$ imply $1 \leq G_0 \leq 2$. Note that the generator of the canonical dual is the function $\frac{b}{G_0} g$ that is also compactly supported and belongs to $C^\infty(\mathbb{R})$. $\square$

**Corollary 5.2.14.** *Suppose that $g$ is a continous function supported on an interval $I$ of length $L > 0$. Then $G(g, a, b)$ is a Bessel sequence for any $a > 0$ and $0 < b \leq \frac{1}{L}$.*

*Concluding remarks.* The material in this section is borrowed from Section 11.2 in [81].

**Exercise 5.2.15.** ([81], Exercise 11.9) Let $f(x) = e^{-\frac{1}{x^2}} \chi_{(0,\infty)}$. Show that for every $n \in \mathbb{N}$ there exists a polynomial $p_n$ of degree $3n$ such that

$$f^{(n)}(x) = p_n(x^{-1}) e^{-\frac{1}{x^2}} \chi_{(0,\infty)}(x).$$

Conclude that $f$ is infinitely differentiable, every derivative of $f$ is bounded, and $f^{(n)}(x) = 0$ for every $x \leq 0$ and $n \geq 0$.

Show that if $0 < a < b$, then $g(x) = f(x - a) f(b - x)$ is infinitely differentiable, is zero outside of $(a, b)$, and is strictly positive on $(a, b)$.

**Exercise 5.2.16.** Prove that for any $g \in C_c(\mathbb{R})$ there exist $a, b > 0$ such that $(M_{mb} T_{na} g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.

## 5.3 Sufficient conditions

We begin our discussion by showing that the necessary part of Theorem 5.2.8 extends to systems generated by any $g \in L^2(\mathbb{R})$ and $a, b > 0$.

**Theorem 5.3.1.** *Suppose that $g \in L^2(\mathbb{R})$ and $a, b > 0$ are such that $G(g, a, b)$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A$ and $B$. Let $G_0$ be as in (32). Then we have*

$$Ab \leq G_0(x) \leq Bb, \quad \text{for a.e. } x. \tag{33}$$

*In particular, $g$ must be essentially bounded.*

**Proof.** Let $f \in L^2(\mathbb{R})$ be any function that is bounded and supported on an interval $I$ of length $\frac{1}{b}$. Then we have $fT_{na}\overline{g} \in L^2(I)$. Since $(\sqrt{b}e^{2\pi imbx})_{m\in\mathbb{Z}}$ is an ONB for $L^2(I)$, it follows, exactly as in the proof of Theorem 5.2.8,

$$b \sum_{m\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 = \int_{-\infty}^{\infty} |f(x)g(x-na)|^2 dx.$$

applying the lower frame bound for $G(g, a, b)$ we find that

$$\int_{-\infty}^{\infty} |f(x)|^2 G_0(x)dx = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} |f(x)g(x-na)|^2 dx = b \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 \geq bA\|f\|^2.$$

Thus, for every bounded $f \in L^2(I)$ we have

$$\int_{-\infty}^{\infty} |f(x)|^2 (G_0(x) - bA)dx \geq 0. \tag{34}$$

Suppose now that $G_0(x) < bA$ on some set $E \subseteq I$ of positive measure. Then we can take $f = \chi_E$ and obtain a contradiction to (34). In a similar way we prove the second inequality in (33). □

**Corollary 5.3.2.** *Suppose that $g \in L^2(\mathbb{R})$ and $a, b > 0$ are such that $G(g, a, b)$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A$ and $B$ and the analysis operator $U$. Then*

*(a)* $Aab \leq \|g\|^2 \leq Bab$.

*(b) If $G(g, a, b)$ is a Parseval frame, then $\|g\|^2 = ab$.*

*(c)* $0 < ab \leq 1$.

*(d)* $\langle g, (U^*U)^{-1}g\rangle = ab$.

*(e) $G(g, a, b)$ is a Riesz basis if and only if $ab = 1$.*

**Proof.** Integrating the function $G_0$ defined by (32) over the interval $[0, a]$ and using (33) we obtain

$$Aab \leq \int_0^a \sum_{n\in\mathbb{Z}} |g(x-na)|^2 dx = \int_{-\infty}^{\infty} |g(x)|^2 dx = \|g\|^2.$$

In a similar way we obtain the second inequality in (a). Moreover, if $G(g,a,b)$ is a Parseval frame we have $A = B = 1$; thus, $\|g\|^2 = ab$.

To prove (c), recall from Proposition 5.2.6 that $G((U^*U)^{-\frac{1}{2}}g, a, b)$ is a Parseval frame. Part (b) therefore implies that $\|(U^*U)^{-\frac{1}{2}}g\|^2 = ab$. On the other hand, the elements of each Parseval frame belong to the closed unit ball; hence $ab = \|(U^*U)^{-\frac{1}{2}}g\|^2 \leq 1$.

To prove (d) we combine the equality $\|(U^*U)^{-\frac{1}{2}}g\|^2 = ab$ with the fact that $(U^*U)^{-\frac{1}{2}}$ is self-adjoint:

$$\langle g, (U^*U)^{-1}g\rangle = \langle (U^*U)^{-\frac{1}{2}}g, (U^*U)^{-\frac{1}{2}}g\rangle = \|(U^*U)^{-\frac{1}{2}}g\|^2 = ab.$$

Finally, $G(g,a,b)$ is a Riesz basis if and only if the associated Parseval frame $G((U^*U)^{-\frac{1}{2}}g,a,b)$ is a Riesz basis - which is in this situation necessarily an ONB - so

$$1 = \|(U^*U)^{-\frac{1}{2}}g\|^2 = \langle g, (U^*U)^{-1}g\rangle \overset{(d)}{=} ab.$$

□

**Remark 5.3.3.** Note that Corollary 5.3.2 (c) implies: if $ab > 1$ then $G(g,a,b)$ cannot be a frame (and this is true for all functions $g$ from $L^2(\mathbb{R})$).

However, $ab \leq 1$ is only a necessary condition and not sufficient. To see this, consider $b = 1$, $\frac{1}{2} < a < 1$ and the system $(M_m T_{na} \chi_{[0,\frac{1}{2}]})_{m,n\in\mathbb{Z}}$. Clearly, $\chi_{[\frac{1}{2},a]} \perp M_m T_{na}\chi_{[0,\frac{1}{2}]}$ for all $m, n$ from $\mathbb{Z}$; hence, the sequence $(M_m T_{na}\chi_{[0,\frac{1}{2}]})_{m,n\in\mathbb{Z}}$ is not even fundamental in $L^2(\mathbb{R})$.

The value $\frac{1}{ab}$ is called the density of the Gabor system $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$. We refer to the density $\frac{1}{ab} = 1$ as the critical density or the Nyquist density.

**Corollary 5.3.4.** *Suppose that $g \in L^2(\mathbb{R})$ and $a, b > 0$ are such that $G(g,a,b)$ is a frame for $L^2(\mathbb{R})$. Then both $g$ and $\hat{g}$ must be essentially bounded.*

**Proof.** Observe that $(\widehat{M_{mb}T_{na}g})_{m,n\in\mathbb{Z}}$ is also a frame for $L^2(\mathbb{R})$ since the Fourier transform is a unitary operator on $L^2(\mathbb{R})$. Now we have

$$\widehat{M_{mb}T_{na}}g = T_{mb}\widehat{T_{na}g} = T_{mb}M_{-na}\hat{g}, \ \ \forall m,n \in \mathbb{Z}.$$

Using Remark 5.2.2 we now conclude that $(M_{na}T_{mb}\hat{g})_{m,n\in\mathbb{Z}}$ is also a frame for $L^2(\mathbb{R})$. An application of Theorem 5.3.1 finishes the proof. □

Suppose now we are given a function $g \in L^2(\mathbb{R})$ and $a, b > 0$. For $k \in \mathbb{Z}$ we define the sequence

$$g_k(x) = \left(g(x - na - \frac{k}{b})\right)_{n\in\mathbb{Z}}, \ \ x \in \mathbb{R}. \tag{35}$$

We claim that $g_k(x) \in \ell^2(\mathbb{Z})$ for a.e. $x$ and all $k$. To see this, observe that

$$\int_0^a \sum_{n\in\mathbb{Z}} |g(x-na-\frac{k}{b})|^2 dx = \sum_{n\in\mathbb{Z}} \int_0^a |g(x-na-\frac{k}{b})|^2 dx = \int_{-\infty}^{\infty} |g(x-\frac{k}{b})|^2 dx = \|T_{\frac{k}{b}}g\|^2 = \|g\|^2.$$

From this we conclude that

$$\sum_{n\in\mathbb{Z}} |g(x-na-\frac{k}{b})|^2 < \infty, \quad \text{for a.e. } x. \tag{36}$$

Now take any $f, g \in L^2(\mathbb{R})$. Since we have, for all $k, l \in \mathbb{Z}$, that the sequences $f_l(x), g_k(x)$ belong to $\ell^2(\mathbb{Z})$, the inner product

$$\langle f_l(x), g_k(x)\rangle = \sum_{n\in\mathbb{Z}} f(x-na-\frac{l}{b})\overline{g(x-na-\frac{k}{b})} \tag{37}$$

is well defined for a.e. $x$. Moreover, we know from the general $\ell^2$ theory that the series in (37) is absolutely convergent. Thus, we have proved

**Lemma 5.3.5.** *Let $f, g \in L^2(\mathbb{R})$, $a, b > 0$, and $k, l \in \mathbb{Z}$ be given. Then the series*

$$\sum_{n\in\mathbb{Z}} f(x-na-\frac{l}{b})\overline{g(x-na-\frac{k}{b})}$$

*converges absolutely a.e. and defines an $a$-periodic function. Moreover, the function*

$$x \mapsto \sum_{n\in\mathbb{Z}} \left| f(x-na-\frac{l}{b})\overline{g(x-na-\frac{k}{b})}\right| \tag{38}$$

*belongs to $L^1([0,a])$.*

**Corollary 5.3.6.** *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. Then for each $k \in \mathbb{Z}$ the series*

$$G_k(x) = \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-\frac{k}{b})}, \quad x \in \mathbb{R}, \tag{39}$$

*converges absolutely a.e. and the function*

$$x \mapsto \sum_{n\in\mathbb{Z}} \left| g(x-na)\overline{g(x-na-\frac{k}{b})}\right|$$

*belongs to $L^1([0,a])$.*

**Proof.** $G_k(x) = \langle g_0(x), g_k(x)\rangle$, where $g_k(x)$ is defined by formula (35) in the preceding considerations. □

It should be observed that (39) for $k = 0$ is in accordance with our earlier definition (32) of the function $G_0$.

**Lemma 5.3.7.** *Let $f, g \in L^2(\mathbb{R})$ and $a, b > 0$. Given $n \in \mathbb{Z}$, let*

$$F_n(x) = \sum_{k\in\mathbb{Z}} f(x-\frac{k}{b})\overline{g(x-na-\frac{k}{b})}. \tag{40}$$

*Then $F_n(x)$ is well defined for a.e. $x$ and defines a function in $L^1([0,\frac{1}{b}])$. Moreover, for any $m \in \mathbb{Z}$ we have*

$$\langle f, M_{mb}T_{na}g\rangle = \int_0^{\frac{1}{b}} F_n(x)e^{-2\pi imbx}dx. \tag{41}$$

*In particular, the $m$-th Fourier coefficient of $F_n(x)$ with respect to the ONB $(\sqrt{b}e^{2\pi imbx})_{m\in\mathbb{Z}}$ for $L^2([0,\frac{1}{b}])$ is*

$$c_m = \sqrt{b}\,\langle f, M_{mb}T_{na}g\rangle. \tag{42}$$

**Proof.** By interchanging $n \leftrightarrow k$ in (40) we see that

$$F_k(x) = \sum_{n\in\mathbb{Z}} f(x-\frac{n}{b})\overline{g(x-ka-\frac{n}{b})}.$$

Now Lemma 5.3.5 with interchanged the roles of $a$ and $\frac{1}{b}$ tells us that $F_n(x)$ is well defined a.e., the series (40) converges absolutely a.e., and that $F_n \in L^1([0,\frac{1}{b}])$. Moreover,

$$\begin{aligned}
\langle f, M_{mb}T_{na}g\rangle &= \int_{-\infty}^{\infty} f(x)\overline{g(x-na)}e^{-2\pi imbx}dx \\
&= \sum_{k\in\mathbb{Z}} \int_0^{\frac{1}{b}} f(x-\frac{k}{b})\overline{g(x-na-\frac{k}{b})}e^{-2\pi imbx}dx \\
&= \int_0^{\frac{1}{b}} \left(\sum_{k\in\mathbb{Z}} f(x-\frac{k}{b})\overline{g(x-na-\frac{k}{b})}\right) e^{-2\pi imbx}dx \\
&= \int_0^{\frac{1}{b}} F_n(x)e^{-2\pi imbx}dx.
\end{aligned}$$

□

**Lemma 5.3.8.** *Let $f$ be a bounded measurable function with compact support. Consider any $g \in L^2(\mathbb{R})$ and the functions $G_k$ defined by (39). If $G_0$ is essentially bounded then*

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 = \frac{1}{b}\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \overline{f(x)}f(x-\frac{k}{b})G_k(x)dx. \tag{43}$$

**Proof.** For $n \in \mathbb{Z}$ consider the $\frac{1}{b}$-periodic function $F_n(x)$ defined by (40). We already know that $F_n \in L^1([0,\frac{1}{b}])$. Since $f$ has compact support, for a given $x \in \mathbb{R}$, $f(x-\frac{k}{b})$ can be non-zero only for finitely many $k$'s. The number of $k$'s for which $f(x-\frac{k}{b}) \neq 0$ is uniformly bounded, i.e. there is a constant $C$ such that at most $C$ values of $k$ appear in the definition of $F_n$. It is now easy to conclude that $F_n$ is bounded; this follows from the Cauchy-Schwarz inequality and boundedness of $f$ and $g$ (notice that $g$ is essentially bounded by the hypothesis on $G_0$). This is enough to conclude that $F_n \in L^2([0,\frac{1}{b}])$. In fact, it follows that

$$x \mapsto \sum_{k\in\mathbb{Z}} \left| f(x-\frac{k}{b})\overline{g(x-na-\frac{k}{b})}\right| \in L^2([0,\frac{1}{b}]).$$

Now the last assertion of Lemma 5.3.7 implies that, by the Parseval equality,

$$\frac{1}{b}\int_0^{\frac{1}{b}} |F_n(x)|^2 dx = \sum_{m\in\mathbb{Z}} \left| \int_0^{\frac{1}{b}} F_n(x) e^{-2\pi i m b x} dx \right|^2 . \tag{44}$$

We now observe that, since $f$ is bounded, measurable, and compactly supported,

$$\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} |\overline{f(x)} f(x-\frac{k}{b})| \sum_{n\in\mathbb{Z}} \left| g(x-na)\overline{g(x-na-\frac{k}{b})} \right| dx < \infty. \tag{45}$$

This is what is needed to justify all interchanges of integration and summation in the computation that follows. In the course of this computation, in order to obtain the second equality, we shall write $|F_n(x)|^2$ in the form

$$|F_n(x)|^2 = \overline{F_n(x)} F_n(x) = \sum_{l\in\mathbb{Z}} \overline{f(x-\frac{l}{b})} g(x-na-\frac{l}{b}) F_n(x).$$

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 &\overset{(41),(44)}{=} \frac{1}{b}\sum_{n\in\mathbb{Z}} \int_0^{\frac{1}{b}} |F_n(x)|^2 dx \\
&= \frac{1}{b}\sum_{n\in\mathbb{Z}} \int_0^{\frac{1}{b}} \sum_{l\in\mathbb{Z}} \overline{f(x-\frac{l}{b})} g(x-na-\frac{l}{b}) F_n(x) dx \\
&= \frac{1}{b}\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \overline{f(x)} g(x-na) F_n(x) dx \\
&= \frac{1}{b}\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \overline{f(x)} g(x-na) \sum_{k\in\mathbb{Z}} f(x-\frac{k}{b}) \overline{g(x-na-\frac{k}{b})} dx \\
&= \frac{1}{b}\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \overline{f(x)} f(x-\frac{k}{b}) G_k(x) dx.
\end{aligned}$$

□

We are now ready to state and prove three theorems that provide us with sufficient conditions on $g$, $a$, and $b$ for $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$.

**Theorem 5.3.9.** *([58]) Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be such that there exist constants $A'$ and $B'$ with the properties*

$$A' \le G_0(x) \le B' \quad \textit{for a.e. } x \tag{46}$$

*and*

$$\sum_{k\neq 0} \|G_k\|_\infty < A'. \tag{47}$$

*Then $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.*

**Theorem 5.3.10.** *([82]) Let $g \in L^2(\mathbb{R})$ and $a > 0$ be such that there exist constants $A'$ and $B'$ for which condition (46) is satisfied. Suppose aditionally that*

$$\lim_{b \to 0} \sum_{k \neq 0} \|G_k\|_\infty = 0. \tag{48}$$

*Then there exists $b_0 > 0$ such that $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ for all $0 < b < b_0$.*

**Theorem 5.3.11.** *([51], Theorem 8.4.4.) Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be such that*

$$B := \frac{1}{b} \sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}} |G_k(x)| < \infty. \tag{49}$$

*Then $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a Bessel sequence with Bessel bound $B$. If, moreover,*

$$A := \frac{1}{b} \inf_{x\in[0,a]} \left( G_0(x) - \sum_{k\neq 0} |G_k(x)| \right) > 0 \tag{50}$$

*then $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A$ and $B$.*

To prove these three theorems we need yet another lemma.

**Lemma 5.3.12.** *Let $f \in L^2(\mathbb{R})$ be a bounded measurable function with compact support. Suppose that for $g \in L^2(\mathbb{R})$ and $a > 0$ the function $G_0$ is essentially bounded. Then*

$$\left| \sum_{k\neq 0} \int_{-\infty}^{\infty} \overline{f(x)} f(x - \frac{k}{b}) G_k(x) dx \right| \leq \int_{-\infty}^{\infty} |f(x)|^2 \sum_{k\neq 0} |G_k(x)| dx. \tag{51}$$

**Proof.** First we have

$$\begin{aligned} \sum_{k\neq 0} \left| T_{-\frac{k}{b}} G_k(x) \right| &= \sum_{k\neq 0} \left| T_{-\frac{k}{b}} \sum_{n\in\mathbb{Z}} T_{na} g(x) \overline{T_{na+\frac{k}{b}} g(x)} \right| \\ &= \sum_{k\neq 0} \left| \sum_{n\in\mathbb{Z}} T_{na-\frac{k}{b}} g(x) \overline{T_{na} g(x)} \right|. \end{aligned}$$

Replacing $k$ with $-k$ from this we obtain

$$\begin{aligned} \sum_{k\neq 0} \left| T_{-\frac{k}{b}} G_k(x) \right| &= \sum_{k\neq 0} \left| \sum_{n\in\mathbb{Z}} T_{na+\frac{k}{b}} g(x) \overline{T_{na} g(x)} \right| \\ &= \sum_{k\neq 0} \left| \sum_{n\in\mathbb{Z}} \overline{T_{na+\frac{k}{b}} g(x)} T_{na} g(x) \right| \\ &= \sum_{k\neq 0} |G_k(x)|. \end{aligned} \tag{52}$$

We now compute:

$$\begin{aligned}
\left|\sum_{k\neq 0}\int_{-\infty}^{\infty}\overline{f(x)}f(x-\frac{k}{b})G_k(x)dx\right| &\leq \sum_{k\neq 0}\int_{-\infty}^{\infty}|f(x)||T_{\frac{k}{b}}f(x)||G_k(x)|dx \\
&= \sum_{k\neq 0}\int_{-\infty}^{\infty}|f(x)|\sqrt{|G_k(x)|}|T_{\frac{k}{b}}f(x)|\sqrt{|G_k(x)|}dx
\end{aligned}$$

(since $f$ is compactly supported, we can now apply the Cauchy-Schwartz inequality in $L^2(\mathbb{R})$)

$$\leq \sum_{k\neq 0}\left(\int_{-\infty}^{\infty}|f(x)|^2|G_k(x)|dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|T_{\frac{k}{b}}f(x)|^2|G_k(x)|dx\right)^{\frac{1}{2}}$$

(now we apply the Cauchy-Schwartz inequality in $\ell^2(\mathbb{Z})$)

$$\leq \left(\sum_{k\neq 0}\int_{-\infty}^{\infty}|f(x)|^2|G_k(x)|dx\right)^{\frac{1}{2}}\left(\sum_{k\neq 0}\int_{-\infty}^{\infty}|T_{\frac{k}{b}}f(x)|^2|G_k(x)|dx\right)^{\frac{1}{2}}$$

(we now replace $x-\frac{k}{b}$ with $x'$ which we again denote by $x$)

$$\begin{aligned}
&\leq \left(\sum_{k\neq 0}\int_{-\infty}^{\infty}|f(x)|^2|G_k(x)|dx\right)^{\frac{1}{2}}\left(\sum_{k\neq 0}\int_{-\infty}^{\infty}|f(x)|^2|T_{-\frac{k}{b}}G_k(x)|dx\right)^{\frac{1}{2}} \\
&\overset{(52)}{=} \left(\int_{-\infty}^{\infty}|f(x)|^2\sum_{k\neq 0}|G_k(x)|dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|f(x)|^2\sum_{k\neq 0}|G_k(x)|dx\right)^{\frac{1}{2}} \\
&= \int_{-\infty}^{\infty}|f(x)|^2\sum_{k\neq 0}|G_k(x)|dx.
\end{aligned}$$

□

**Proof of Theorem 5.3.9.** It is enough to obtain the conclusion for all $f$ from a dense subspace of $L^2(\mathbb{R})$. Thus, we may assume that $f$ is bounded, measurable, and compactly supported. By Lemma 5.3.8 we have

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle f, M_{mb}T_{na}g\rangle|^2 &\leq \frac{1}{b}\int_{-\infty}^{\infty}|f(x)|^2G_0(x)dx+\frac{1}{b}\left|\sum_{k\neq 0}\int_{-\infty}^{\infty}\overline{f(x)}f(x-\frac{k}{b})G_k(x)dx\right| \\
&\overset{(51)}{\leq} \frac{1}{b}\|f\|^2\|G_0\|_\infty+\frac{1}{b}\int_{-\infty}^{\infty}|f(x)|^2\sum_{k\neq 0}|G_k(x)|dx \\
&\overset{(46),(47)}{\leq} \frac{1}{b}B'\|f\|^2+\frac{1}{b}A'\|f\|^2 \\
&= \frac{1}{b}(A'+B')\|f\|^2.
\end{aligned}$$

Similarly,

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle f, M_{mb}T_{na}g\rangle|^2 \overset{(51),(46),(47)}{\geq} \|f\|^2\left(A' - \sum_{k\neq 0}\|G_k\|_\infty\right)$$

□

**Proof of Theorem 5.3.10.** Exactly as in the preceding proof we take arbitrary bounded, measurable, and compactly supported $f$ and obtain

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle f, M_{mb}T_{na}g\rangle|^2 \overset{(46)}{\leq} \frac{1}{b}\left(B' + \sum_{k\neq 0}\|G_k\|_\infty\right)\|f\|^2.$$

Analogously, using (46) and (51) we get

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle f, M_{mb}T_{na}g\rangle|^2 \geq \frac{1}{b}\left(A' - \sum_{k\neq 0}\|G_k\|_\infty\right)\|f\|^2.$$

By (48) we can now find $b_0$ such that $b < b_0$ implies $\sum_{k\neq 0}\|G_k\|_\infty < A'$. □

**Proof of Theorem 5.3.11.** Again, consider any bounded, measurable, and compactly supported $f$. Then, using (43) and (51), we obtain

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle f, M_{mb}T_{na}g\rangle|^2 &\leq \frac{1}{b}\int_{-\infty}^{\infty}|f(x)|^2\left(G_0(x) + \sum_{k\neq 0}|G_k(x)|\right)dx \\
&= \frac{1}{b}\int_{-\infty}^{\infty}|f(x)|^2\sum_{k\in\mathbb{Z}}|G_k(x)|dx \text{ (since } \sum_{k\in\mathbb{Z}}|G_k(x)| \text{ is } a-\text{periodic)} \\
&\leq \frac{1}{b}\left(\sup_{x\in[0,a]}\sum_{k\in\mathbb{Z}}|G_k(x)|\right)\|f\|^2 \\
&\overset{(49)}{\leq} B\|f\|^2.
\end{aligned}$$

Analogously we also obtain

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle f, M_{mb}T_{na}g\rangle|^2 \geq \frac{1}{b}\int_{-\infty}^{\infty}|f(x)|^2\left(G_0(x) - \sum_{k\neq 0}|G_k(x)|\right)dx \overset{(50)}{\geq} A\|f\|^2.$$

□

**Remark 5.3.13.** It should be noted that the advantage of Theorem 5.3.11 is that we compare the functions $G_0(x)$ and $\sum_{k\neq 0}|G_k(x)|$ pointwise rather than requiring that the supremum of $\sum_{k\neq 0}|G_k(x)|$ is smaller than the infimum of $G_0(x)$ as is the case in Theorem 5.3.9. There are situations where we cannot apply Theorem 5.3.9, but Theorem 5.3.11 applies.

**Example 5.3.14.** Let

$$g(x) = \begin{cases} x+1, & x \in [0,1] \\ \frac{1}{2}x, & x \in [1,2] \\ 0, & \text{otherwise} \end{cases}$$

and $a = b = 1$. One finds that

$$\sum_{n\in\mathbb{Z}} g(x-n)g(x-n-k) = \begin{cases} \frac{1}{2}(x+1)^2, & k=-1 \\ \frac{5}{4}(x+1)^2, & k=0 \\ \frac{1}{2}(x+1)^2, & k=1 \\ 0, & \text{otherwise} \end{cases}$$

From this we conclude that

$$G_0(x) = \frac{5}{4}(x+1)^2,\ x \in [0,1] \ \text{ and } \sum_{k\neq 0} |G_k(x)| = (x+1)^2,\ x \in [0,1].$$

Clearly, condition (46) is satisfied with $A' = \frac{5}{4}$ and $B' = 5$; that is $\frac{5}{4} \leq G_0(x) \leq 5$. Since $\sum_{k\neq 0} |G_k(x)|_\infty = 2$, (47) is not fulfilled and we cannot apply Theorem 5.3.9. However, the conditions from Theorem 5.3.11 are satisfied and we conclude that $(M_m T_n g)_{m,n\in\mathbb{Z}}$ is a frame (actually, a Riesz basis) with frame bounds $\frac{1}{4}$ and 9.

*Concluding remarks.* The material in this section is a combination of the expositions in Section 8.4 from [51] and 11.3 from [81]. The interested reader should also consult [82] as well as the related references quoted in the aforementioned sources. For the analogous results in $d$ dimensions we refer the reader to [73].

**Exercise 5.3.15.** Prove inequality (45) and justify the subsequent steps in the proof of Lemma 5.3.8.

### 5.4 The Walnut representation and dual frames

Consider the function $\chi_{[0,1]}$ and compare it with the function $g$ constructed in the following way. Devide the interval $[0,1)$ into infinitely many pieces $[0,\frac{1}{2}), [\frac{1}{2},\frac{3}{4}), [\frac{3}{4},\frac{7}{8})\dots$ and send those pieces to "infinity". Let

$$g = \chi_{[0,\frac{1}{2})} + \chi_{[1+\frac{1}{2},1+\frac{3}{4})} + \chi_{[2+\frac{3}{4},2+\frac{7}{8})} + \dots \tag{53}$$

Clearly, $g$ does not decay at infinity. However, $G(g,1,1)$ is also an ONB for $L^2(\mathbb{R})$. On the other hand, we cannot distinguish between $g$ and $\chi_{[0,1]}$ by considering their $L^p$-norms.

The amalgam spaces $W(L^p,\ell^q)$ which are first considered by Wiener are determined by a norm which mixes a local criterion with a global behavior. Here we restrict ourselves only to $W(L^\infty,\ell^1)$.

**Definition 5.4.1.** *The Wiener amalgam space $W(L^\infty,\ell^1)$ consists of those functions $f \in L^\infty(\mathbb{R})$ for which the norm*

$$\|f\|_{W(L^\infty,\ell^1)} = \sum_{n\in\mathbb{Z}} \|f\chi_{[n,n+1]}\|_\infty < \infty. \tag{54}$$

Thus, a function in $W(L^\infty,\ell^1)$ is locally an $L^\infty$ function and globally decays as an $L^1$ function.

**Theorem 5.4.2.** *(a) $W(L^\infty,\ell^1)$ is contained in $L^p(\mathbb{R})$ for $1 \le p \le \infty$ and is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$.*

*(b) $W(L^\infty,\ell^1)$ is invariant for all translations $T_b$, $b \in \mathbb{R}$, and*

$$\|T_b f\|_{W(L^\infty,\ell^1)} \le 2\|f\|_{W(L^\infty,\ell^1)}, \;\; \forall f \in W(L^\infty,\ell^1). \tag{55}$$

*(c) $W(L^\infty,\ell^1)$ is an ideal in $L^\infty$ with respect to pointwise products and*

$$\|fg\|_{W(L^\infty,\ell^1)} \le \|f\|_\infty \|g\|_{W(L^\infty,\ell^1)}, \;\; \forall f \in L^\infty(\mathbb{R}), \forall g \in W(L^\infty,\ell^1). \tag{56}$$

*(d) Given $a > 0$,*

$$\|f\|_{W(L^\infty,\ell^1),a} = \sum_{n\in\mathbb{Z}} \|f\chi_{[an,a(n+1)]}\|_\infty \tag{57}$$

*is an equivalent norm for $W(L^\infty,\ell^1)$ with*

$$\frac{1}{C_{\frac{1}{a}}}\|f\|_{W(L^\infty,\ell^1),a} \le \|f\|_{W(L^\infty,\ell^1)} \le C_a\|f\|_{W(L^\infty,\ell^1),a}, \;\; \forall f \in W(L^\infty,\ell^1) \tag{58}$$

*where $C_a = \max\{1+a, 2\}$.*

**Proof.** We will prove only the second inequality in (58); the rest is left as na exercise. Let

$$I_k = \{n \in \mathbb{Z} : [k,k+1) \cap [an,a(n+1)] \ne \emptyset\}, \;\; k \in \mathbb{Z},$$

$$J_n = \{k \in \mathbb{Z} : [k,k+1) \cap [an,a(n+1)] \ne \emptyset\}, \;\; n \in \mathbb{Z}.$$

If $a \geq 1$ then $|J_n| \leq 1+a$, while if $0 < a < 1$ then $|J_n| \leq 2$. Hence $|J_n| \leq C_a$ for all $n \in \mathbb{Z}$. Therefore

$$\begin{aligned} \|f\|_{W(L^\infty,\ell^1)} &= \sum_{k\in\mathbb{Z}} \|f\chi_{[k,k+1]}\|_\infty \\ &\leq \sum_{k\in\mathbb{Z}} \sum_{n\in I_k} \|f\chi_{[an,a(n+1)]}\|_\infty \\ &= \sum_{n\in\mathbb{Z}} \sum_{k\in J_n} \|f\chi_{[an,a(n+1)]}\|_\infty \\ &\leq C_a \sum_{n\in\mathbb{Z}} \|f\chi_{[an,a(n+1)]}\|_\infty. \end{aligned}$$

□

**Corollary 5.4.3.** *If $f \in W(L^\infty, \ell^1)$ and $a > 0$, then*

$$\sum_{k\in\mathbb{Z}} \|T_{ak} f\chi_{[0,a]}\|_\infty \leq C_{\frac{1}{a}} \|f\|_{W(L^\infty,\ell^1)}. \tag{59}$$

**Proof.** Observe that $\sum_{k\in\mathbb{Z}} \|T_{ak} f\chi_{[0,a]}\|_\infty = \sum_{k\in\mathbb{Z}} \|f\chi_{[ak,a(k+1)]}\|_\infty$. □

An important property of functions in $W(L^\infty, \ell^1)$ is that a periodization of a function $g \in W(L^\infty, \ell^1)$ is bounded. Observe that a periodization of a general function from $L^1(\mathbb{R})$ is only integrable over the period.

**Corollary 5.4.4.** *Let $a > 0$ and $g \in W(L^\infty, \ell^1)$. Then the $a$-periodization of $g$*

$$\varphi(x) = \sum_{n\in\mathbb{Z}} g(x - an) = \sum_{n\in\mathbb{Z}} T_{na} g(x) \tag{60}$$

*is $a$-periodic, bounded, and satisfies*

$$|\varphi(x)| \leq \sum_{n\in\mathbb{Z}} |g(x - na)| \leq C_{\frac{1}{a}} \|g\|_{W(L^\infty,\ell^1)} \quad a.e. \tag{61}$$

**Proof.** Fix $x$ and observe that for any given $n \in \mathbb{Z}$ there exists exactly one value of $k \in \mathbb{Z}$ such that

$$x - na \in [ka, (k+1)a].$$

Moreover, different values of $n$ lead to different values of $k$. Therefore

$$\sum_{n\in\mathbb{Z}} |g(x - na)| \leq \sum_{k\in\mathbb{Z}} \|g\chi_{[ka,(k+1)a]}\|_\infty = \|g\|_{W(L^\infty,\ell^1),a} \leq C_{\frac{1}{a}} \|g\|_{W(L^\infty,\ell^1)}.$$

□

**Corollary 5.4.5.** *If $g, h \in W(L^\infty, \ell^1)$, $a > 0$, and $0 < b \leq \frac{1}{a}$, then*

$$\sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} g(x-na)\overline{h(x-na-\frac{k}{b})} \right| \leq C^2_{\frac{1}{a}} \|g\|_{W(L^\infty,\ell^1)} \|h\|_{W(L^\infty,\ell^1)} \quad a.e.$$

**Proof.** First observe that

$$\sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} g(x-na)\overline{h(x-na-\frac{k}{b}} \right| \leq \sum_{n\in\mathbb{Z}} |g(x-na)| \sum_{k\in\mathbb{Z}} |h(x-na-\frac{k}{b})|.$$

We now apply inequality (61) from Corollary 5.4.4 twice; first to $g$ and $a$ and then to $h$ and $\frac{1}{b}$. In this way we obtain

$$\sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} g(x-na)\overline{h(x-na-\frac{k}{b}} \right| \leq C_{\frac{1}{a}} \|g\|_{W(L^\infty,\ell^1)} C_b \|h\|_{W(L^\infty,\ell^1)}.$$

Finally, observe that $b \leq \frac{1}{a}$ implies $C_b = \max\{1+b, 2\} \leq \max\{1+\frac{1}{a}, 2\} = C_{\frac{1}{a}}$ which yields the desired conclusion. □

The following two propositions show that functions from $W(L^\infty, \ell^1)$ serve as natural candidates for generating Gabor frames.

**Proposition 5.4.6.** *For each $g \in W(L^\infty, \ell^1)$ and all $a, b > 0$ the sequence $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is Bessel. If $ab \leq 1$ then $C^2_{\frac{1}{a}} \|g\|^2_{W(L^\infty,\ell^1)}$ is its Bessel bound.*

**Proof.** We will apply Theorem 5.3.11; thus, we need to show that condition (49) is satisfied. If $ab \leq 1$ we have $b \leq \frac{1}{a}$. Now Corollary 5.4.5 gives us

$$B := \frac{1}{b} \sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}} |G_k(x)| = \frac{1}{b} \sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-\frac{k}{b})} \right| \leq \frac{1}{b} C^2_{\frac{1}{a}} \|g\|^2_{W(L^\infty,\ell^1)}.$$

Consider now the case $ab > 1$. We can find $N \in \mathbb{N}$ large enough to have $\frac{a}{N}b \leq 1$. By the first part of the proof we know that the sequence $(M_{mb}T_{n\frac{a}{N}}g)_{m,n\in\mathbb{Z}}$ is Bessel. In particular, its subsequence that is obtained by taking only those $n \in \mathbb{Z}$ that are of the form $n = pN$, $p \in \mathbb{Z}$, is also Bessel. □

**Proposition 5.4.7.** *Let $g \in W(L^\infty, \ell^1)$ and $a > 0$ be given. Suppose that there exists a constant $C > 0$ such that $C \leq G_0(x)$ a.e. Then $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ for any $b > 0$ sufficiently small.*

**Proof.** By the preceding proposition the sequence $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a Bessel sequence for all $b > 0$. Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $\sum_{|n|\geq N} \|g\chi_{[na,(n+1)a]}\|_\infty < \epsilon$. Let $g_0 = g\chi_{[-aN,aN]}$ and $g_1 = g - g_0$. We first observe that

$$\begin{aligned} \|g_1\|_{W(L^\infty,\ell^1),a} &= \sum_{n\in\mathbb{Z}} \|(g - g\chi_{[-aN,aN]}g)\chi_{[an,a(n+1)]}\|_\infty \\ &\leq \sum_{|n|\geq N} \|g\chi_{[na,(n+1)a]}\|_\infty < \epsilon. \end{aligned} \quad (62)$$

We now have

$$\begin{aligned}
\sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-\frac{k}{b})}\right| &= \sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}} (g_0+g_1)(x-na)\overline{(g_0+g_1)(x-na-\frac{k}{b})}\right| \\
&\leq \sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}} g_0(x-na)\overline{g_0(x-na-\frac{k}{b})}\right| \\
&+ \sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}} g_0(x-na)\overline{g_1(x-na-\frac{k}{b})}\right| \\
&+ \sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}} g_1(x-na)\overline{g_0(x-na-\frac{k}{b})}\right| \\
&+ \sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}} g_1(x-na)\overline{g_1(x-na-\frac{k}{b})}\right|
\end{aligned}$$

The function $g_0$ has support in an interval of length $2aN$, so if choose $b < \frac{1}{2aN}$ we will have $\frac{1}{b} > 2aN$ and the first of the above four terms is equal to 0. To estimate the remaining three terms we use Corollary 5.4.5:

$$\begin{aligned}
\sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-\frac{k}{b})}\right| &\leq 2C^2_{\frac{1}{a}}\|g_0\|_{W(L^\infty,\ell^1)}\|g_1\|_{W(L^\infty,\ell^1)} + C^2_{\frac{1}{a}}\|g_1\|^2_{W(L^\infty,\ell^1)} \\
&\overset{(58),(62)}{<} 2C^2_{\frac{1}{a}}\|g_0\|_{W(L^\infty,\ell^1)}C_a\epsilon + C^2_{\frac{1}{a}}C_a^2\epsilon^2.
\end{aligned}$$

We now choose $\epsilon > 0$ such that the last term in the above computation is smaller than $C$. This gives us condition 50 from Theorem 5.3.11. $\square$

We now proceed towards the Walnut representation which is a particularly useful form of the frame operator $U^*U$ of Bessel sequences of the form $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$. Here again it turns out that a natural class of functions in our considerations is the Wiener space $W(L^\infty,\ell^1)$.

Recall the functions $G_k$, $k\in\mathbb{Z}$, defined by (39) for each $g\in L^2(\mathbb{R})$ and $a,b>0$. Observe that

$$G_k(x) = \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-\frac{k}{b})} = \sum_{n\in\mathbb{Z}} T_{na}gT_{na+\frac{k}{b}}\overline{g} = \sum_{n\in\mathbb{Z}} T_{na}(gT_{\frac{k}{b}}\overline{g}). \tag{63}$$

The last expression tells us that $G_k$ is the $a$-periodization of $gT_{\frac{k}{b}}\overline{g}$.

Suppose now that $g\in W(L^\infty,\ell^1)$. This implies that $g$ is bounded; thus, Theorem 5.4.2 (c) implies that $gT_{\frac{k}{b}}\overline{g}$ belongs to $W(L^\infty,\ell^1)$. This enables us to show that the corresponding functions $G_k$ are not only integrable, but also bounded.

**Lemma 5.4.8.** *If $g\in W(L^\infty,\ell^1)$ then, for all $a,b>0$, $G_k\in L^\infty(\mathbb{R})$ for all $k\in\mathbb{Z}$ and*

$$\sum_{k\in\mathbb{Z}}\|G_k\|_\infty \leq 2C_{\frac{1}{a}}C_b\|g\|^2_{W(L^\infty,\ell^1)}.$$

**Proof.** By formula (61) in Corollary 5.4.4 we have

$$\|G_k\|_\infty = \|\sum_{n\in\mathbb{Z}} T_{na}(gT_{\frac{k}{b}}\overline{g})\|_\infty \le C_{\frac{1}{a}}\|gT_{\frac{k}{b}}\overline{g}\|_{W(L^\infty,\ell^1)}.$$

This implies

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\|G_k\|_\infty &\le C_{\frac{1}{a}}\sum_{k\in\mathbb{Z}}\|gT_{\frac{k}{b}}\overline{g}\|_{W(L^\infty,\ell^1)} \\
&= C_{\frac{1}{a}}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\|g\chi_{[n,n+1]}T_{\frac{k}{b}}g\chi_{[n,n+1]}\|_\infty \\
&\le C_{\frac{1}{a}}\sum_{n\in\mathbb{Z}}\|g\chi_{[n,n+1]}\|_\infty\left(\sum_{k\in\mathbb{Z}}T_{\frac{k}{b}}g\chi_{[n,n+1]}\|_\infty\right) \\
&= C_{\frac{1}{a}}\|g\|_{W(L^\infty,\ell^1)}\left(\sum_{k\in\mathbb{Z}}T_{\frac{k}{b}}g\chi_{[n,n+1]}\|_\infty\right). \qquad (64)
\end{aligned}$$

Observe that the series in the parenthesis in the last line is not the $W(L^\infty,\ell^1)$-norm of $T_{\frac{k}{b}}g$ since we have the summation over $k$ instead of over $n$. Hence some additional work is needed similar to the proof of Theorem 5.4.2 (d).

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}T_{\frac{k}{b}}g\chi_{[n,n+1]}\|_\infty &= \sum_{k\in\mathbb{Z}}\|g\chi_{[-\frac{k}{b}+n,-\frac{k}{b}+n+1]}\|_\infty \\
&\le 2C_b\sum_{l\in\mathbb{Z}}\|g\chi_{[l,l+1]}\|_\infty \\
&= 2C_b\|g\|_{W(L^\infty,\ell^1)} \qquad (65)
\end{aligned}$$

The main point in the above computation is the observation that an interval of the form $[l,l+1]$ intersects at most $2C_b$ intervals of the form $[-\frac{k}{b}+n,-\frac{k}{b}+n+1]$.

Putting together (64) and (65) we obtain the desired conclusion. □

**Theorem 5.4.9.** *(Walnut representation.) Let $g\in W(L^\infty,\ell^1)$ and $a,b>0$. Denote by $U$ the analysis operator of the Bessel sequence $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$. Then*

$$U^*Uf = \frac{1}{b}\sum_{k\in\mathbb{Z}}T_{\frac{k}{b}}fG_k,\quad \forall f\in L^2(\mathbb{R}). \qquad (66)$$

**Proof.** We already know from Proposition 5.4.6 that $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a Bessel sequence. Lemma 5.4.8 tells us that the series

$$Lf = \frac{1}{b}\sum_{k\in\mathbb{Z}}T_{\frac{k}{b}}fG_k$$

converges absolutely for each $f$ in $L^2(\mathbb{R})$. Moreover,

$$\|Lf\|_2 \le \frac{1}{b}\sum_{k\in\mathbb{Z}}\|T_{\frac{k}{b}}f\|_2\|G_k\|_\infty \le B\|f\|_2,$$

where

$$B = \frac{2}{b} C_{\frac{1}{a}} C_b \|g\|^2_{W(L^\infty,\ell^1)}.$$

This shows us that $L$ is a bounded operator on $L^2(\mathbb{R})$. It remains to show that $U^*U = L$ and to do that it suffices to see that $U^*U$ and $L$ coincide on a dense subspace of $L^2(\mathbb{R})$. Take any continuous compactly supported function $f$. Then, using Lemma 5.3.8, we have

$$\begin{aligned}
\langle U^*Uf, f\rangle &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}g\rangle|^2 \\
&= \frac{1}{b} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \overline{f(x)} f(x - \frac{k}{b}) G_k(x) dx \\
&= \left\langle \frac{1}{b} \sum_{k\in\mathbb{Z}} T_{\frac{k}{b}} f G_k, f \right\rangle \\
&= \langle Lf, f\rangle
\end{aligned}$$

□

We emphasize the contrast between the original form of $U^*U$, i.e.

$$U^*Uf = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, M_{mb}T_{na}g\rangle M_{mb}T_{na}g \tag{67}$$

and the Walnut representation (66). The Walnut representation contains a single summation and, what is more important, it contains no complex exponentials. If $f$ is real, all terms on the right hand side of (66) are real valued, while the terms on the right hand side of (67) need not be.

In [39] it is proved that each $g \in W(L^\infty, \ell^1)$ satisfies condition (50) from Theorem 5.3.11. There it is also proved that the Walnut representation is valid for all functions satisfying (50). Since there are functions (see Exercise 5.4.14) which satisfy (50) and are not in $g \in W(L^\infty, \ell^1)$, this result is more general. However, there are examples which show that (50) is not necessary for $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ to be a frame (see [39]).

We end this section by three results which we include without proof. The first one is known as the Wexler-Raz theorem; it characterizes functions which generate a dual Gabor frame of a frame $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$. It is followed by another result of this type proved by Ron and Shen ([105]) which we include in the form presented by Janssen in [88]. The last one describes Parseval Gabor frames.

**Theorem 5.4.10.** *Let $g, h \in L^2(\mathbb{R})$ and $a, b > 0$ be such $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ and $(M_{mb}T_{na}h)_{m,n\in\mathbb{Z}}$ are Bessel sequences. Then they are dual frames $L^2(\mathbb{R})$ if and only if*

$$\langle h, M_{\frac{m}{a}} T_{\frac{n}{b}} g\rangle = 0,\ \forall (m,n) \neq 0, \quad \textit{and}\ \langle h, g\rangle = ab.$$

**Theorem 5.4.11.** *Let $g, h \in L^2(\mathbb{R})$ and $a, b > 0$ be such $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ and $(M_{mb}T_{na}h)_{m,n\in\mathbb{Z}}$ are Bessel sequences. Then they are dual frames for $L^2(\mathbb{R})$ if and only if the equations*

$$\sum_{n\in\mathbb{Z}} g(x-na)\overline{h(x-na)} = b$$

*and*

$$\sum_{n\in\mathbb{Z}} g(x-na)\overline{h(x-na-\frac{k}{b})} = 0, \;\; k \in \mathbb{Z}\setminus\{0\}$$

*hold a.e.*

**Theorem 5.4.12.** *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. The following conditions are equivalent:*

*(a) $(M_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a Parseval frame.*

*(b) $G_0(x) = b$ a.e. and $G_k(x) = 0$ a.e. for all $k \neq 0$.*

*(c) $g \perp M_{\frac{m}{a}}T_{\frac{n}{b}}g$ for all $(m,n) \neq 0$ and $\|g\|^2 = ab$.*

*(d) $(M_{\frac{m}{a}}T_{\frac{n}{b}}g)_{m,n\in\mathbb{Z}}$ is an orthogonal sequence and $\|g\|^2 = ab$.*

*Concluding remarks.* This section is again a combination of the material from [51] and [81]. We refer the reader to [51] for the proofs of Theorem 5.4.10 and 5.4.12.

As Theorem 5.4.10 suggests, there is a closed relationship between frame properties for a function $g$ with respect to the lattice $\{(m,n) : m,n \in \mathbb{Z}\}$ and with respect to the dual lattice $\{(\frac{n}{b}, \frac{m}{a}) : m,n \in \mathbb{Z}\}$. Ron and Shen were the first to obtain some important results along this line. We refer the reader to [105] for the Ron-Shen duality principle and related results. The interested reader should consult [60].

**Exercise 5.4.13.** Prove Theorem 5.4.2.

**Exercise 5.4.14.** Show that the function $g$ defined by (53) satisfies condition (50).

**Exercise 5.4.15.** ([73], Theorem 6.3.2, Walnut representation of the mixed frame operator.) Let $g, h \in W(L^\infty, \ell^1)$ and $a, b > 0$. Denote by $U$ and $V$ the analysis operators of the Bessel sequences $G(g,a,b)$ and $G(h,a,b)$, respectively. Show that

$$V^*Uf = \frac{1}{b}\sum_{k\in\mathbb{Z}} T_{\frac{k}{b}}f\left(\sum_{n\in\mathbb{Z}} h(x-na)\overline{g(x-na-\frac{k}{b})}\right), \;\; \forall f \in L^2(\mathbb{R}).$$

### 5.5 The Zak transform and the Balian-Low theorem

Consider the Gabor system $G(\chi_{[0,1]}, 1, 1)$ which is an ONB for $L^2(\mathbb{R})$. Let $Q$ denote the unit square $[0,1]^2$ in $\mathbb{R}^2$. It is easy to see that the sequence

$$(E_{mn})_{m,n\in\mathbb{Z}},\ \ E_{mn}(x,\xi) = e^{2\pi imx}e^{-2\pi in\xi}$$

is an ONB for the Hilbert space $L^2(Q)$.

**Definition 5.5.1.** *The Zak transform is the unitary operator $Z : L^2(\mathbb{R}) \to L^2(Q)$ defined by*

$$Z(M_mT_n\chi_{[0,1]}) = E_{mn},\ \ m,n\in\mathbb{Z}.$$

**Theorem 5.5.2.** *Given $f \in L^2(\mathbb{R})$, we have*

$$Zf(x,\xi) = \sum_{j\in\mathbb{Z}} f(x-j)e^{2\pi ij\xi},\ \ (x,\xi)\in Q. \tag{68}$$

*where this series converges unconditionally in $L^2(Q)$.*

**Proof.** First we observe that for each $f \in L^2(\mathbb{R})$ and all $j \neq l$ the functions $f(x-j)e^{2\pi ij\xi}$ and $f(x-l)e^{2\pi il\xi}$ are orthogonal vectors in $L^2(Q)$. Let us now take any finite set $F \subseteq \mathbb{Z}$. Then

$$\begin{aligned}
\left\|\sum_{j\in F} f(x-j)e^{2\pi ij\xi}\right\|^2 &= \sum_{j\in F}\left\|f(x-j)e^{2\pi ij\xi}\right\|^2 \\
&= \sum_{j\in F}\int_0^1\int_0^1 \left|f(x-j)e^{2\pi ij\xi}\right|^2 dxd\xi \\
&= \sum_{j\in F}\int_0^1 |f(x-j)|^2 dx. \qquad (69)
\end{aligned}$$

Since $f$ belongs to $L^2(\mathbb{R})$, the series $\sum_{j\in\mathbb{Z}}\int_0^1 |f(x-j)|^2dx$ converges unconditionally to $\|f\|^2$. Thus, the series on the right hand side of (68) converges unconditionally and defines a linear operator $T : L^2(\mathbb{R}) \to L^2(Q)$, $Tf = \sum_{j\in\mathbb{Z}} f(x-j)e^{2\pi ij\xi}$, which is by (69) an isometry. To prove that $T = Z$ we only need to show that $T(M_mT_n\chi_{[0,1]}) = E_{mn}$ for all $m,n \in \mathbb{Z}$. Take any $m$ and $n$. Then

$$\begin{aligned}
T(M_mT_n\chi_{[0,1]})(x,\xi) &= \sum_{j\in\mathbb{Z}} M_mT_n\chi_{[0,1]}(x-j)e^{2\pi ij\xi} \\
&= \sum_{j\in\mathbb{Z}} e^{2\pi im(x-j)}\chi_{[0,1]}(x-j-n)e^{2\pi ij\xi} \\
&= e^{2\pi imx}e^{-2\pi in\xi} \\
&= E_{mn}
\end{aligned}$$

Observe that the penultimate equality follows from the fact that $x \in [0,1]$, so $\chi_{[0,1]}(x-j-n)$ vanishes for all $j \neq -n$. $\square$

In general, one can consider the Zak transform on domains other than $L^2(\mathbb{R})$. Again, it turns out that natural domains are precisely the Wiener amalgam spaces $W(L^p, \ell^1)$. Here we restrict ourselves to the case $p = \infty$.

**Theorem 5.5.3.** *For each $f \in W(L^\infty, \ell^1)$ the series $Zf(x,\xi) = \sum_{j\in\mathbb{Z}} f(x-j)e^{2\pi ij\xi}$, $(x,\xi) \in Q$, converges absolutely in $L^\infty(Q)$ and $Z : W(L^\infty, \ell^1) \to L^\infty(Q)$ is a bounded linear operator.*

**Proof.**
$$\sum_{j\in\mathbb{Z}} \left\| f(x-j)e^{2\pi ij\xi} \right\|_\infty = \sum_{j\in\mathbb{Z}} \|f\chi_{[j,j+1]}\|_\infty = \|f\|_{W(L^\infty,\ell^1)}.$$

□

We will use the Zak transform to analyze Gabor systems at the critical density $ab = 1$. Using Lemma 5.2.3, by delating $g$, if necessary, we can assume that $a = b = 1$, i.e. we can restrict ourselves to Gabor systems of the form $G(g, 1, 1)$.

We begin with a simple but useful lemma.

**Lemma 5.5.4.** *If $g \in L^2(\mathbb{R})$ then*
$$Z(M_m T_n g) = E_{mn} Zg, \quad \textit{a.e. for all } m, n \in \mathbb{Z}. \tag{70}$$

**Proof.** We compute (using the fact that $e^{-2\pi il} = 1$ for all integers $l$):
$$\begin{aligned}
Z(M_m T_n g)(x,\xi) &= \sum_{j\in\mathbb{Z}} (M_m T_n g)(x-j)e^{2\pi ij\xi} \\
&= \sum_{j\in\mathbb{Z}} e^{2\pi im(x-j)} g(x-j-n)e^{2\pi ij\xi} \text{ (changing } j+n \to j'\text{)} \\
&= \sum_{j'\in\mathbb{Z}} e^{2\pi im(x-j'+n)} g(x-j')e^{2\pi i(j'-n)\xi} \\
&= e^{2\pi imx}e^{-2\pi in\xi} \sum_{j'\in\mathbb{Z}} g(x-j')e^{2\pi ij'\xi} \\
&= E_{mn} Zg.
\end{aligned}$$

□

The following corollary should be compared to Proposition 4.2.6 (a). It shows that $L^2(\mathbb{R})$ cannot be generated, as a shift-invariant space, by a single function.

**Corollary 5.5.5.** *If $g \in L^2(\mathbb{R})$ and $a > 0$, then $(g(x-na))_{n\in\mathbb{Z}}$ is not fundamental in $L^2(\mathbb{R})$.*

**Proof.** By dilating $g$ it suffices to consider the case $a = 1$. So, our sequence is $(T_n g)_{n\in\mathbb{Z}}$. Taking $m = 0$ in the preceding lemma we see that
$$\{Z(T_n g) : n \in \mathbb{Z}\} = \{E_{0n} Zg : n \in \mathbb{Z}\} = \{e^{-2\pi in\xi} Zg : n \in \mathbb{Z}\}.$$

Taking finite linear combinations and limits with respect to $\|\cdot\|_2$-norm, we conclude that each function in $\overline{\text{span}}\,\{e^{-2\pi in\xi} Zg : n \in \mathbb{Z}\}$ has the form $f(\xi)Zg(x,\xi)$. Clearly, there are elements in $L^2(Q)$ which are not of that form. □

**Theorem 5.5.6.** *Let $g \in L^2(\mathbb{R})$.*

*(a) $G(g,1,1)$ is fundamental in $L^2(\mathbb{R})$ if and only if $Zg \neq 0$ a.e.*

*(b) $G(g,1,1)$ is a Bessel sequence in $L^2(\mathbb{R})$ if and only if $Zg \in L^\infty(Q)$.*

*(c) $G(g,1,1)$ is a frame for $L^2(\mathbb{R})$ if and only if there exist constants $A, B > 0$ such that $A \leq |Zg(x,\xi)|^2 \leq B$ a.e. In this case $G(g,1,1)$ is in fact a Riesz basis, $A, B$ are its frame bounds, and, if $\tilde{g}$ denotes the function which generates the canonical dual, we have*

$$Z(M_m T_n \tilde{g}) = \frac{e^{2\pi i m x} e^{-2\pi i n \xi}}{\overline{Zg(x,\xi)}}. \tag{71}$$

**Proof.** (a). Suppose that $Zg \neq 0$ a.e. Since $Z$ is a unitary operator, it suffices, by Lemma 5.5.4, to show that $(E_{mn} Zg)_{m,n\in\mathbb{Z}}$ is fundamental in $L^2(Q)$. Suppose that $\langle F, E_{mn} Zg\rangle = 0$, for all $m, n \in \mathbb{Z}$. Take $H = F\overline{Zg}$. Then we see that $H \in L^1(Q)$ and that the Fourier coefficients of $H$ with respect to the ONB $(E_m n)_{m,n\in\mathbb{Z}}$ of $L^2(Q)$ are

$$\langle H, E_{mn}\rangle = \int_0^1 \int_0^1 F(x,\xi)\overline{Zg(x,\xi)E_{mn}(x,\xi)}dxd\xi = \langle F, E_{mn} Zg\rangle = 0.$$

By the uniqueness theorem, from this we conclude that $H = 0$. As $H = F\overline{Zg}$ and $Zg \neq 0$ a.e., it follows that $F = 0$ a.e. The converse is proved similarly.

(b), (c). Consider any $f \in L^2(\mathbb{R})$. Then we have

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle f, M_m T_n g\rangle|^2 &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle Zf, ZM_m T_n g\rangle|^2 \\
&\overset{(70)}{=} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \left|\int_0^1 \int_0^1 Zf(x,\xi) e^{-2\pi i m x} e^{2\pi i n\xi}\overline{Zg(x,\xi)}dxd\xi\right|^2 \\
&= \|Zf\overline{Zg}\|^2 \\
&= \int_0^1\int_0^1 |Zf(x,\xi)|^2 |Zg(x,\xi)|^2 dxd\xi. \qquad (72)
\end{aligned}$$

The penultimate equality in the above computation is obtained in the following way. If we assume that $G(g,1,1)$ is a Bessel sequence, the double series $\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle f, M_m T_n g\rangle|^2$ is convergent. Hence, the double series $\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \left|\int_0^1\int_0^1 Zf(x,\xi)\overline{Zg(x,\xi)}e^{-2\pi i m x}e^{2\pi i n\xi}dxd\xi\right|^2$ is convergent. This tells us that the function $Zf\overline{Zg}$ belongs in fact to $L^2(Q)$, so we could apply the Parseval equality with respect to the basis $(E_{mn})_{m,n\in\mathbb{Z}}$.

If we assume that $G(g,1,1)$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A$ and $B$, we conclude from (72) that for each $f \in L^2(\mathbb{R})$ we have

$$A\langle Zf, Zf\rangle = A\|f\|^2 \leq \int_0^1\int_0^1 |Zf(x,\xi)|^2 |Zg(x,\xi)|^2 dxd\xi \leq B\|f\|^2 = B\langle Zf, Zf\rangle. \tag{73}$$

As $f$ ranges throughout $L^2(\mathbb{R})$, $|Zf(x,\xi)|$ ranges throughout all positive functions in $L^2(Q)$. Thus,

$$0 < A \leq |Zg(x,\xi)|^2 \leq B < \infty \quad \text{a.e.}$$

The converse is proved in a similar way.

Denoting the analysis operator by $U$ and writing $\varphi = |Zg|^2$, we rewrite (72) in the form

$$\|Uf\|^2 = \langle Uf, Uf\rangle = \int_0^1 \int_0^1 \varphi(x,\xi) Zf(x,\xi)\overline{Zf(x,\xi)} dx d\xi = \langle M_\varphi Zf, Zf\rangle \tag{74}$$

where $M_\varphi : L^2(Q) \to L^2(Q)$ is the multiplication operator defined by $M_\varphi Zf = \varphi Zf$ on elements of the form $Zf$, $f \in L^2(\mathbb{R})$ (and hence on all elements from $L^2(Q)$).

We now observe that, since $Z$ is unitary, we have $\langle U^*Uf, f\rangle = \langle Z(U^*Uf), Zf\rangle$. Thus, (74) can be rewritten as

$$\langle Z(U^*Uf), Zf\rangle = \langle M_\varphi Zf, Zf\rangle, \ \forall f \in L^2(\mathbb{R}).$$

By polarization we obtain

$$\langle Z(U^*Uf), Zh\rangle = \langle M_\varphi Zf, Zh\rangle, \ \forall f, h \in L^2(\mathbb{R}).$$

This gives us

$$Z(U^*Uf) = M_\varphi Zf, \ \forall f \in L^2(\mathbb{R}). \tag{75}$$

Writing $U^*Uf = h$, i.e. $f = (U^*U)^{-1}h$ we obtain

$$Zh = \varphi Z((U^*U)^{-1}h), \ \forall h \in L^2(\mathbb{R})$$

and consequently

$$\frac{1}{\varphi} Zh = Z((U^*U)^{-1}h), \ \forall h \in L^2(\mathbb{R}).$$

In particular, this gives us

$$\frac{1}{\varphi} Zg = Z\tilde{g}$$

and this finally implies

$$Z\tilde{g} = \frac{1}{\overline{Zg}}. \tag{76}$$

Applying Lemma 5.5.4 we now have

$$Z(M_b T_n \tilde{g}) = E_{mn} Z\tilde{g} = \frac{e^{2\pi i m x} e^{-2\pi i n \xi}}{\overline{Zg(x,\xi)}}.$$

□

**Remark 5.5.7.** If for some $g \in L^2(\mathbb{R})$ the sequence $G(g, 1, 1, )$ is a frame for $L^2(\mathbb{R})$, we know from Corollary 5.3.2 that this is in fact a Riesz basis and that $\langle g, \tilde{g}\rangle = ab = 1$. This last fact we see also from the following computation:

$$\langle g, M_m T_n \tilde{g}\rangle = \langle Zg, Z(M_m T_n \tilde{g})\rangle \overset{(71)}{=} \int_0^1 \int_0^1 Zg(x,\xi) e^{-2\pi i m x} e^{2\pi i n \xi} \frac{1}{Zg(x,\xi)} dx d\xi = \delta_{m,0}\delta_{n,0}. \tag{77}$$

Consider now the operators $X$ and $P$ defined by

$$(Xf)(x) = xf(x), \;\; (Pf)(x) = \frac{1}{2\pi i} f'(x). \tag{78}$$

Clearly, the Schwartz class $S(\mathbb{R})$ is a common domain for $X, P, XP$, and $PX$. The largest common domain for these operators is the subspace

$$\{f \in L^2(\mathbb{R}) : xf(x), f'(x), xf'(x) \in L^2(\mathbb{R})\}.$$

It can be shown that $X$ and $P$ are self-adjoint. Recall that $\widehat{f'}(\omega) = 2\pi i \omega \hat{f}(\omega)$ and observe that

$$\int_{-\infty}^{\infty} |(Xg)(x)|^2 dx = \int_{-\infty}^{\infty} x^2 |g(x)|^2 dx \tag{79}$$

and

$$\begin{aligned}
\int_{-\infty}^{\infty} |(Pg)(x)|^2 dx &= \|Pg\|^2 \\
&= \|\widehat{Pg}\|^2 \\
&= \int_{-\infty}^{\infty} \left| \widehat{\frac{1}{2\pi i} f'(\omega)} \right|^2 d\omega \\
&= \int_{-\infty}^{\infty} \frac{1}{|2\pi i|^2} |2\pi i|^2 |\omega|^2 |\hat{f}(\omega)|^2 d\omega \\
&= \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega.
\end{aligned} \tag{80}$$

**Lemma 5.5.8.** *For any $h \in L^2(\mathbb{R})$, if $Xh \in L^2(\mathbb{R})$ and $Ph \in L^2(\mathbb{R})$ we have*

$$Z(Xh)(x,\xi) = xZh(x,\xi) - \frac{1}{2\pi i} \frac{\partial}{\partial \xi} Zh(x,\xi) \tag{81}$$

*and*

$$Z(Ph)(x,\xi) = \frac{1}{2\pi i} \frac{\partial}{\partial x} Zh(x,\xi). \tag{82}$$

**Proof.** Using formula (68) from Theorem 5.5.2 for $f = Xh$ we obtain

$$\begin{aligned}
Z(Qh)(x,\xi) &= \sum_{j\in\mathbb{Z}} (x-j)h(x-j)e^{2\pi i j\xi} \\
&= x \sum_{j\in\mathbb{Z}} h(x-j)e^{2\pi i j \xi} - \frac{1}{2\pi i} \sum_{j\in\mathbb{Z}} 2\pi i j h(x-j) e^{2\pi i j \xi} \\
&= xZh(x,\xi) - \frac{1}{2\pi i} \frac{\partial}{\partial \xi} Zh(x,\xi).
\end{aligned}$$

Similarly, if $f = Ph$ we obtain

$$Z(Ph)(x,\xi) = \sum_{j\in\mathbb{Z}} \frac{1}{2\pi i} f'(x-j) e^{2\pi i j \xi} = \frac{1}{2\pi i} \frac{\partial}{\partial x} Zh(x,\xi).$$

$\square$

**Corollary 5.5.9.** *Suppose that $g \in L^2(\mathbb{R})$ is such that the sequence $G(g,1,1)$ is a frame for $L^2(\mathbb{R})$ and denote by $\tilde{g}$ the generator of its canonical dual. If $Xg, Pg \in L^2(\mathbb{R})$ then $X\tilde{g}, P\tilde{g} \in L^2(\mathbb{R})$.*

**Proof.** If $Xg \in L^2(\mathbb{R})$, (81) shows that $\frac{\partial}{\partial \xi} Zg \in L^2(Q)$. Since by Theorem 5.5.6 (c) we have $Z\tilde{g} = \frac{1}{\overline{Zg}}$, it follows that

$$\frac{\partial}{\partial \xi} Z\tilde{g} = -\frac{1}{(\overline{Zg})^2}\frac{\partial}{\partial \xi}(\overline{Zg}).$$

The inequalities in Theorem 5.5.6 (c) show that $|Zg|$ is bounded away from zero, so that $\frac{\partial}{\partial \xi} Z\tilde{g} \in L^2(Q)$.

Applying the preceding lemma to $h = \tilde{g}$ we deduce that $X\tilde{g} \in L^2(\mathbb{R})$.

A similar argument, using Lemma 5.5.8, shows that $P\tilde{g} \in L^2(\mathbb{R})$. $\square$

Suppose again, as in the preceding corollary that $G(g,1,1)$ is a frame for $L^2(\mathbb{R})$ with $\tilde{g}$ denoting the generator of the canonical dual. Then we also have, for all $m, n \in \mathbb{Z}$,

$$\begin{aligned}
\langle Xg, M_m T_n \tilde{g}\rangle &\overset{(77)}{=} \langle Xg, M_m T_n \tilde{g}\rangle - n\langle g, M_m T_n \tilde{g}\rangle \\
&= \int_{-\infty}^{\infty} (x-n) g(x) e^{-2\pi i m x}\overline{\tilde{g}(x-n)}dx \quad (\text{changing } x - n = y) \\
&= \int_{-\infty}^{\infty} g(y+n) y \overline{\tilde{g}(y)} e^{-2\pi i m y} \\
&= \langle M_{-m} T_{-n} g, X\tilde{g}\rangle. \qquad (83)
\end{aligned}$$

Similarly, we have

$$\langle Pg, M_m T_n \tilde{g}\rangle = \frac{1}{2\pi i}\int_{-\infty}^{\infty} g'(x) e^{-2\pi i m x}\overline{\tilde{g}(x-n)}dx. \qquad (84)$$

Using integration by parts we obtain

$$\begin{aligned}
&= m\int_{-\infty}^{\infty} g(x) e^{-2\pi i m x}\overline{\tilde{g}(x-n)}dx - \frac{1}{2\pi i}\int_{-\infty}^{\infty} g(x) e^{-2\pi i m x}\overline{\tilde{g}'(x-n)}dx \\
&= m\langle M_m T_n \tilde{g}\rangle - \frac{1}{2\pi i}\int_{-\infty}^{\infty} g(y+n) e^{-2\pi i m y}\overline{\tilde{g}'(y)}dy \\
&\overset{(77)}{=} \int_{-\infty}^{\infty} e^{-2\pi i m y} g(y+n)\overline{\left(\frac{1}{2\pi i}\right)}\overline{\tilde{g}'(y)}dy \\
&= \langle M_{-m} T_{-n} g, P\tilde{g}\rangle. \qquad (85)
\end{aligned}$$

Putting together (84) and (85) we obtain for all $m, n \in \mathbb{Z}$, analogously to (83),

$$\langle Pg, M_m T_n \tilde{g}\rangle = \langle M_{-m} T_{-n} g, P\tilde{g}\rangle. \qquad (86)$$

We are now ready to state and prove the Balian-Low theorem for frames. Roughly speaking, the Balian-Low theorem tells us that the generator $g \in L^2(\mathbb{R})$ of a Gabor frame $G(g,1,1)$ cannot bi "nice", i.e. localized both in time and frequency.

**Theorem 5.5.10.** *Let $g \in L^2(\mathbb{R})$. If $G(g,1,1)$ is a frame for $L^2(\mathbb{R})$ (in which case it is necessarily a Riesz basis), then either*

$$\int_{-\infty}^{\infty} x^2 |g(x)|^2 dx = \infty \quad \text{or} \quad \int_{-\infty}^{\infty} \xi^2 |\hat{g}(\xi)|^2 dx = \infty.$$

**Proof.** By (79) and (80) we must show that $Xg$ and $Pg$ cannot belong to $L^2(\mathbb{R})$. Suppose the opposite: let $Xg, Pg \in L^2(\mathbb{R})$. Then Corollary 5.5.9 implies $X\tilde{g}, P\tilde{g} \in L^2(\mathbb{R})$. Now we have

$$\langle Xg, P\tilde{g}\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle Xg, M_m T_n \tilde{g}\rangle \langle M_m T_n g, P\tilde{g}\rangle \tag{87}$$

and

$$\langle Pg, X\tilde{g}\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle Pg, M_m T_n \tilde{g}\rangle \langle M_m T_n g, X\tilde{g}\rangle. \tag{88}$$

We now observe that (83), (86), (87), and (88) imply

$$\langle Xg, P\tilde{g}\rangle = \langle Pg, X\tilde{g}\rangle \tag{89}$$

Using the formula for the integration by parts

$$\int_{-\infty}^{\infty} f' g dx = -\int_{-\infty}^{\infty} f g' dx$$

we obtain

$$\begin{aligned}
\langle Xg, P\tilde{g}\rangle &= \int_{-\infty}^{\infty} x g(x) \overline{\frac{1}{2\pi i}\overline{\tilde{g}'(x)}} dx \\
&= +\frac{1}{2\pi i}\int_{-\infty}^{\infty} (g(x) + x g'(x)) \overline{\tilde{g}(x)} dx \\
&= \frac{1}{2\pi i}\langle g, \tilde{g}\rangle + \langle Pg, X\tilde{g}\rangle.
\end{aligned}$$

But now (89) implies $\langle g, \tilde{g}\rangle = 0$ which is impossible because we know that $\langle g, \tilde{g}\rangle = 1$. $\square$

The Balian-Low theorem should be compared with the classical uncertainty principle which is refered to as the Heisenberg-Pauli-Weyl inequality.

**Theorem 5.5.11.** *If $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$ are arbitrary, then*

$$\left(\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} (\omega-a)^2 |\hat{f}(\omega)|^2 d\omega\right)^{\frac{1}{2}} \geq \frac{1}{4\pi}\|f\|^2. \tag{90}$$

*Equality holds if and only if $f$ is multiple of $T_a M_b \phi_c(x) = e^{2\pi i b(x-a)} e^{-\frac{\pi(x-a)^2}{c}}$ for some $c > 0$. In particular, we have*

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega\right)^{\frac{1}{2}} \geq \frac{1}{4\pi}\|f\|^2, \ \ \forall f \in L^2(\mathbb{R}). \tag{91}$$

Observe that for general $f \in L^2(\mathbb{R})$ the left hand side of (90) may be infinite, but then there is nothing to prove.

To prove Theorem 5.5.11 we need the following lemma.

**Lemma 5.5.12.** *Let $A$ and $B$ be (possibly unbounded) self-adjoint operators on a Hilbert space $H$ Then*

$$2\|(A-aI)x\| \cdot \|(B-bI)x\| \geq |\langle [A,B]x, x\rangle|, \tag{92}$$

*for all $a, b \in \mathbb{R}$ and all $x$ in the domain of $AB$ and $BA$, where $[A,B] = AB - BA$ is the commutator of $A$ and $B$.*

*Equality holds if and only if $(A-aI)x = ic(B-bI)x$ for some $c \in \mathbb{R}$.*

We first have, using the self-adjointness of $A$ and $B$,

$$\begin{aligned}
\langle [A,B]x, x\rangle &= \langle (A-aI)(B-bI) - (B-bI)(A-aI)x, x\rangle \\
&= \langle (B-bI)x, (A-aI)x\rangle - \langle (A-aI)x, (B-bI)x\rangle \\
&= 2i\mathrm{Im}\,\langle (B-bI)x, (A-aI)x\rangle.
\end{aligned}$$

By applying the Cauchy-Schwarz inequality we obtain

$$|\langle [A,B]x, x\rangle| \leq 2|\langle (B-bI)x, (A-aI)x\rangle| \leq 2\|(B-bI)x\| \cdot \|(A-aI)x\|. \tag{93}$$

Equality holds in the first inequality in (93) if and only if $\langle (B-bI)x, (A-aI)x\rangle$ is purely imaginary and in the second inequality of (93) equality holds if and only if $(B-bI)x = \lambda(A-aI)x$ for some $\lambda \in \mathbb{C}$. This two facts together imply $\lambda = ic$ for some $c \in \mathbb{R}$.

**Proof of Theorem 5.5.11.** Consider again the operators $X$ and $P$ defined by (78). Then we have

$$[X,P]f(x) = \frac{1}{2\pi i}(xf'(x) - (xf)'(x)) = \frac{1}{2\pi i}(xf'(x) - xf'(x) - f(x)) = \frac{1}{2\pi i}f(x).$$

Thus, Lemma 5.5.12 implies

$$\|f\|^2 = 2\pi|\langle [X,P]f, f\rangle| \leq 4\pi\|(X-aI)f\| \cdot \|(P-bI)f\|. \tag{94}$$

Notice that (cf. (79), (80))

$$\|(X-aI)f\| = \left(\int_{-\infty}^{\infty} (x-a)^2|f(x)|^2 dx\right)^{\frac{1}{2}}$$

and

$$\|(P-bI)f\| = \|\widehat{(P-bI)f}\| = \left(\int_{-\infty}^{\infty} (\omega-b)^2|\hat{f}(\omega)|^2 d\omega\right)^{\frac{1}{2}}.$$

Finally, equality in (94) holds if and only if $(P-bI)f = ic(X-aI)f$ for some $c \in \mathbb{R}$. This condition is in fact the differential equation

$$f' - 2\pi i b f = -2\pi c(x-a)f.$$

It turns out that its solutions are precisely all scalar multiples of $T_a M_b \phi_{\frac{1}{c}}$, $c \in \mathbb{R}$, where $\phi_r(x) = e^{\pi \frac{x^2}{r}}$.

Since we require that the solution $f = \lambda T_a M_b \phi_{\frac{1}{c}}$ belongs to $L^2(\mathbb{R})$, we must have $c > 0$. $\square$

In the light of the Heisenberg-Pauli-Weyl uncertainty principle the Balian-Low theorem tells us that, if $g \in L^2(\mathbb{R})$ is such that the sequence $G(g, 1, 1)$ is a frame (a Riesz basis, actually), then not only do we have the bound given in inequality (91) (which we have for all $L^2$-functions), but the left hand side of that inequality must actually be infinite. Thus, the generator $g$ of a Gabor Riesz basis must in a sense maximize uncertainty.

Observe also that the property of the Fourier transform is that it interchanges the roles of smoothness and decay. As a rule of thumb smoothness of $f$ implies a decay of $\hat{f}$ and vise versa (cf. Lemma 1.2.3 in [73]). If $g$ decays fast at infinity then we should have $\int_{-\infty}^{\infty} |xg(x)|^2 dx < \infty$. For example, if $g$ is bounded and satisfies $|g(x)| \leq C \frac{1}{|x|^p}$ with $p > \frac{3}{2}$ for $x$ large enough, then $\int_{-\infty}^{\infty} |xg(x)|^2 dx$ will be finite. If such a function $g$ is a generator of a Gabor Riesz basis, then the Balian-Low theorem forces $\int_{-\infty}^{\infty} |\xi \hat{g}(\xi)|^2 d\xi = \infty$ which shows that $\hat{g}$ does not decay well and hence $g$ cannot be very smooth. (Think about $\chi_{[0,1]}$.)

At the end we observe that an immediate consequence of the Balian-Low theorem is that $\phi(x) = e^{-\pi x^2}$ cannot be a Gabor generator for a Riesz basis. This is simply because $\hat{\phi} = \phi$ ad $\int_{-\infty}^{\infty} |x\phi(x)|^2 dx < \infty$.

*Concluding remarks.* For more properties of the Zak transform we refer the reader to Section 11.6. in [81]. We should also mention the amalgam version of the Balian-Low theorem (Theorem 11.33 in [81]). The first publication of the Balian-Low theorem in the form presented here contained a technical gap that was closed by Daubechies in [58]. Our proof is adopted from [84]. The short discussion on the Heisenberg-Pauli-Weyl theorem is borrowed from [73].

**Exercise 5.5.13.** Show that there are elements of $L^2(Q)$ that are not of the form $f(\xi)Zg(x, \xi)$, where $f$ is from $\text{Ł}^2([0, 1])$ and $g$ is some fixed function from $L^2(\mathbb{R})$.

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